Z. Wahrscheinlichkeitstheorie verw. Gebiete
47, 195–203 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On the Probability of a Symmetric Stable Process Crossing a Bottom Outer Boundary*

Michael J. Wichura
Department of Statistics, University of Chicago, Chicago, Illinois 60637, USA

§1. Introduction and Summary

Let $X=(X_t)_{t\geqq 0}$ be the standard symmetric stable process in R^N of index α, with $0<\alpha\leqq 2$. That is, X has stationary independent increments whose continuous densities, relative to Lebesgue measure in R^N, are given by the Fourier inversion formula

$$p(x;u)=\frac{1}{(2\pi)^N}\int_{R^N} e^{-i\langle x,\xi\rangle}\exp(-u|\xi|^\alpha)\,d\xi, \qquad x\in R^N,\ u>0; \tag{1.1}$$

moreover, $X_0=0$, and X has sample paths which are right continuous and have left limits everywhere. For $\alpha=2$, $X_{\cdot/2}$ is standard N-dimensional Brownian motion. We write $R=(R_t)_{t\geqq 0}$ for the corresponding radial process:

$$R_t=|X_t|. \tag{1.2}$$

It is well known (confer Fristedt (1974)) that X is point recurrent if $\alpha>N$, neighborhood recurrent but not point recurrent if $\alpha=N$, and transient if $\alpha<N$. For the most part we shall be dealing with the non point recurrent case, $\alpha\leqq N$. When $\alpha<N$, with probability one, the sample paths of X wander off to ∞ as

* Support for this research was provided in part by National Science Foundation Grant No. MCS76-81435 and by U.S. Energy Research and Development Administration Contract No. EY76-S-02-2751.*000. By acceptance of this article, the publisher and/or recipient acknowledges the U.S. Government's right to retain a non-exclusive, royalty-free license in and to any copyright covering this paper

Notice. This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the U.S. Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights

0044-3719/79/0047/0195/$01.80

$t \to \infty$. When $\alpha = N$, with probability one, the sample paths have the interesting property of coming arbitrarily close to 0 as $t \to \infty$ without ever actually returning to 0. These path properties are elaborated on by the following result, which, from the point of view of almost sure convergence, gives a definitive answer to the question of how slowly X can escape to ∞, or approach 0, as the case may be:

Theorem 1.0. *Let $R = (R_t)_{t \geqq 0}$ be the radial part of the standard symmetric stable process in R^N of index α, with $0 < \alpha \leqq \min(2, N)$. Let*

$$g: t \leadsto (t/\psi(t))^{1/\alpha}$$

be a function such that $\psi(t)$ is nondecreasing in t. Then

$$P\{R_t \leqq g(t) \quad \text{infinitely often as } t \to \infty\} = {0 \atop 1} \tag{1.3}$$

according to whether

$$\int^{\infty} h_{\alpha,N}(t, \psi(t))\, dt \; {< \infty \atop = \infty}; \tag{1.4}$$

here

$$1/h_{\alpha,N}(t, \psi(t)) = \begin{cases} t(\psi(t))^{N/\alpha - 1}, & \text{if } \alpha < N \\ t \log(\psi(t)), & \text{if } \alpha = N. \end{cases} \tag{1.5}$$

Theorem (1.0) is due to Dvoretzky and Erdös (1951) for $\alpha = 2 < N$, to Spitzer (1958) for $\alpha = 2 = N$, to Takeuchi (1964a) for $\alpha < \min(2, N)$, and to Takeuchi and Watanabe (1964) for $\alpha = 1 = N$. The case $\alpha = 2 \leqq N$ is treated also by Motoo (1959). A function g for which the probability in (1.3) is 0 (resp. 1) is called a bottom outer (resp. inner) bound on R. Now to say that g is a bottom outer bound is just to say

$$\rho_g(t) = P\{R_u \leqq g(u) \text{ for some } u \geqq t\} \to 0 \tag{1.6}$$

as $t \to \infty$. The main result of this paper gives the rate of convergence in (1.6):

