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On the Limiting Behavior of Normed Sums of Independent Random Variables

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1. Introduction

Consider a sequence $X_1, X_2, \ldots$ of independent random variables (rv); define $S_n = \sum_{i=1}^{n} X_i$. This paper is concerned with determining upper and lower bounds on the (almost surely (a.s.) constant) value of $\limsup_{n\to\infty} S_n/C_n$, where $C_n \uparrow \infty$ is a real sequence.

When $E(X_n^2) < \infty$ for every n, it is natural to consider a norming sequence of the form $C_n = (2s_n^2 \operatorname{loglog} s_n^2)^{1/2}$, where $s_n^2 = \operatorname{Var}(S_n)$; indeed, the law of the iterated logarithm (LIL) is said to hold in its classical form if

$$\limsup_{n\to\infty} \frac{S_n - E(S_n)}{(2s_n^2 \operatorname{loglog} s_n^2)^{1/2}} = 1 \text{ a.s.}$$

But strong limit theorems such as the LIL depend (in principle) on probabilities rather than moments. This fact is borne out by a number of published results, among which are those of Feller [7], Klass [10] and [11], Klass and Teicher [12] and Kesten [9] in the independent, identically distributed (i.i.d) case, and those of Martikainen and Petrov [14] and Tomkins [20] in the general independent case. As an illustration, consider a rv X with $P[X = \pm k^k] = Ak^{-3}$ for $k \geqq 1$, where $A = \left(2\sum_{k=1}^{\infty} k^{-3}\right)^{-1}$. Let $\{X_n\}$ be independent rv such that X_n has the same distribution as $XI(|X| \geqq n^n)$, where $I(E)$ denotes the indicator function of an event E. It is easy to see that $E|X_n^r| = \infty$ for every $n \geqq 1$ and every $r > 0$. But $P[X_n \neq 0] = P[|X| \geqq n^n] = 2A \sum_{k=n}^{\infty} k^{-3} = O(n^{-2})$, so $P[X_n \neq 0$ i.o.$] = 0$ by the Borel-Cantelli lemma (as usual, "i.o." means "infinitely often").

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Therefore S_n converges a.s., notwithstanding the fact that all moments are infinite, so $S_n/C_n \to 0$ a.s. for every sequence $C_n \uparrow \infty$.

It is helpful to consider the case in which $X_n/d_n \to 0$ a.s. for some real sequence d_n; by the Borel-Cantelli lemma this is tantamount to assuming $\sum_{n=1}^{\infty} P[|X_n| > \varepsilon d_n] < \infty$ for every $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} X_n I(|X_n| > \varepsilon d_n)$ converges a.s., the choice of $\{C_n\}$ should depend only on $\{Y_j = X_j I(|X_j| \leqq \varepsilon d_j)\}$. Thus it seems preferable to consider $C_n = (2g_n^2 \log\log g_n^2)^{1/2}$, where $g_n^2 = \operatorname{Var}\left(\sum_{j=1}^{n} Y_j\right)$, instead of $(2s_n^2 \log\log s_n^2)^{1/2}$. The difficulty with using s_n instead of g_n stems from the fact that expectations can overinflate the effects of events of low probability, perhaps to the point where $s_n/g_n \to \infty$. For instance, suppose $P[X_n = 1] = 1 - \frac{1}{n^2}$ and $P[X_n = -n^2 + 1] = 1/n^2$ for $n \geqq 1$. Then $E(X_n) = 0$, $E(X_n^2) = n^2 - 1$ so $s_n^2 \sim n^3/3$ (we will write "$a_n^* \sim a_n$" when $a_n^*/a_n \to 1$). Moreover, $P[X_n \neq 1 \text{ i.o.}] = 0$, so $S_n/n \to 1$ a.s. But then $S_n/(2s_n^2 \log\log s_n^2)^{1/2} \sim S_n/((2/3)\, n^3 \log\log n)^{1/2} \to 0$ a.s. Notice that $X_n/d_n \to 0$ a.s. for every sequence $d_n \uparrow \infty$ in this example.

The main results of the paper will be stated in Sects. 2 and 3, and proved in Sect. 4. These theorems, which assume nothing about the existence of any moments of the X_n's, present hypotheses involving only properties of the individual X_n's (rather than those of S_n, as in [20]) under which the value of $\limsup_{n\to\infty} |S_n|/C_n$ or $\limsup_{n\to\infty} S_n/C_n$ may be ascertained.

2. Two-Sided Limit Theorems

This section addresses the problem stated at the beginning of the paper by presenting hypotheses under which bounds on $\limsup_{n\to\infty} |S_n|/C_n$ can be determined. These hypotheses involve only properties of each X_n, but do not require any moments of X_n to be finite. All theorems of this section will be proved in Sect. 4.

The following theorem was motivated by a theorem of Teicher [16].

