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A Maximal Inequality for Upcrossings of a Continuous Martingale

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Let M_t be a continuous martingale, with $M_0=0$. Set $M_t^* = \sup_{s \leqq t} |M_s|$, and let $\langle M \rangle_t$ be the increasing process associated with M (the unique increasing process such that $M^2 - \langle M \rangle$ is a local martingale). The classical Burkholder-Gundy inequality states that there exist universal constants c_p, C_p, such that

$$c_p \|M_\infty^*\|_p \leqq \|\langle M \rangle_\infty^{1/2}\|_p \leqq C_p \|M_\infty^*\|_p, \qquad 0<p<\infty. \tag{1}$$

We shall summarise an inequality like (1) by saying that the quantities M_∞^* and $\langle M \rangle_\infty^{1/2}$ are equivalent.

It is natural to look for other functionals of M which are equivalent to M_∞^*. Let $L_t^a(M)$ be the local time of M at a; we may, and shall, take $L_t^a(M)$ to be jointly continuous in (a, t) - see [9]. In [2] it was shown that the quantity $L_\infty^*(M) = \sup_a L_\infty^a(M)$ is equivalent to M_∞^* and $\langle M \rangle_\infty^{1/2}$: we have

$$c_p \|M_\infty^*\|_p \leqq \|L_\infty^*\|_p \leqq C_p \|M_\infty^*\|_p. \tag{2}$$

Now let $U_t(M, a, b)$ be the number of upcrossings made by M across the interval (a, b) in the time interval $[0, t]$, and let

$$U_t^*(M, \varepsilon) = \sup_a U_t(M, a, a+\varepsilon),$$

$$V_t(M) = \sup_{\varepsilon>0} \varepsilon\, U_t^*(M, \varepsilon).$$

In [1] it was shown that, for each $\varepsilon>0$,

$$c_p \|M_\infty^*\|_p - \varepsilon \leqq \|\varepsilon\, U_\infty^*(M, \varepsilon)\|_p \leqq C_p \|M_\infty^*\|_p, \tag{3}$$

so that $\varepsilon\, U_\infty^*(M, \varepsilon)$ is very nearly equivalent to M_∞^*. Also, M. Yor (private communication), proved that, for each a,

$$\|\sup_{\varepsilon>0} \varepsilon\, U_\infty(M, a, a+\varepsilon)\|_p \leqq C_p \|M_\infty^*\|_p. \tag{4}$$

In this paper we show that V_∞ is equivalent to M_∞^*.

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Theorem 1. *Let M be a continuous local martinagale, with $M_0=0$. There exist universal constants c_p, C_p, $0<p<\infty$, such that*

$$c_p \|M^*_\infty\|_p \leqq \|V_\infty(M)\|_p \leqq C_p \|M^*_\infty\|_p. \tag{5}$$

The left hand side of (5) is easy to prove. On the other hand, as $V_\infty(M) + V_\infty(-M)$ is greater than each of M^*_∞, $L^*_\infty(M)$ and $\langle M\rangle^{1/2}_\infty$, the right hand side is stronger than the right hand inequality in each of (1)-(4). For, by Lévy's downcrossing theorem, $L^a_\infty(M)=\lim_{\varepsilon\downarrow 0} \varepsilon U_\infty(M, a, a+\varepsilon)$, so that

$$L^*_\infty(M) \leqq V_\infty(M). \tag{6}$$

As $U_\infty(M, 0, \sup_s M_s-\delta)=1$ for $0<\delta<\sup_s M_s$, we deduce that

$$\sup_s M_s \leqq V_\infty(M), \tag{7}$$

$$M^*_\infty \leqq V_\infty(M)+V_\infty(-M). \tag{8}$$

Finally, $\langle M\rangle_\infty = \int_{-\infty}^{\infty} L^a_\infty(M)\, da \leqq 2 M^*_\infty L^*_\infty(M)$, and therefore, since $L^*_\infty(M) \leqq V_\infty(-M) \wedge V_\infty(-M) \leqq \frac{1}{2}(V_\infty(M)+V_\infty(-M))$, we have

$$\langle M\rangle^{1/2}_\infty \leqq V_\infty(M)+V_\infty(-M). \tag{9}$$

Let B_t be a Brownian motion with $B_0=0$, and let $(\mathscr{F}_t)$ be the usual filtration of B - that is, the usual augmentation of the filtration $\sigma(B_s, s\leqq t)$. Our main tool in the proof of (5) is a decomposition of the path of B, which was also used in [1]. We define the skeleton of B on the grid $2^{-m}\mathbb{Z}$, denoted $B^{(m)}$, as follows.

