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A Central Limit Theorem for Trigonometric Series with Small Gaps

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1. Introduction

It is well known that lacunary subsequences of the trigonometric system behave like independent random variables. For instance, a theorem of Salem and Zygmund (see [4]) states that if

$$n_{k+1}/n_k \geqq q > 1 \tag{1}$$

then $\cos 2\pi n_k x$ obeys the central limit theorem i.e.

$$\lim_{N\to\infty} \lambda \left\{ x: \sum_{k=1}^{N} \cos 2\pi n_k x < t\sqrt{N/2} \right\} = \Phi(x).$$

(Here λ is the Lebesgue measure on [0, 1].) A sharpening of this result (see Erdös [2]) states that $\cos 2\pi n_k x$ satisfies the central limit theorem also in the case when we replace (1) by the weaker condition

$$n_{k+1}/n_k \geqq 1 + c_k/\sqrt{k} \qquad c_k \to \infty. \tag{2}$$

And this latter condition cannot be weakened any further: for any constant $c>0$ there is a sequence $\{n_k\}$ of integers such that $n_{k+1}/n_k \geqq 1 + c/\sqrt{k}$ and the sequence $\cos 2\pi n_k x$ does not satisfy the central limit theorem. Consequently, (2) is the optimal growth condition for the central limit theorem. This does not mean, however, that in the absence of (2) the central limit theorem never holds. From a result of Salem and Zygmund ([5], Theorem (3.1.1)) it follows easily that if $\{\varepsilon_n\}$ are independent r.v.'s taking the values ± 1 with probability $1/2 - 1/2$ each and $n_1(\omega) < n_2(\omega) < \dots$ denote the indices n such that $\varepsilon_n = +1$ then the sequence $\cos 2\pi n_k x$ obeys the central limit theorem with probability one. Since $\lim_{k\to\infty} n_k/k = 2$ a.s. by the law of large numbers, we see that there are sequences $\{n_k\}$ with $n_k = O(k)$ such that $\cos 2\pi n_k x$ obeys the central limit theorem. The growth order $n_k = O(k)$ is much slower than that dictated by (2) (it is easily seen that (2) implies $n_k/e^{\sqrt{k}} \to \infty$) and of course it is the slowest possible (except the constant in O)

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since $n_k \geqq k$ holds for any strictly increasing sequence $\{n_k\}$ of integers. As to the size of the "gaps" $n_{k+1}-n_k$, however, the situation is different. For the sequence $\{n_k\}$ above we have $\overline{\lim_{k\to\infty}}\,(n_{k+1}-n_k)/\log k=1$ a.s. by the well known "pure heads" theorem of Erdös and Rényi (see [3]). Our purpose is to show that there is a sequence $\{n_k\}$ such that $\cos 2\pi n_k x$ obeys the central limit theorem and the gaps $n_{k+1}-n_k$ are of smaller order of magnitude than $\log k$, and in fact these gaps can grow as slowly as desired. More exactly we shall prove the following

Theorem. *Let $f(k)\to\infty$ be any function. Then there is a strictly increasing sequence $\{n_k\}$ of positive integers such that $n_{k+1}-n_k=O(f(k))$ and the sequence $\cos 2\pi n_k x$ satisfies the central limit theorem.*

Whether there exists a sequence $\{n_k\}$ such that $n_{k+1}-n_k$ is bounded and $\cos 2\pi n_k x$ obeys the central limit theorem we were unable to decide.

The proof of our theorem depends also on a random construction for the sequence $\{n_k\}$. To give an explicit (nonrandom) construction producing the same effect seems to be much harder. In [1] we constructed a large class of sequences $\{n_k\}$ growing slower than $\exp((\log k)^3)$ such that $\cos 2\pi n_k x$ obeys the central limit theorem but the problem of constructing an explicit sequence $n_k=O(k^r)$ ($r>0$ is a constant) with the same property remains open.

It can be asked if there is a sequence $\{n_k\}$ of integers for any given $f(k)\to\infty$ such that $n_{k+1}-n_k=O(f(k))$ and $\cos 2\pi n_k x$ obeys the law of the iterated logarithm. The answer is in the affirmative; indeed, the random sequence $\{n_k\}$ constructed in the proof of our theorem will be such that $\cos 2\pi n_k x$ obeys the law of the iterated logarithm with probability one. (This will follow trivially from the classical law of the iterated logarithm and Fubini's theorem; see the Remark at the end of our paper.)