Theorem 2.1. *Let $X_1, X_2, \ldots$ be a sequence of independent rv and suppose $0 < B_1 \leqq B_2 \leqq \ldots \uparrow \infty$ is a real sequence. For $n \geqq 1$, define $S_n = \sum_{i=1}^{n} X_i$, $b_n^2 = 2 \log\log B_n^2$ and, for $\varepsilon > 0$,*

$$T_n(\varepsilon) = B_n^{-2} \sum_{i=1}^{n} \operatorname{Var}(X_{i,\varepsilon}) \tag{2.1}$$

where

$$X_{n,\varepsilon} \equiv (X_n \vee (-\varepsilon B_n b_n^{-1})) \wedge (\varepsilon B_n b_n^{-1}), \qquad n \geqq 1. \tag{2.2}$$

Define the non-negative numbers $T-$ and $T+$ by

$$T_-^2 = \lim_{\varepsilon \downarrow 0} \liminf_{n\to\infty} T_n(\varepsilon), \qquad T_+^2 = \lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} T_n(\varepsilon). \tag{2.3}$$

Let $\{a_n\}$ be a positive real sequence. Assume $a_n = O(B_n b_n)$,

$$\sum_{n=1}^{\infty} P[|X_n| > a_n] < \infty, \tag{2.4}$$

$$(B_n b_n)^{-1} \sum_{k=1}^{n} E\{X_k I(|X_k| \leqq a_k)\} \to 0 \quad as \quad n \to \infty, \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} (B_n b_n)^{-2\beta} E\{X_n^{2\beta} I(\varepsilon B_n b_n^{-1} < |X_n| \leqq a_n)\} < \infty \qquad for\ every\ \varepsilon > 0 \tag{2.6}$$

and some $\beta > 0$. If $\beta > 1$, assume moreover, that

$$\sum_{k=1}^{\infty} (B_{n_k} b_{n_k})^{-2\beta} \Big(\sum_{n_k \leqq n < n_{k+1}} E\{X_n^2 I(\varepsilon B_n b_n^{-1} < |X_n| \leqq a_n)\}\Big)^{\beta} < \infty \tag{2.6'}$$

for every $\varepsilon > 0$ and some integral sequence $\{n_k\}$ obeying $n_{k+1} = \min\{n: B_n \geqq c B_{n_k}\}$ for some $c > 1$ and all $k \geqq 1$. Then

$$T_- \leqq \limsup_{n \to \infty} \frac{S_n}{B_n b_n} \leqq T_+ \quad \text{a.s.} \tag{2.7}$$

and, if $T_+ < \infty$,

$$S_n/(B_n b_n) \xrightarrow{P} 0. \tag{2.8}$$

(Here, "$\xrightarrow{P}$" denotes convergence in probability). If, moreover,

$$\limsup_{n \to \infty} B_{n+1}/B_n < \infty \tag{2.9}$$

then this theorem remains true with $T_n(\varepsilon)$ replaced by

$$T_n'(\varepsilon) = B_n^{-2} \sum_{i=1}^{n} \operatorname{Var}((X_i \vee (-\varepsilon B_n b_n^{-1})) \wedge \varepsilon B_n b_n^{-1}). \tag{2.10}$$

With $a_n = B_n b_n$ and $\beta = 1$, (2.4) and (2.5) and the definitions of T_- and T_+ are, in a sense, reminiscent of the Degenerate Convergence Criterion ([13], p. 317). In fact, Theorem 2.1 has the following partial converse.

Theorem 2.2. *Let X_n, S_n, B_n, b_n and $T_n(\varepsilon)$ be as given in Theorem 2.1; define T_- and T_+ by (2.3). If $S_n/(B_n b_n) \xrightarrow{P} 0$ and $\Lambda \equiv \limsup_{n \to \infty} |S_n|/(B_n b_n) < \infty$ a.s., then (2.4) and (2.5) hold with $a_n = \delta B_n b_n$ for any $\delta > \Lambda$. If, moreover, (2.6) holds for some $\beta > 0$ and (2.6′) holds if $\beta > 1$, then (2.7) is also true.*

Remark. 1. Theorems 2.1 and 2.2 clearly remain valid for the sequence $\{-X_n\}$.

2. If a sequence $C_n \uparrow \infty$ is given with a view to finding the value of $\limsup_{n \to \infty} S_n/C_n$, one might test the hypotheses of Theorem 2.1 using $B_n = C_n (2 \log\log C_n^2)^{-1/2}$.

3. Theorems 2.1 and 2.2 give some clues in the search for an appropriate choice of the sequence $\{B_n\}$. One approach is to define B_n by the equation $T_n(1) = 1$ or (cf. (2.10)) $T_n'(1) = 1$.

4. Theorem 2.1 uses the truncation $X_{n,\varepsilon}$ defined by (2.2) instead of the simpler truncation $X_n I(|X_n| \leqq \varepsilon B_n b_n^{-1})$ because, unlike $\operatorname{Var}(X_n I(|X_n| \leqq \varepsilon B_n b_n^{-1}))$, $\operatorname{Var}(X_{n,\varepsilon})$ is a non-decreasing function of $\varepsilon > 0$ by Corollary 4 of [1] and, hence, so is $T_n(\varepsilon)$. This fact is crucial to our proof of Theorem 2.1. It will be clear from their proofs (see Sect. 4) that Theorems 2.1 and 2.2 remain true with $T_n(\varepsilon)$ replaced throughout by

$$T_n^*(\varepsilon) = B_n^{-2} \sum_{i=1}^{n} \operatorname{Var}(X_i I(|X_i| \leqq \varepsilon B_i b_i^{-1}))$$

if T_n^* is non-decreasing in ε (in particular, if the X_n's are all symmetrically distributed).

3. Some Asymmetrical Strong Limit Theorems

Theorem 2.1, as noted earlier, applies equally to the sequences $\{-X_n\}$ and $\{X_n\}$; therefore, its usefulness is limited to circumstances in which $\limsup_{n\to\infty} |S_n|/(B_n b_n) < \infty$ a.s. However, Klass and Teicher [12] have shown that it is possible for $\limsup_{n\to\infty} S_n/C_n = 1$ a.s. and $\limsup_{n\to\infty} -S_n/C_n = \infty$ a.s., even for i.i.d. rv with zero means. The results in this section establish sufficient conditions for $\limsup_{n\to\infty} S_n/(B_n b_n) < \infty$ a.s., without making any implications about the limiting behavior of $\{-S_n\}$.