Theorem 1.1. *Let R be the radial part of the standard symmetric stable process in R^N of index α, with $0 < \alpha \leqq \min(2, N)$. Put*

$$L_{\alpha,N} = K_{\alpha,N}\, \nu_{\alpha,N}, \tag{1.7}$$

with

$$K_{\alpha,N} = \frac{\Gamma\left(\frac{N}{\alpha}\right)}{2^{N-\alpha}\Gamma^2\left(\frac{N-\alpha}{2} + 1\right)} \tag{1.8}$$

and

$$\nu_{\alpha,N} = \begin{cases} \dfrac{N-\alpha}{\alpha}, & \text{if } \alpha < N \\ 1, & \text{if } \alpha = N. \end{cases} \tag{1.9}$$

Let

$$g: t \rightsquigarrow (t/\psi(t))^{1/\alpha}$$

be a function such that the integral in (1.4) is convergent, and for which one has

$$\lim_{s,t\to\infty,\, t/s\to 1} \frac{\xi(t)}{\xi(s)} = 1, \qquad \text{with } \xi(t) = \begin{cases} \psi(t), & \text{if } \alpha < N \\ \log(\psi(t)), & \text{if } \alpha = N. \end{cases} \tag{1.10}$$

Then as $t \to \infty$

$$\rho_g(t) = P\{R_u \leqq g(u) \text{ for some } u \geqq t\} = (1+o(1))\, I_g(t), \tag{1.11}$$

where

$$I_g(t) = L_{\alpha,N} \int_t^\infty h_{\alpha,N}(u, \psi(u))\, du, \tag{1.12}$$

h being defined by (1.5).

For example, in the transient case, one has

$$P\{R_u \leqq u^{1/\alpha}/u^\zeta \text{ for some } u \geqq t\} \sim \frac{K_{\alpha,N}}{\alpha\zeta} \frac{1}{t^{(N-\alpha)\zeta}}$$

for each $\zeta > 0$. Theorem 1.1 provides the following heuristic interpretation of the integral test of Theorem 1.0. For any bottom bound g, be it outer or inner, one can conceive of the tail integrals $I_g(t)$ of (1.12) as trying to approximate the tail probabilities $\rho_g(t)$ of (1.11). One then has the following string of implications: g is an outer bound $\Leftrightarrow \rho_g(t) \to 0 \Leftrightarrow I_g(t) \to 0 \Leftrightarrow$ the integral in (1.4) is finite. That a similar phenomenon takes place in connection with most other so-called strong forms of the iterated logarithm is one of the themes of Wichura (1979). In the case $\alpha = 2$, Kono (1975) has recently generalized Theorem 1.0 in several directions while Robbins and Siegmund (1973) used martingale techniques to obtain exact, but for the most part unwieldy, expressions for $\rho_g(t)$ for a restricted class of g's. Also for $\alpha = 2$, Wichura (1973) obtained (1.11) for a rather narrow class of g's using an elaboration of Motoo's technique (Motoo (1959)). Theorems 1.0 and 1.1 have counterparts for t tending to 0; we shall not dwell on this point (confer Takeuchi (1964b)).

As will be shown in Section 3, Theorem 1.1 follows in a fairly elementary manner from the probabilistic estimates below. For $t < v$ and $\varepsilon > 0$, put

$$H(t, v; \varepsilon) = P\{R_u \leqq \varepsilon^{1/\alpha} \text{ for some } t \leqq u \leqq v\}. \tag{1.13}$$

Theorem 1.2. *Let* $0<\alpha\leqq\min(2,N)$, *and let* $K=K_{\alpha,N}$ *and* $\nu=\nu_{\alpha,N}$ *be defined by* (1.8) *and* (1.9). *Then as* $\varepsilon\downarrow 0$, *t and v remaining fixed, one has*

$$H(t,v;\varepsilon)=(1+o(1))\,K\frac{1/t^\nu-1/v^\nu}{1/\varepsilon^\nu} \tag{1.14}$$

for $\alpha<N$, *and*

$$H(t,v;\varepsilon)=(1+o(1))\,K\frac{\log(1/t)-\log(1/v)}{\log(1/\varepsilon)} \tag{1.15}$$

for $\alpha=N$.