2. Proof of the Theorem

Without loss of generality we may assume that $f(k)$ is positive, non-decreasing, integer-valued and $f(k+1)\leqq 2f(k)$. Let

$$U_1=\{j\colon 1\leqq j\leqq f(1)\},$$
$$U_2=\{j\colon f(1)<j\leqq f(1)+f(2)\},\ \ldots,$$
$$U_k=\{j\colon f(1)+\ldots+f(k-1)<j\leqq f(1)+\ldots+f(k)\},\ \ldots.$$

Let $n_1, n_2, \ldots$ be independent random variables on some probability space $(\Omega, \mathcal{F}, P)$ such that n_1 is uniformly distributed over U_1 (i.e. it has its values from U_1, each element of U_1 having the same probability), n_2 is uniformly distributed over U_2 etc. Then evidently

$$n_{k+1}-n_k\leqq f(k+1)+f(k)\leqq 2f(k+1)\leqq 4f(k).$$

We are going to show that, for almost all $\omega\in\Omega$, the sequence $\cos 2\pi n_k(\omega)x$ obeys the central limit theorem. Let

$$\lambda_k(x)=E(\cos 2\pi n_k(\omega)x)=\frac{1}{f(k)}\sum_{j\in U_k}\cos 2\pi jx \tag{3}$$

$$\varphi_k(x)=\cos 2\pi n_k x-\lambda_k(x)\,^1. \tag{4}$$

Clearly, $\lambda_k(x)=a_k b_k$ where $a_k=f(k)^{-1}$, $b_k=\sum_{j\in U_k}\cos 2\pi jx$ and here $a_k\to 0$ decreasingly and the partial sums of b_k remain bounded for all $x\in(0,1)$ $\left(\text{since } \sup_{k\geqq 1}\left|\sum_{j=1}^{k}\cos 2\pi jx\right|<\infty \text{ for } x\in(0,1)\right)$. Hence, using summation by parts we see that $\sum_{k=1}^{\infty}\lambda_k(x)=\sum_{k=1}^{\infty}a_k b_k$ is convergent for $x\in(0,1)$ and thus, for any fixed $\omega\in\Omega$, $\cos 2\pi n_k x$ obeys the central limit theorem if $\varphi_k(x)$ does and conversely. Let

$$I_N=I_N(\lambda,\omega)=\int_0^1\exp\left(\frac{i\lambda}{\sqrt{N/2}}\sum_{k=1}^{N}\varphi_k(x)\right)dx$$

$$J_N=J_N(\lambda,\omega)=\int_0^1\prod_{k=1}^{N}\left(1+\frac{i\lambda}{\sqrt{N/2}}\varphi_k(x)\right)dx.$$

We are going to show that $J_{N^3}\to 1$ a.s. for every fixed $-1\leqq\lambda\leqq 1$ and also that

$$I_N=e^{-\lambda^2/2}J_N+o(1)\qquad (N\to\infty) \tag{5}$$

for any fixed λ, ω. These two statements, together with Fubini's theorem, imply that T_{N^3} is asymptotically normal (as $N\to\infty$) for almost all $\omega\in\Omega$ where

$$T_N=\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\varphi_k(x).$$

From the last fact we immediately get the asymptotic normality of the whole sequence T_N since, for every fixed ω, the $L_2(0,1)$ norm of $|T_M-T_{N^3}|$ is $\leqq 12/\sqrt{N}$ for $N^3\leqq M\leqq(N+1)^3$ by an easy calculation (using the fact that the φ_k's are orthogonal over $[0,1]$ (which is evident from (3), (4)) and $|\varphi_k(x)|\leqq 2$).

To see (5) we fix ω and expand the integrand of I_N by using $\exp z=(1+z)\exp(z^2/2+o(z^2))$ $(z\to 0)$ to get

$$I_N=\int_0^1\prod_{k=1}^{N}\left(1+\frac{i\lambda}{\sqrt{N/2}}\varphi_k(x)\right)\exp\left\{-(1+o(1))\frac{\lambda^2}{N}\sum_{k=1}^{N}\varphi_k^2(x)\right\}dx \tag{6}$$

where the $o(1)$ is uniform in $0\leqq x\leqq 1$ and $-1\leqq\lambda\leqq 1$. It is easy to see (using e.g. the orthogonality of the sequence $\{\cos^2 2\pi nx-1/2\}$ and the Rademacher-Mensov convergence theorem) that $N^{-1}\sum_{k=1}^{N}\cos^2 2\pi n_k x\to 1/2$ for a.e. $x\in(0,1)$

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[1] In the sequel we shall not indicate the variable ω in n_k or $\varphi_k(x)$. We also emphasize here – although it will be clear from the formulas – that the symbol E (for expectation) will always be meant with respect to ω

and hence by the closeness of $\cos 2\pi n_k x$ and $\varphi_k(x)$ we also have $N^{-1} \sum_{k=1}^{N} \varphi_k^2(x) \to 1/2$ a.e. which shows that the exponential in the integral (6) tends to $e^{-\lambda^2/2}$ for a.e. $x \in (0, 1)$. Since this exponential is $\leqq 1$ and the absolute value of the product in the integral (6) is

$$\prod_{k=1}^{N} (1+2\lambda^2 \varphi_k^2(x)/N)^{1/2} \leqq e^{4\lambda^2}$$

by $1+x \leqq e^x$, (5) follows from the dominated convergence theorem.