Theorem 3.1. *Let $X_1, X_2, \ldots$ be independent rv and $S_n = \sum_{i=1}^{n} X_i$. Let $a_n \uparrow \infty$ and $B_n \uparrow \infty$ be real sequences and let $b_n^2 = 2 \log\log B_n^2$. For any $\eta > 0$, define the constants $C(\eta)$ and $\alpha_+(\eta) \geqq 0$ as follows:*

$$C(\eta) = \limsup_{n\to\infty} (B_n b_n)^{-1} \sum_{i=1}^{n} E\{(X_i \vee (-\eta B_i b_i^{-1}))\, I(X_i \leqq a_i)\},$$

and

$$\alpha_+(\eta) = \lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} B_n^{-2} \sum_{i=1}^{n} \operatorname{Var}((X_i \vee (-\eta B_i b_i^{-1}))\, I(X_i \leqq \varepsilon B_i b_i^{-1})).$$

Suppose $a_n = O(B_n b_n)$,

$$\sum_{n=1}^{\infty} P(X_n > a_n) < \infty \tag{3.1}$$

and, for some $\beta > 0$,

$$\sum_{n=1}^{\infty} (B_n b_n)^{-2\beta} E\{X_n^{2\beta} I(\varepsilon B_n b_n^{-1} < X_n \leqq a_n)\} < \infty \quad \text{for every } \varepsilon > 0. \tag{3.2}$$

If $\beta > 1$, then assume, moreover, that

$$\sum_{k=1}^{\infty} (B_{n_k} b_{n_k})^{-2\beta} \Big(\sum_{n_k \leqq n < n_{k+1}} E(X_n^2 I(\varepsilon B_n b_n^{-1} < X_n \leqq a_n)\Big)^{\beta} < \infty \quad \text{for every } \varepsilon > 0 \tag{3.2'}$$

where $n_{k+1}=\min\{n: B_n \geqq cB_{n_k}\}$ *for some* $c>1$ *and all* $k \geqq 1$. *Then*

$$\limsup_{n\to\infty} S_n/(B_n b_n) \leqq 2^{-1/2} K_\gamma^* \alpha_+(\eta)+C(\eta) \quad \text{a.s.,} \tag{3.3}$$

where $\gamma=\eta/(2^{1/2}\alpha_+(\eta))$ *and*

$$K_\gamma^* = \min_{b>0}((1+(e^{\gamma b}-1-\gamma b)/\gamma^2)/b). \tag{3.4}$$

If, moreover, $\alpha_+ \equiv \lim_{\eta\downarrow 0} \alpha_+(\eta)<\infty$ *or* $C \equiv \lim_{\eta\downarrow 0} C(\eta)>-\infty$, *then*

$$\limsup_{n\to\infty} S_n/(B_n b_n) \leqq \alpha_+ + C \qquad \text{a.s.} \tag{3.5}$$

A more standard formulation may be desirable. Modifying Theorem 3.1 slightly, simpler truncation conditions can be obtained.

Theorem 3.2. *Assume* $X_1, X_2, \ldots$ *are independent rv and* $S_n=\sum_{i=1}^n X_i$. *Let* $a_n \uparrow \infty$ *and* $B_n \uparrow \infty$ *be real sequences and define* $b_n=(2\operatorname{loglog} B_n^2)^{1/2}$. *Assume* $a_n=O(B_n b_n)$, (3.1) *holds, that* (2.6) *holds for some* $\beta>0$, *and, if* $\beta>1$, *that* (2.6′) *holds. For any* $\eta>0$, *define the constants*

$$C^*(\eta)=\limsup_{n\to\infty}(B_n b_n)^{-1}\sum_{i=1}^n E\{X_i I(-\eta B_i b_i^{-1} \leqq X_i \leqq a_i)\}$$

and

$$\alpha_{++}(\eta)=\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} B_n^{-2}\sum_{i=1}^n E\{X_i^2 I(-\eta B_i b_i^{-1}<X_i \leqq \varepsilon B_i b_i^{-1})\},$$

where B_n *is a real sequence such that* $B_n \uparrow \infty$, *and* $b_n=(2\operatorname{loglog} B_n^2)^{1/2}$.

If $C^*(\eta)>-\infty$ *or* $\alpha_{++}(\eta)<\infty$ *then*

$$\limsup_{n\to\infty} S_n/(B_n b_n) \leqq 2^{1/2}\alpha_{++}(\eta) K_\gamma^* + C^*(\eta) \tag{3.6}$$

where $\gamma=\eta/(2^{1/2}\alpha_{++}(\eta))$ *and* K_γ^* *is defined by* (3.4). *If, in addition,* $C^* \equiv \lim_{n\downarrow 0} C^*(\eta)>-\infty$ *or* $\alpha_{++} \equiv \lim_{\eta\downarrow 0}\alpha_{++}(\eta)<\infty$, *then*

$$\limsup_{n\to\infty} S_n/(B_n b_n) \leqq \alpha_{++} + C^* \quad \text{a.s.} \tag{3.7}$$

4. Proofs of the Theorems

The following lemmas are presented for ease of reference.

Lemma 4.1 (Egorov [3]). *Let* $\{a_n(\varepsilon), n \geqq 1\}$ *be a sequence of non-negative functions defined for all* $\varepsilon>0$. *If* $\sum_{n=1}^\infty a_n(\varepsilon)<\infty$ *for every* $\varepsilon>0$, *then there exists a sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n \downarrow 0$ *and* $\sum_{n=1}^\infty a_n(\varepsilon_n)<\infty$.