Set

$$\begin{aligned} \tau_0 &= 0, \\ \tau_{n+1} &= \inf\{t\geqq \tau_n: |B_t - B_{\tau_n}| = 2^{-m}\}, \\ B^{(m)}_r &= B_{\tau_r}. \end{aligned}$$

Thus $B^{(m)}$ is a simple symmetric random walk on $2^{-m}\mathbb{Z}$.

It is intuitively clear, and was proved in Lemma 2.1 of [1] that, conditional on whether $B_{\tau_{n+1}}-B_{\tau_n}$ equals $+2^{-m}$ or -2^{-m}, the path $B_{\tau_n+t}-B_{\tau_n}$, $0\leqq t\leqq \tau_{n+1}-\tau_n$, is independent of the process $B^{(m)}$.

Before stating the precise independence result we will use, we need some notation. Set $\mathscr{G}_m=\sigma(B^{(m)})$. Let $a=k2^{-m}$, $k\in\mathbb{Z}$, let $T_y=\inf\{s\geqq 0: |B_s|=y\}$, $N = U_{T_y}(B, a, a+2^{-m})$, and let $R_1, S_1, R_2, S_2, \dots, R_N, S_N$ be the endpoints of the upcrossings B makes from a to $a+2^{-m}$ before time T_1. Thus we have

$$\begin{aligned} &B_{R_i}=a, \qquad S_i=\inf\{t\geqq R_i: B_t=a+2^{-m}\}, \\ &a-2^{-m}<B_t\leqq a+2^{-m} \qquad \text{for } R_i\leqq t\leqq S_i, \end{aligned}$$

and the process $W^i_t=B_{R_i+t}-B_{R_i}$ is a Brownian motion started at 0, and conditioned to hit 2^{-m} before -2^{-m}. Note that while the S_i are stopping times, the R_i are not. Set

$$K_i^m(a)=L_{S_i}^a(B)-L_{R_i}^a(B)=L_{S_i-R_i}^0(W^i). \tag{10}$$

Since the conditioning of W^i does not affect the value of $L^0(W^i)$, by Tanaka's formula we have $EK_i^m(a)=EL_{S_i-R_i}^0(W^i)=2^{-m}$. Lemma 2.1 of [1] states that W^i is independent of $\mathscr{G}_m$, and thus we have $E(K_i^m(a)|\mathscr{G}_m)=2^{-m}$. Hence, if A is any $\mathscr{G}_m$ measurable random variable with range $2^{-m}\mathbb{Z}$, we have

$$E(K_i^m(A)|\mathscr{G}_m)=2^{-m}, \qquad 1\leqq i\leqq U_{T_1}(B,A,A+2^{-m}). \tag{11}$$

(Note that $U_{T_1}(B,A,A+2^{-m})$ is $\mathscr{G}_m$ measurable.)

Proposition 2. *For each* $x>0$

$$P(V_{T_1}(B)>x)\leqq\frac{1}{x}4\,E\,L_{T_1}^*(B).$$

Proof. Let

$$Z_m=\max_{a\in 2^{-m}\mathbb{Z}} U_{T_1}(B,a,a+2^{-m}),$$

and let A_m be the smallest a at which this maximum is attained: Z_m and A_m are both $\mathscr{G}_m$ measurable. Let $x>0$ be fixed, and let

$$N=\inf\{m\geqq 0\colon Z_m>2^m x\},$$

so that N is a stopping time/$(\mathscr{G}_m)$.

Now

$$\begin{aligned}L_{T_1}^*(B)&\geqq 1_{(N<\infty)}L_{T_1}^{A_N}(B)\\ &\geqq 1_{(N<\infty)}\sum_{i=1}^{Z_N}K_i^N(A_N).\end{aligned}$$

Hence

$$\begin{aligned}E\,L_{T_1}^*(B)&\geqq E\left[\sum_{m=0}^{\infty}1_{(N=m)}\sum_{i=1}^{Z_m}K_i^m(A_m)\right]\\ &=E\left[\sum_{n=0}^{\infty}1_{(N=m)}\sum_{i=1}^{Z_m}E(K_i^m(A_m)|\mathscr{G}_m)\right]\\ &\qquad\text{as } Z_m,\ \{N=m\} \text{ are } \mathscr{G}_m \text{ measurable}\\ &=E\left[\sum_{m=0}^{\infty}1_{(N=m)}Z_m 2^{-m}\right] \quad \text{by (11)}\\ &=E\,1_{(N<\infty)}2^{-N}Z_N\\ &\geqq x\,P(N<\infty).\end{aligned}$$

Thus $P(\sup_m 2^{-m}Z^m>x)\leqq x^{-1}EL_{T_1}^*(B)$. To conclude the proof, we note the pathwise inequalities

$$U^*_{T_1}(B, 2^{-m}) \leqq \max_{a \in 2^{-(m+1)}\mathbb{Z}} U_{T_1}(B, a, a+2^{-(m+1)}),$$

$$V_{T_1}(B) \leqq 2 \sup_{m \geqq 0} 2^{-m} U^*_{T_1}(B, 2^{-m}),$$

which together imply that $V_{T_1}(B) \leqq 4 \sup_{m \geqq 0} 2^{-m} Z_m$.