Using Laplace transform techniques, Spitzer (1958) established (1.15) for $\alpha=2$, and Takeuchi and Watanabe (1964) established it for $\alpha=1$. A similar approach may be made to yield (1.14) for $\alpha=2$; however when $\alpha<2$ the necessary estimates do not seem to be available (confer Takeuchi and Watanabe (1964), p. 209). In Section 2 we shall derive (1.14) (for all $\alpha<N$) using a result of Port (1967).

§ 2. Proof of Theorem 1.2

We embark now on the proof of Theorem 1.2. As remarked in the introduction, only the transient case (1.14) need be treated here. Accordingly, suppose until further notice that

$$0<\alpha\leqq 2, \qquad \alpha<N.$$

Put

$$h(t)=h_{\alpha,N}(t)=P\{R_u\leqq 1 \text{ for some } u\geqq t\}.$$

The following lemma identifies the asymptotic behavior of h (and, incidentally, establishes Theorem 1.1 for a horizontal boundary g):

Lemma 2.0. *One has*

$$\lim_{t\to\infty} t^\nu h(t)=K \tag{2.1}$$

with $K=K_{\alpha,N}$ *and* $\nu=\nu_{\alpha,N}$ *being defined by* (1.8) *and* (1.9).

Proof of Lemma 2.0. It follows from Port (1967, p. 162) that the limit in (2.1) is

$$p(0;1)\,C(B)\Big/\left(\frac{N}{\alpha}-1\right),$$

where by (1.1)

$$p(0;1)=\left(\frac{1}{2\pi}\right)^N\frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)}\,\frac{\Gamma\left(\frac{N}{\alpha}\right)}{\alpha}$$

and where

$$C(B) = \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-\alpha}{2}+1\right)\Gamma\left(\frac{\alpha}{2}\right)} \Bigg/ \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^{\alpha} \pi^{N/2}}$$

is the capacity of the unit ball B in R^N relative to the potential kernel for X (cf. Blumenthal and Getoor (1968) p. 71, and Takeuchi (1964a) p. 111). Combining these expressions produces the constant $K_{\alpha,N}$. □

We need one additional pair of estimates. For this set

$$h(t,v) = h_{\alpha,N}(t,v) = P\{R_u \leqq 1 \text{ for some } t \leqq u \leqq v\}.$$

Lemma 2.1. *Let* $0<t<v<w<\infty$ *be given. Then*

(i) $h(t,v) \geqq h(t) - h(v)$,

(ii) $h(t,v) \leqq \dfrac{h(t)-h(w)}{1-h(w-v)}$.

Proof. (i) is immediate. To get (ii), introduce the events

$$A = \{|X_u| \leqq 1 \text{ for some } t \leqq u \leqq v\}, \qquad B = \{|X_z| > 1 \text{ for all } z > w\}$$

and for $t \leqq u \leqq v$, $|y| \leqq 1$ set

$$f(u,y) = P(B \mid X_u = y).$$

For fixed y, f is clearly decreasing in u. For fixed u, f is increasing in $|y|$; this intuitively plausible fact is established in Lemma 2.2. Hence

$$\inf\{f(u,y) : t \leqq u \leqq v, |y| \leqq 1\} = f(v,0).$$

Putting

$$\tau = \inf\{u \geqq t : |X_u| \leqq 1\},$$

we therefore obtain

$$h(t,v) = P(A) \leqq \int_A \frac{f(\tau, X_\tau)}{f(v,0)} \, dP = \frac{P(A \cap B)}{f(v,0)} \leqq \frac{h(t)-h(w)}{1-h(w-v)}. \quad \square$$

We can now establish (1.14). From the scaling property $c^{1/\alpha} R_{\cdot/c} \stackrel{\mathscr{D}}{=} R$, we get

$$H(t,v;\varepsilon) = P\{R_u \leqq \varepsilon^{1/\alpha} \text{ for some } t \leqq u \leqq v\} = H(t/c, v/c; \varepsilon/c) \tag{2.2}$$

for any $c>0$. In particular

$$H(t,v;\varepsilon) = h(t/\varepsilon, v/\varepsilon)$$

and Lemmas 2.0 and 2.1 provide the estimates

$$(1+o(1))\, K\, \varepsilon^{\nu}(1/t^{\nu} - 1/v^{\nu}) \leqq H(t,v;\varepsilon) \leqq \frac{(1+o(1))\, K\, \varepsilon^{\nu}(1/t^{\nu} - 1/w^{\nu})}{(1-(1+o(1))\, K\, \varepsilon^{\nu}/(w-v)^{\nu})},$$

valid for any $w > v$. (1.14) follows upon first letting $\varepsilon \downarrow 0$, then $w \downarrow v$.