Lemma 4.2 (Tomkins [21]). *Let* $\{a_n(\varepsilon)\}$ *be a sequence of non-negative functions, defined for all* $\varepsilon>0$. *Define* $a^*=\liminf_{\varepsilon\downarrow 0}\liminf_{n\to\infty} a_n(\varepsilon)$. *Then there exists a sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n\downarrow 0$ *and* $\liminf_{n\to\infty} a_n(\varepsilon_n)\geqq a^*$. *Moreover, if* $a_n(\varepsilon)$ *is a non-decreasing function of* ε *for each* $n\geqq 1$, *then* $\liminf_{n\to\infty} a_n(\delta_n)\leqq a^*$ *for every real sequence* $\{\delta_n\}$ *satisfying* $\delta_n\downarrow 0$.

Lemma 4.3 (Tomkins [21]). *Let* $(M_n, F_n, n\geqq 1)$ *be a submartingale and let* $\{\alpha_n\}$, $\{B_n\}$ *and* $\{c_n\}$ *be positive real sequences. Suppose* $B_n\uparrow\infty$ *and define* $\alpha=\limsup_{n\to\infty}\alpha_n$, $\gamma=\limsup_{n\to\infty}(\log\log B_n^2)^{1/2} c_n$, *and* $g(x)=x^{-2}(e^x-1-x)$. *Assume* $\alpha<\infty$ *and* $\gamma<\infty$.

If positive numbers C, N *and* T *exist such that*

$$E\exp\{tM_n/(\alpha_n B_n)\}\leqq C\exp\{t^2 g(tc_n)\} \tag{4.1}$$

whenever $n\geqq N$ *and* $0\leqq tc_n<T$, *then*

$$\limsup_{n\to\infty}\frac{M_n}{(B_n^2\log\log B_n^2)^{1/2}}\leqq \alpha K_\gamma \quad \text{a.s.}$$

where

$$K_0=2^{1/2} \quad \text{and} \quad K_\gamma=\min_{0<b\leqq\gamma T^{-1}}(b^{-1}+bg(\gamma b)) \quad \text{for } \gamma>0. \tag{4.2}$$

Lemma 4.4. *Let* $X_1, X_2, \ldots$ *be independent rv with zero means. Suppose* $\{\alpha_n\}$, $\{B_n\}$, *and* $\{c_n\}$ *are real sequences such that, for* $n\geqq 1$, $\sum_{i=1}^n E(X_i^2)\leqq(\alpha_n B_n)^2$ *and* $X_n\leqq c_n\alpha_n B_n$ a.s. *Then* (4.1) *holds with* $C=N=1$, $M_n=\sum_{i=1}^n X_i$ *and every* $T>0$.

Proof. Let $s_n^2=\operatorname{Var}\left(\sum_{i=1}^n X_i\right)$ and $c_n^*=c_n\alpha_n B_n/s_n$. By dint of Lemma 1(i) of Teicher [18],

$$E\exp\{tM_n/s_n\}\leqq\exp\{t^2 g(tc_n^*)\}$$

for all $t>0$ and $n\geqq 1$. Replacing t by $ts_n/(\alpha_n B_n)$,

$$\begin{aligned}E\exp\{tM_n/(\alpha_n B_n)\}&\leqq\exp\{t^2 s_n^2\alpha_n^{-2}B_n^{-2}g(tc_n)\}\\&\leqq\exp\{t^2 g(tc_n)\}\end{aligned}$$

as required. □

Lemma 4.5. *Let* $\{X_n\}$ *be a sequence of independent rv and let* $a_n\uparrow\infty$ *be a real sequence. If* $\sum_{n=1}^\infty P[|X_n|>a_n]<\infty$ *and* $\sum_{i=1}^n X_i/a_n\xrightarrow{P}0$ *then* $a_n^{-1}\sum_{i=1}^n E(X_i I(|X_i|\leqq a_i))\to 0$.

Proof. Let $X_n^*=X_n I(|X_n|\leqq a_n)$. Then the hypotheses and the Borel-Cantelli lemma imply $a_n^{-1}\sum_{i=1}^n X_i^*\xrightarrow{P}0$. But then

$$a_n^{-1}\sum_{i=1}^{n} E(X_i^*)=a_n^{-1}\sum_{i=1}^{n} E(X_i^*(|X_i^*|\leqq a_n))\to 0$$

by the Degenerate Convergence Criterion ([13], p. 317). □

The following strong law of large numbers will be used repeatedly.

Lemma 4.6. *Let $Y_1, Y_2, \ldots$ be independent rv with finite means and define $T_n = \sum_{i=1}^{n} Y_i$. Let $B_n \uparrow \infty$ and $b_n \uparrow$ be real sequences and suppose that $\sum_{n=1}^{\infty} (B_n b_n)^{-2\beta} E|Y_n|^{2\beta} < \infty$ for some $\beta > 0$.*

If (i) *$\beta < 1/2$ and $|Y_n| \leqq K B_n b_n$ a.s. for some $K > 0$, or* (ii) *$1/2 \leqq \beta \leqq 1$ or* (iii) *$\beta > 1$ and $\sum_{k=1}^{\infty} (B_{n_k} b_{n_k})^{-2\beta} (E(T_{m_{k+1}} - T_{m_k})^2)^{\beta} < \infty$, where n_k is any integral sequence such that $n_{k+1} = \min\{n: B_n \geqq c B_{n_k}\}$ for some $c > 1$ and all $k \geqq 1$, $m_k = n_k - 1$, and $b_{m_{k+1}} \sim b_{n_k}$ as $k \to \infty$, then $(T_n - E(T_n))/(B_n b_n) \to 0$ a.s.*

Proof. Under the assumptions of (i), $E|Y_n|/(B_n b_n) \leqq K^{1-2\beta} E|Y_n|^{2\beta}/(B_n b_n)^{2\beta}$; clearly, then, $\sum_{n=1}^{\infty} (B_n b_n)^{-1} E|Y_n| < \infty$ so the hypotheses of (ii) hold with $\beta = 1/2$ when the assumptions in (i) hold. But the desired result follows when $1/2 \leqq \beta \leqq 1$ by a result of Loève ([13], p. 214). It remains only to consider part (iii).