To show $J_{N^3} \to 1$ a.s. for a fixed $-1 \leqq \lambda \leqq 1$ we compute the mean and variance of J_N. Evidently

$$E(J_N) = \int_0^1 E \prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N/2}} \varphi_k(x)\right) dx = \int_0^1 1\, dx = 1$$

since the random variables $1 + i\lambda\varphi_k(x)/\sqrt{N/2}$ are independent and $E\varphi_k(x) = 0$ for all x. Furthermore, putting $\psi_k(x, y) = E\varphi_k(x)\,\varphi_k(y)$ we get

$$\begin{aligned} E|J_N - 1|^2 &= E(J_N \bar{J}_N) - 1 \\ &= E \int_0^1 \int_0^1 \prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N/2}} \varphi_k(x)\right) \prod_{k=1}^{N} \left(1 - \frac{i\lambda}{\sqrt{N/2}} \varphi_k(y)\right) dx\, dy - 1 \\ &= E \int_0^1 \int_0^1 \prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N/2}} \varphi_k(x) - \frac{i\lambda}{\sqrt{N/2}} \varphi_k(y) + \frac{2\lambda^2}{N} \varphi_k(x)\,\varphi_k(y)\right) dx\, dy - 1 \\ &= \int_0^1 \int_0^1 \prod_{k=1}^{N} \left(1 + \frac{2\lambda^2}{N} \psi_k(x, y)\right) dx\, dy - 1 \end{aligned} \tag{7}$$

where in the last step we exchanged the expectation with the integral and the product sign in the third line of (7) (the terms of the product are clearly independent r.v.'s) and used again $E\varphi_k(x) = E\varphi_k(y) = 0$. Let

$$G_N(x, y) = \sum_{k=1}^{N} \frac{2\lambda^2}{N} \psi_k(x, y)$$

then using $1 + x = \exp(x + O(x^2))$ for $|x| \leqq 1$ (notice $|\psi_k(x, y)| \leqq 4$) we can write the product in the last integral of (7) in the form

$$\begin{aligned} \prod_{k=1}^{N} \left(1 + \frac{2\lambda^2}{N} \psi_k(x, y)\right) &= \exp\left\{G_N(x, y) + N \cdot O\left(\frac{\lambda^4}{N^2}\right)\right\} \\ &= 1 + O(|G_N(x, y)|) + O\left(\frac{1}{N}\right) \end{aligned} \tag{8}$$

where the constants in O are absolute (provided $-1 \leqq \lambda \leqq 1$). By the definition of $\psi_k(x, y)$ we have

$$\psi_k(x, y) = \frac{1}{f(k)} \sum_{j \in U_k} (\cos 2\pi j x - \lambda_k(x))(\cos 2\pi j y - \lambda_k(y)) \tag{9}$$

whence we can see that the system $\{\psi_k(x,y)\}$ is orthogonal over $[0,1]^2$. This fact and $|\psi_k(x,y)| \leqq 4$ show that the integral of $G_N^2(x,y)$ over $[0,1]^2$ is $\leqq 16/N$, hence integrating (8) and using Schwarz's inequality we see that the last integral in (7) is $1+O(N^{-1/2})$ i.e. we have

$$E|J_N - 1|^2 = O\left(\frac{1}{\sqrt{N}}\right)$$

which evidently implies $J_{N^3} \to 1$ a.s. by the Beppo Levi theorem. Q.E.D.

Remark. Using a standard trick (cf. e.g. [5]) one can see immediately that the sequence $\{\varphi_k(x)\}$ (and by the convergence of $\sum_{k=1}^{\infty} \lambda_k(x)$ also the sequence $\{\cos 2\pi n_k x\}$) obeys the law of the iterated logarithm for almost every $\omega \in \Omega$. Indeed, for any fixed $x \in (0,1)$ the ordinary law of the iterated logarithm (for independent r.v.'s) applies to the sequence $\{\varphi_k(x,\omega)\}$; clearly $|\varphi_k(x)| \leqq 2$, $E\varphi_k(x) = 0$ and by an easy calculation we get (using (3), (9) and $\lambda_k(t) \to 0$ for $0<t<1$)

$$\begin{aligned} E\varphi_k^2(x) &= \psi_k(x,x) = f(k)^{-1} \sum_{j \in U_k} \cos^2 2\pi j x + o(1) \\ &= 1/2 + \lambda_k(2x)/2 + o(1) = 1/2 + o(1) \qquad \text{if } x \in (0,1),\ x \neq 1/2. \end{aligned}$$

Hence, for every fixed $x \in (0,1)$, $x \neq 1/2$ the relation

$$\overline{\lim_{N\to\infty}} (N \log\log N)^{-1/2} \sum_{k=1}^{N} \varphi_k(x,\omega) = 1$$

holds for a.s. $\omega \in \Omega$; by Fubini's theorem the last relation also holds a.e. in x for almost all $\omega \in \Omega$, proving our statement above.

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