The following result was appealed to in the course of proving Lemma 2.1, and will be needed again in the next section. Here we do not require $\alpha < N$.

Lemma 2.2. *Let* $0<\alpha\leqq 2$, $N\geqq 1$. *Let* $0\leqq t<w\leqq\infty$, *let* $r>0$, *and for* $y\in R^N$ set

$$F(y)=F_{t,w;r}(y)=P\{|X_v+y|\leqq r \text{ for some finite } v \text{ satisfying } t\leqq v\leqq w\}.$$

Then F *is a decreasing function of* $|y|$.

Proof. Put $u=w-t$ and note that

$$F(y)=E(\Phi(y+X_t))=\int\Phi(y+z)\,p(z;t)\,dz$$

where $p(\cdot;t)$ is the density of X_t and $\Phi(x)=F_{0,u;r}(x)$. It is well known (cf. Blumenthal and Getoor (1968) p. 19) that $p(\cdot;t)$ is a mixture of spherically symmetric multivariate normal densities, and is therefore spherically symmetric and decreasing along rays from the origin; in particular it is unimodal in the sense of Anderson (1955). The function Φ is clearly spherically symmetric; that it is also decreasing in $|x|$ follows from the scaling property of X:

$$\begin{aligned}\Phi(x_1)&=P\{|X_s+x_1|\leqq r \text{ for some } s\leqq u\}\\&=P\{|X_s+x_2|\leqq r\,\rho \text{ for some } s\leqq u\,\rho^\alpha\}\geqq\Phi(x_2)\end{aligned}$$

for $\rho=|x_2|/|x_1|\geqq 1$. The desired conclusion follows from Anderson (1955). □

§ 3. Proof of Theorem 1.1

It will be convenient to introduce the function $\kappa=\kappa_{\alpha,N}$ defined for $x>0$ by

$$\kappa(x)=\begin{cases}x^\nu, & \text{if } \alpha<N\\ \log(x), & \text{if } \alpha=N\end{cases}$$

where $\nu=\nu_{\alpha,N}$ is given by (1.9). In terms of κ we may restate Theorem 1.2 as

$$H(t,u;\varepsilon)\sim K_{\alpha,N}\frac{(\kappa(1/t)-\kappa(1/u))}{\kappa(1/\varepsilon)} \tag{3.1}$$

as $\varepsilon\downarrow 0$. Set

$$\mathscr{I}_g(\tau)=\int_\tau^\infty\frac{1}{t\,\kappa(\psi(t))}\,dt=I_g(\tau)/L_{\alpha,N} \tag{3.2}$$

(confer (1.12)).

We will prove Theorem 1.1 under a growth condition on ψ which is slightly weaker than (1.10), namely

$$\lim_{c\downarrow 1}\limsup_{\tau\uparrow\infty}V_{\tau,c}(\psi)=1, \tag{3.3}$$

where

$$V_{\tau,c}(\psi)=V_{\tau,c}=\sup\left\{\frac{\max(\kappa(\psi(s)),\kappa(\psi(t)))}{\min(\kappa(\psi(s)),\kappa(\psi(t)))}:\ s,t\geqq\tau,\ 1\leqq t/s\leqq c\right\}.$$

Suppose henceforth that (3.3) holds. Since we are assuming $\int^{\infty} \frac{1}{t\,\kappa(\psi(t))}\,dt<\infty$, we necessarily have

$$\psi(t)\to\infty \quad \text{as} \quad t\to\infty. \tag{3.4}$$

To avoid trivialities we assume $\psi(t)>1$ for all t of interest.