If $\beta > 1$, then $E|Y_n - E(Y_n)|^{2\beta} \leqq 2^{2\beta} E|Y_n|^{2\beta}$ by the c_r-inequality and the Hölder's inequality. Since $E(Y_n - E(Y_n))^2 \leqq E(Y_n^2)$, it is evident that the hypotheses of part (iii) hold with $Y_n - E(Y_n)$ in place of Y_n. Therefore, it can and will be assumed that $E(Y_n) = 0$ in the remainder of the proof.

For brevity, let $I_k = \{n: n_k \leqq n < n_{k+1}\}$, $k \geqq 1$. Then, for any $\varepsilon > 0$,

$$\begin{aligned} P_k &\equiv P[\max_{n\in I_k} |T_n - T_{m_k}| \geqq \varepsilon B_{n_k} b_{n_k}] \\ &\leqq (\varepsilon B_{n_k} b_{n_k})^{-2\beta} E|T_{m_{k+1}} - T_{m_k}|^{2\beta} \quad \text{by Doob's inequality ([2], p. 314)} \\ &\leqq C_\beta (\varepsilon B_{n_k} b_{n_k})^{-2\beta} \{\sum_{n\in I_k} E|Y_n|^{2\beta} + (\sum_{n\in I_k} E(Y_n^2))^{\beta}\} \end{aligned}$$

for some constant C_β (depending only on β) by an inequality of Rosenthal [15]. Since $B_{n_k} > B_n/c$ for $n \in I_k$ and $b_{m_{k+1}}/b_{n_k} \to 1$ by hypothesis, it is clear from (i) and (ii) that $\sum_{k=1}^{\infty} P_k < \infty$. Consequently, the Borel-Cantelli lemma ensures the existence of an integer-valued rv L such that

$$\max_{n\in I_k} |T_n - T_{m_k}| < \varepsilon B_{n_k} b_{n_k} \quad \text{for all } k \geqq L. \tag{4.3}$$

Notice that $B_{n_k} \geqq c B_{n_{k-1}} \geqq \ldots \geqq c^{k-i} B_{n_i}$ for all $i \leqq k$, so

$$\sum_{i=1}^{k} B_{n_i} \leqq B_{n_k} \sum_{i=1}^{k} c^{i-k} \leqq c B_{n_k}/(c-1). \tag{4.4}$$

Now, for $n \in I_k$, where $k \geqq L$,

$$|T_n| = \left| T_{m_L} + \sum_{i=L}^{k-1} (T_{m_{i+1}} - T_{m_i}) + T_n - T_{m_k} \right|$$
$$\leqq |T_{m_L}| + \sum_{i=L}^{k} \varepsilon B_{n_i} b_{n_i} \quad \text{by (4.3)}$$
$$\leqq |T_{m_L}| + \varepsilon c B_{n_k} b_{n_k}/(c-1) \quad \text{by (4.4).}$$

It follows readily that $\limsup_{n\to\infty} |T_n|/(B_n b_n) \leqq \varepsilon c/(c-1)$ a.s. for every $\varepsilon > 0$. The desired conclusion is established. □

Proof of Theorem 2.1. Lemmas 4.1 and 4.2 ensure the existence of real sequences $\{\varepsilon_{n,1}\}$, $\{\varepsilon_{n,2}\}$ and $\{\varepsilon_{n,3}\}$, where $\varepsilon_{n,i} \downarrow 0$ as $n \to \infty$ for $i = 1, 2, 3$, such that

$$\limsup_{n\to\infty} T_n(\varepsilon_{n,1}) = T_-^2, \tag{4.5}$$

$$\sum_{n=1}^{\infty} (B_n b_n)^{-2\beta} E\{X_n^{2\beta} I(\varepsilon_{n,2} B_n b_n^{-1} < |X_n| \leqq a_n)\} < \infty, \tag{4.6}$$

and, if $\beta > 1$,

$$\sum_{k=1}^{\infty} (B_{n_k} b_{n_k})^{-2\beta} \Big(\sum_{n_k \leqq n < n_{k+1}} E\{X_n^2 I(\varepsilon_{n,3} B_n b_n^{-1} < |X_n| \leqq a_n)\}\Big)^{\beta} < \infty \tag{4.6'}$$

in view of hypotheses (2.3), (2.6) and (2.6′). Corollary 4 of Chow and Studden [1] shows that, for each $n \geqq 1$, $T_n(\varepsilon)$ is a non-decreasing function of $\varepsilon > 0$; by dint of Lemma 4.2, therefore, (4.5) remains true with $\varepsilon_n \equiv \max_{1\leqq i \leqq 3} (\varepsilon_{n,i})$ in place of $\varepsilon_{n,1}$. Moreover, the series in (2.6) and (2.6′) are clearly non-increasing functions of ε, so (4.6) and (4.6′) are also still valid with $\varepsilon_{n,2}$ and $\varepsilon_{n,3}$ replaced by ε_n.