We now use some standard machinery for proving martingale inequalities to deduce Theorem 1. The following lemma is due to Lenglart, Lepingle and Pratelli [7], and is given here in the slightly improved form used in [3]. Here $\|B_T\|_\infty = \operatorname{ess\,sup} |B_T|$.

Lemma 3. *Let A_t, B_t be two previsible increasing processes, with $A_0 = B_0 = 0$. Suppose there exists $q > 0$, $a > 0$ such that, for every couple of stopping times S, T, with $S \leqq T$,*

$$E(A_T - A_S)^q \leqq a \|B_T\|^q_\infty P(S < T).$$

Then for every moderate function F, there exists a constant C_F depending only on a and F, such that

$$EF(A_T) \leqq C_F EF(B_T) \quad \textit{for all stopping times } T.$$

In particular,

$$\|A_T\|_p \leqq C_p \|B_T\|_p \quad 0 < p < \infty.$$

Proof of Theorem 1. The left hand side of (5) is an immediate consequence of (6) and the left hand side of (2).

To establish the right hand side, it is enough to show that, for any stopping time T,

$$\|V_T(B)\|_p \leqq C_p \|B^*_T\|_p. \tag{12}$$

For, by the Dubins-Schwarz theorem [4], any continuous martingale M is the time change of a stopped Brownian motion, and the quantities $V(M)$ and M^* are preserved under time change.

By scaling the Brownian motion B_t, we have $V_{T_y}(B) = y V_{T_y}(B/y) \sim y V_{T_1}(B)$. By Proposition 2, $E(V_{T_1}(B))^{1/2} = c < \infty$, and therefore $E(V_{T_y}(B))^{1/2} = c y^{1/2}$. Hence, if T is any stopping time, setting $y = \|B^*_T\|_\infty$, we have $T \leqq T_y$, and so $E V_T(B)^{1/2} \leqq c y^{1/2} = c \|B^*_T\|^{1/2}_\infty$.

Now let $S \leqq T$ be two stopping times, $Q = P|_{(S<T)}$, and $\tilde{B}_t = B_{S+t} - B_S$. Then

$$V_T(B) \leqq V_S(B) + V_{T-S}(\tilde{B}) + B^*_T 1_{(S<T)},$$

and so

$$\begin{aligned} E(V_T(B) - V_S(B))^{1/2} &\leqq E[(V_{T-S}(\tilde{B}) + B^*_T)^{1/2} 1_{(S<T)}] \\ &\leqq E^Q(V_{T-S}(\tilde{B})^{1/2}) P(S<T) + \|B^*_T\|^{1/2}_\infty P(S<T) \\ &\leqq (c\|\tilde{B}^*_{T-S}\|^{1/2}_\infty + \|B^*_T\|^{1/2}_\infty) P(S>T) \\ &\leqq (1+2c)\|B^*_T\|^{1/2}_\infty P(S<T) \quad \text{as } \tilde{B}^*_{T-S} \leqq 2 B^*_T. \end{aligned}$$

Applying Lemma 3 we obtain (12), which completes the proof of the Theorem.

Remarks. 1. The proof of Theorem 1 also gives (5) for moderate functions F:

$$c_F EF(M_\infty^*) \leqq EF(V_\infty(M)) \leqq C_F EF(M_\infty^*).$$

2. Although (5) is stronger than the L^* inequality (2), its proof rests on the finiteness of $EL_{T_1}^*(B)$, and proving this is the hard part of the L^* inequality. It may be done either by using the Ray-Knight theorem [6, 8], as in [2], or by using the Garsia-Rodemich-Rumsey lemma [5], as in [3, Cor 5.2.1]. A direct proof of (5) would of course also establish (2), via (6), and it is possible that some variation of the method of Proposition 2 would yield such a proof. For each fixed $a=k2^{-m}$, the process $Y_n^a = 2^{-n} U_{T_1}(B, a, a+2^{-n})$ is a martingale, which converges to $L_{T_1}^a(B)$, and this is essentially the method used by Yor to obtain (3). However, the process $\max_k Y_m^{k2^{-m}}$ is not a martingale, and appears to have quite complicated behaviour.

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