A. The Upper Bound

We will first derive an asymptotic upper bound on

$$\rho_g(\tau)=P\{R_t\leqq(t/\psi(t))^{1/\alpha} \text{ for some } t\geqq\tau\}.$$

Let τ be given. Let $c>1$ and set

$$n_k=\tau\,c^k, \qquad k\geqq 0.$$

For $k\geqq 1$, denote the interval $[n_{k-1},n_k]$ by J_k, and set

$$\gamma_k=\inf\{\psi(t)\colon t\in J_k\}, \qquad \Gamma_k=\sup\{\psi(t)\colon t\in J_k\}. \tag{3.5}$$

Notice

$$1\leqq\sup\left\{\frac{\kappa(\Gamma_k)}{\kappa(\gamma_k)}\colon k\geqq 1\right\}\leqq V_{\tau,c}.$$

Clearly

$$\rho_g(\tau)\leqq\sum_{k=1}^{\infty}P\{R_t\leqq(n_k/\gamma_k)^{1/\alpha} \text{ for some } t\in J_k\}$$

$$=\sum_{k=1}^{\infty}H(n_{k-1},n_k;n_k/\gamma_k)=\sum_{k=1}^{\infty}H(1/c,1;1/\gamma_k),$$

the last step holding by the scaling property (2.2). From (3.4) it follows that $1/\gamma_k\to 0$ uniformly in k as $\tau\to\infty$, so by (3.1)

$$H(1/c,1;1/\gamma_k)=(1+o(1))\,K\frac{\kappa(c)-\kappa(1)}{\kappa(\gamma_k)}$$

uniformly in k as $\tau\to\infty$. Accordingly as $\tau\to\infty$,

$$\rho_g(\tau)\leqq(1+o(1))\,V_{\tau,c}\,K\frac{\kappa(c)-\kappa(1)}{1-1/c}\sum_{k=1}^{\infty}\frac{1}{\kappa(\Gamma_k)}\,\frac{n_k-n_{k-1}}{n_k}$$

$$\leqq(1+o(1))\,V_{\tau,c}\,K\frac{\kappa(c)-\kappa(1)}{(c-1)/c}\,\mathscr{I}_g(\tau).$$

Upon letting $\tau\uparrow\infty$ and then $c\downarrow 1$, we obtain

$$\rho_g(\tau)\leqq(1+o(1))\,\kappa'(1)\,K\,\mathscr{I}_g(\tau) \tag{3.6}$$

with

$$\kappa'(1)=\begin{cases}\nu, & \text{if } \alpha<N\\ 1, & \text{if } \alpha=N.\end{cases}$$

Notice $\kappa'(1)K=L_{\alpha,N}$ (confer (1.7)).

B. The Lower Bound

We will now show that the right hand side of (3.6) is also an asymptotic lower bound on $\rho_g(\tau)$. For this let τ be given. Let $1<b<c$, and define times n_k, $k\geqq 0$, inductively by the rule $n_0=\tau$, and

$$n_{k+1}=\begin{cases}c\,n_k, & \text{if } k \text{ is even}\\ b\,n_k, & \text{if } k \text{ is odd}\end{cases}.$$

Throughout the rest of the argument, restrict the indices k and l to *even* values. For $k\geqq 0$, let J_k now denote the interval $[n_k, n_{k+2}]$; define γ_k and Γ_k by (3.5). One has

$$\begin{aligned}&\{R_t\leqq(t/\psi(t))^{1/\alpha} \text{ for some } t \text{ in } [n_k, n_{k+1}]\}\\ &\quad\supseteq\{R_t\leqq(n_k/\Gamma_k)^{1/\alpha} \text{ for some } t \text{ in } [n_k, n_{k+1}]\};\end{aligned}$$

call this last event A_k. We will use the elementary inequality

$$\rho_g(\tau)\geqq P(\bigcup_{k=0} A_k)\geqq\sum_{k=0}^{\infty}[P(A_k)(1-\sum_{l>k}P(A_l|A_k))]. \tag{3.7}$$