Now, following Teicher [16], define $w_n = \varepsilon_n B_n b_n^{-1}$, $X_n' = X_n I(|X_n| \leqq w_n)$, $X_n''' = X_n I(|X_n| > a_n)$, $X_n'' = X_n - X_n' - X_n'''$ and $X_n^* = X_{n,\varepsilon_n}$ (cf. (2.2)). Let $\{S_n'\}$, $\{S_n''\}$ and $\{S_n'''\}$ be the respective partial sums of the sequences $\{X_n'\}$, $\{X_n''\}$ and $\{X_n'''\}$, and let $S_n^* = \sum_{k=1}^{n} X_k^*$.

Since (2.4) holds, the Borel-Cantelli lemma implies $P[X_n''' \neq 0 \text{ i.o.}] = 0$. Therefore, trivially,

$$\frac{S_n'''}{B_n b_n} \to 0 \quad \text{a.s. as} \quad n \to \infty. \tag{4.7}$$

Note that (2.5) is tantamount to

$$\frac{E(S_n' + S_n'')}{B_n b_n} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.8}$$

Observe that $B_{n_k} \leqq B_{m_{k+1}} < c B_{n_k}$, whence it follows that $b_{n_k}/b_{m_{k+1}} \to 1$. So, in light of (4.6) and (4.6′), Lemma 4.6 implies

$$(S_n'' - E(S_n''))/(B_n b_n) \to 0 \quad \text{a.s.} \tag{4.9}$$

Now let $Y_n = X_n^* - X_n' = w_n(I(X_n > w_n) - I(X_n < -w_n))$, $Y_n'' = Y_n I(|X_n| > a_n)$ and $Y_n' = Y_n - Y_n''$. Since $P[Y_n'' \neq 0 \text{ i.o.}] = P[|X_n| > a_n \text{ i.o.}] = P[X_n''' \neq 0 \text{ i.o.}] = 0$, obviously $(B_n b_n)^{-1} \sum_{k=1}^{n} Y_k'' \to 0$ a.s. Moreover, applying Kronecker's lemma to the series in (2.4), $(B_n b_n)^{-1} \sum_{k=1}^{n} B_k b_k P[|X_k| > a_k] \to 0$ from which it follows readily that $\left| \sum_{k=1}^{n} E(Y_k'') \right| \leqq \sum_{k=1}^{n} \varepsilon_k B_k b_k^{-1} P[|X_k| > a_k] = o(B_n b_n)$. Furthermore, $Y_n' = Y_n I(w_n < |X_n| \leqq a_n)$ so $|Y_n'| \leqq w_n \leqq B_n b_n$. In view of (4.6) and (4.6′), Lemma 4.6 yields

$$\frac{S_n^* - S_n' - E(S_n^* - S_n')}{B_n b_n} = \frac{\sum_{i=1}^{n} (Y_i - E(Y_i))}{B_n b_n} = 0 \quad \text{a.s.} \tag{4.10}$$

In light of (4.7), (4.8), (4.9) and (4.10), it remains only to prove that

$$\frac{S_n^* - E(S_n^*)}{B_n b_n} \xrightarrow{P} 0 \quad \text{if } T_+ < \infty \tag{4.11}$$

and

$$T_- \leqq \limsup_{n \to \infty} \frac{S_n^* - E(S_n^*)}{B_n b_n} \leqq T_+ \quad \text{a.s.} \tag{4.12}$$

Let $v_n^2 = \operatorname{Var}(S_n^*)$. For any $\varepsilon > 0$, an integer $m = m(\varepsilon)$ exists such that $\varepsilon_k < \varepsilon$ if $k \geqq m$. Again using Corollary 4 of Chow and Studden [1], it follows that, for $n \geqq m$,

$$\begin{aligned} v_n^2 &\leqq \sum_{i=1}^{m-1} \operatorname{Var}(X_{i,\varepsilon_i}) + \sum_{i=m}^{n} \operatorname{Var}(X_{i,\varepsilon}) \\ &= \sum_{i=1}^{m-1} \{\operatorname{Var}(X_{i,\varepsilon_i}) - \operatorname{Var}(X_{i,\varepsilon})\} + B_n^2 T_n(\varepsilon). \end{aligned}$$

Therefore, since m and ε are fixed,

$$\limsup_{n \to \infty} v_n^2 / B_n^2 \leqq \limsup_{n \to \infty} T_n(\varepsilon).$$

Now let $\varepsilon \downarrow 0$ to get $\limsup_{n \to \infty} v_n / B_n \leqq T_+$, so, by virtue of (4.5),

$$0 \leqq T_- = \liminf_{n \to \infty} v_n / B_n \leqq \limsup_{n \to \infty} v_n / B_n \leqq T_+. \tag{4.13}$$

In the case when $T_- = 0$, it is possible that v_n converges. But then $S_n^* - E(S_n^*)$ converges a.s. by the Kolmogorov Convergence Theorem ([13], p. 236). Therefore, since $B_n \uparrow \infty$, $(S_n^* - E(S_n^*))/(B_n b_n) \to 0$ a.s., so (4.11) and (4.12) both hold if v_n converges.

For the rest of the proof, assume that $v_n \to \infty$. By Čebyšev's inequality and (4.13), for every $\eta > 0$,

$$P[|S_n^* - E(S_n^*)| \geqq \eta B_n b_n] \leqq v_n^2 / (\eta B_n b_n)^2 \leqq T_+^2 / (\eta b_n)^2$$

so (4.11) is clear. The right-hand inequality of (4.12) is trivial if $T_+ = \infty$, so suppose $T_+ < \infty$. For $n \geqq 1$, define $\alpha_n = \sup_{k \geqq n} v_k/B_k$ and $c_n = 2\varepsilon_n/(b_n \alpha_n)$. Since $|X_n^* - E(X_n^*)| \leqq 2w_n = c_n \alpha_n B_n$, Lemma 4.4 shows that (4.1) holds for every $T>0$ with $N=C=1$ and $M_n = S_n^* - E(S_n^*)$. Since $\alpha \equiv \limsup_{n\to\infty} \alpha_n \leqq T_+$ by (4.13) and $\gamma \equiv \limsup_{n\to\infty} 2^{-1/2} b_n c_n = 0$, Lemma 4.3 implies

$$\limsup_{n\to\infty} \frac{S_n^* - E(S_n^*)}{B_n b_n} \leqq T_+ \quad \text{a.s.}$$

It remains to prove the left-hand inequality of (4.12). Define $u_n = (2 \operatorname{loglog} v_n^2)^{1/2}$. Then $(S_n^* - E(S_n^*))/(v_n u_n) \xrightarrow{P} 0$ by Čebyšev's inequality, whence it follows that $\limsup_{n\to\infty} \frac{S_n^* - E(S_n^*)}{v_n u_n} \geqq 0$ a.s. Since $B_n b_n > 0$ for all large n, (4.12) is true if $T_- = 0$.

Now suppose $T_- > 0$. Define $f(x) = (2 \operatorname{loglog} x^2)^{1/2}/x$, and note that f is decreasing on the interval $(3, \infty)$. If n is so large that $v_n > (T_-/2) B_n > 3$, then $u_n/v_n = f(v_n) < f(B_n T_-/2)$ and, hence

$$\frac{u_n |X_n^* - E(X_n^*)|}{v_n} \leqq 2u_n w_n/v_n \leqq 2f(B_n T_-/2)\, w_n \sim 4\varepsilon_n/T_- = 0(\varepsilon_n).$$

Since $v_n \to \infty$, Kolmogorov's law of the iterated logarithm yields

$$\limsup_{n\to\infty} \frac{S_n^* - E(S_n^*)}{v_n u_n} = 1 \quad \text{a.s.}$$

The left-hand part of (4.12) now follows from (4.13).

Now define T'_- and T'_+ by replacing $T_n(\varepsilon)$ by $T'_n(\varepsilon)$ in (2.3). The proof of Theorem 2.1 will be completed by showing that $T'_- = T_-$ and $T'_+ = T_+$ under (2.9).

Since $y = x/(\operatorname{loglog} x^2)^{1/2}$ is increasing on the interval $(3, \infty)$, there is no harm in assuming that $d_j \leqq d_{j+1}$, where $d_j \equiv B_j/b_j$. For $n \geqq i \geqq 1$ and $\varepsilon > 0$, define $X_{i,n,\varepsilon} = (X_i \vee (-\varepsilon B_n b_n^{-1})) \wedge \varepsilon B_n b_n^{-1}$. Then it is clear from Corollary 4 of [1] that $T_n(\varepsilon) \leqq T'_n(\varepsilon)$ and, hence, that $T_- \leqq T'_-$, $T_+ \leqq T'_+$.

In view of (2.9), a constant A exists such that $\limsup_{n\to\infty} B_{n+1}/B_n < A < \infty$. Let $0 < \delta < 1$. Choose an integer N so large that $B_1 \leqq \delta B_N$ and $B_{n+1} < AB_n$ for all $n \geqq N$. For $n \geqq N$, let $j_n = \min\{i \mid d_i > \delta d_n\}$; clearly $j_n \to \infty$. Note that, for every $\varepsilon > 0$,

$$B_n^{-2} \sum_{i=1}^{N-1} \operatorname{Var}(X_{i,n,\varepsilon}) \leqq B_n^{-2} \varepsilon^2 d_n^2 (N-1) = O(b_n^{-2}) \to 0.$$

Therefore, using Corollary 4 of [1] again, if $j_n > N$

$$\begin{aligned} T_n(\varepsilon) &\geqq B_n^{-2} \sum_{i=j_n+1}^{n} \operatorname{Var}(X_{i,\varepsilon}) \\ &\geqq B_n^{-2} \sum_{i=j_n+1}^{n} \operatorname{Var}(X_{i,n,\varepsilon\delta}) \end{aligned}$$

$$\begin{aligned}
&= T_n'(\varepsilon\delta) - B_n^{-2} \sum_{i=N}^{j_n} \operatorname{Var}(X_{i,n,\varepsilon\delta}) + o(1) \\
&\geqq T_n'(\varepsilon\delta) - B_n^{-2} B_{j_n}^{-2} T_{j_n}'(\varepsilon) + o(1) \\
&\geqq T_n'(\varepsilon\delta) - (B_{j_n-1}^2/B_n^2)(B_{j_n}^2/B_{j_n-1}^2)\, T_{j_n}(\varepsilon).
\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} T_n'(\varepsilon\delta) \leqq \limsup_{n\to\infty} T_n(\varepsilon) + \delta^2 A^2 \limsup_{n\to\infty} T_{j_n}'(\varepsilon).$$

Letting $\varepsilon\downarrow 0$ yields $T_-' \leqq T_- + \delta^2 A^2 T_-'$ and $T_+' \leqq T_+ + \delta^2 A^2 T_+'$. Now let $\delta\downarrow 0$ to complete the proof. □

Proof of Theorem 2.2. Since $\Lambda < \infty$ and $S_n/(B_n b_n) \xrightarrow{P} 0$, it follows from Theorem 2(ii) of Tomkins [20] that (2.4) holds with $a_n = \delta B_n b_n$ for any $\delta > \Lambda$. But then (2.5) holds by dint of Lemma 4.5. Finally, (2.7) results under (2.6) and (2.6′) from an application of Theorem 2.1. □

Proof of Theorem 3.1. Let $\eta > 0$. There is no loss of generality in assuming that $\alpha_+(\eta) < \infty$ when proving (3.3). For $\varepsilon > 0$ and $n \geqq 1$, define $d_n = B_n b_n^{-1}$, $X_n' = \max(X_n, -\eta d_n)\, I(X_n \leqq \varepsilon d_n)$ and $X_n'' = X_n I(\varepsilon d_n < X_n \leqq a_n)$.