We begin by estimating the sum of conditional probabilities. It follows from Lemma 2.2 and the method of argument employed in the proof of Lemma 2.1 that

$$\begin{aligned}P(A_l|A_k)&\leqq H(n_l-n_{k+1}, n_{l+1}-n_k;\, n_l/\Gamma_l)\leqq H(1-1/b, c; 1/\Gamma_l)\\ &=(1+o(1))\,K\frac{n_{l+2}-n_l}{n_{l+2}\,\kappa(\Gamma_l)}\,\frac{\kappa(1/(1-1/b))-\kappa(1/c)}{1-1/(b\,c)},\end{aligned}$$

whence

$$\sum_{l>k}P(A_l|A_k)=O(\mathscr{I}_g(n_{k+2}))=o(1)$$

uniformly in k, as $\tau\to\infty$. This and (3.7) imply

$$\rho_g(\tau)\geqq(1+o(1))\sum_{k=0}^{\infty}P(A_k).$$

But

$$\begin{aligned}P(A_k)&=H(n_k, n_{k+1}; n_k/\Gamma_k)=H(1, c; 1/\Gamma_k)\\ &\geqq(1+o(1))\frac{K}{V_{\tau,cb}}\,\frac{\kappa(1)-\kappa(1/c)}{b\,c-1}\,\frac{n_{k+2}-n_k}{n_k\,\kappa(\gamma_k)},\end{aligned}$$

whence

$$\rho_g(\tau) \geqq (1+o(1)) \frac{K}{V_{\tau,cb}} \frac{\kappa(1)-\kappa(1/c)}{(b-1/c)\,c} \mathscr{I}_g(\tau);$$

the $o(1)$ term here tends to 0 as $\tau \to \infty$, b and c remaining fixed. Upon taking limits first as $\tau \uparrow \infty$, then as $b \downarrow 1$, and finally as $c \downarrow 1$, we get the desired

$$\rho_g(\tau) \geqq (1+o(1))\, K\, \kappa'(1)\, \mathscr{I}_g(\tau).$$

References

1. Anderson, T.W.: The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Amer. Math. Soc. **6**, 170–176 (1955)
2. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968
3. Dvoretzky, A., Erdös, P.: Some problem on random walk in space. Proc. Second Berkeley Sympos. Math. Statist. Probab. 353–367. Univ. California Press (1951)
4. Fristedt, B.: Sample functions of stochastic processes with stationary independent increments. In Advances in Probability and Related Topics **3**, 241–384. New York: Dekker 1974
5. Kono, N.: Sur la minoration asymptotique et le caractère transitoire des trajectoires des fonctions aléatoires gaussiennes à valuers dans R^d. Z. Wahrscheinlichkeitstheorie verw. Gebiete **33**, 95–112 (1975)
6. Motoo, M.: Proof of the law of iterated logarithm through diffusion equation. Ann. Inst. Statist. Math. **10**, 21–28 (1959)
7. Port, S.: Hitting times for transient stable processes. Pacific J. Math. **21**, 161–165 (1967)
8. Robbins, H., Siegmund, D.: Statistical tests of power one and the integral representation of solutions of certain partial differential equations. Inst. Math. Academia Sinica **1**, 93–120 (1973)
9. Spitzer, F.: Some theorems concerning 2-dimensional Brownian motion. Trans. Amer. Math. Soc. **87**, 187–197 (1958)
10. Takeuchi, J.: On the sample paths of the symmetric stable processes in space. J. Math. Soc. Japan **16**, 109–127 (1964)
11. Takeuchi, J.: A local asymptotic law for the transient stable process. Proc. Japan Acad. **40**, 141–144 (1964b)
12. Takeuchi, J., Watanabe, S.: Spitzer's test for the Cauchy process on the line. Z. Wahrscheinlichkeitstheorie verw. Gebiete. **3**, 204–210 (1964)
13. Wichura, M.J.: Boundary crossing probabilities associated with Motoo's law of the iterated logarithm. Ann. Probability **1**, 437–456 (1973)
14. Wichura, M.J.: Convergence rates for laws of the iterated logarithm. [To appear 1979]

Received January 4, 1978; in revised form May 4, 1978