Now let $\alpha_n^2 = \sup_{k\geqq n} B_k^{-2} \sum_{i=1}^{k} \operatorname{Var}(X_i')$ and $c_n = (\varepsilon+\eta)/(b_n \alpha_n)$ for $n \geqq 1$. Since $X_n' - E(X_n') \leqq (\varepsilon+\eta)\, d_n = c_n \alpha_n B_n$, Lemma 4.4 ensures the validity of the hypotheses of Lemma 4.3 with $M_n = \sum_{j=1}^{n} (X_j' - E(X_j'))$, $C = N = 1$, any $T > 0$, $\alpha = \alpha(\varepsilon) = \limsup_{n\to\infty} \alpha_n$ and $\gamma = \gamma(\varepsilon) = \limsup_{n\to\infty} 2^{-1/2} b_n c_n = (\varepsilon+\eta)/(2^{1/2}\alpha)$. Consequently, by Lemma 4.3,

$$\limsup_{n\to\infty} \sum_{i=1}^{n} (X_i' - E(X_i'))/(B_n b_n) \leqq 2^{-1/2} \alpha K_\gamma \quad \text{a.s.} \tag{4.14}$$

where K_γ is given by (4.2). Since any $T > 0$ may be used to determine K_γ, it is clear that K_γ may be replaced in (4.14) by K_γ^*, as defined by (3.4).

It follows from (3.2), (3.2′), and Lemma 4.6 that

$$\sum_{i=1}^{n} (X_i'' - E(X_i''))/(B_n b_n) \to 0 \quad \text{a.s.} \tag{4.15}$$

By virtue of (3.1) and the Borel-Cantelli Lemma,

$$\limsup_{n\to\infty} S_n/(B_n b_n) = \limsup_{n\to\infty} \sum_{i=1}^{n} X_i I(X_i \leqq a_i)/(B_n b_n).$$

But $X_n I(X_n \leqq a_n) \leqq X_n' + X_n''$ so, using (4.14) and (4.15),

$$\limsup_{n\to\infty} S_n/(B_n b_n) \leqq 2^{-1/2} \alpha K_\gamma^* + \limsup_{n\to\infty} E(X_n' + X_n'')/(B_n b_n) \tag{4.16}$$

for every $\varepsilon > 0$. But $\alpha \to \alpha_+$ and $\gamma \to \eta/(2^{1/2}\alpha_+)$ as $\varepsilon\downarrow 0$, so, keeping the definition of $C(\eta)$ in mind, (3.3) follows by letting $\varepsilon\downarrow 0$ in (4.16).

Moreover, $\gamma \downarrow 0$ and, hence, $K_\gamma^* \to K_0^*$ as $\eta \downarrow 0$, so (3.5) obtains by letting $\eta \downarrow 0$ in (3.3). □

Proof of Theorem 3.2. For $\eta > 0$, $\varepsilon > 0$ and $n \geqq 1$, define $d_n = B_n b_n^{-1}$, $X_n' = X_n I(-\eta d_n \leqq X_n \leqq \varepsilon d_n)$ and $X_n'' = X_n I(\varepsilon d_n < X_n \leqq a_n)$. Note that $X_n I(X_n \leqq a_n) \leqq X_n' + X_n''$.

There is no loss of generality in assuming that $\alpha_{++}(\eta) < \infty$. Let $\alpha_n = \sup_{k \geqq n} B_k^{-2} \sum_{i=1}^{k} \operatorname{Var}(X_i')$ and $c_n = (\varepsilon + \eta)/(b_n \alpha_n)$. But $X_n' - E(X_n') \leqq \alpha_n B_n c_n$, so, by Lemmas 4.3 and 4.4,

$$\limsup_{n \to \infty} (B_n b_n)^{-1} \sum_{i=1}^{n} (X_i' - E(X_i')) \leqq \alpha(\varepsilon) K_\gamma^* 2^{-1/2} \quad \text{a.s.} \tag{4.17}$$

where $\alpha(\varepsilon) = \lim_{n \to \infty} \alpha_n$, $\gamma = \gamma(\varepsilon) = \limsup_{n \to \infty} 2^{-1/2} b_n c_n = (\varepsilon + \eta)/(2^{1/2} \alpha)$, and K_γ^* is defined by (3.4).

Moreover, by dint of (2.6), (2.6′) and Lemma 4.6,

$$\lim_{n \to \infty} (B_n b_n)^{-1} \sum_{j=1}^{n} (X_j'' - E(X_j'')) = 0 \quad \text{a.s.} \tag{4.18}$$

Therefore, in view of (3.1), (4.17) and (4.18),

$$\begin{aligned} \limsup_{n \to \infty} S_n/(B_n b_n) &= \limsup_{n \to \infty} \sum_{i=1}^{n} X_i I(X_i \leqq a_i)/(B_n b_n) \\ &\leqq \limsup_{n \to \infty} \sum_{i=1}^{n} (X_i' + X_i'')/(B_n b_n) \\ &\leqq \alpha(\varepsilon) K_\gamma^* 2^{-1/2} + \limsup_{n \to \infty} \sum_{i=1}^{n} E(X_i' + X_i'')/(B_n b_n) \end{aligned}$$

for every $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ yields (3.6) which, in turn, yields (3.7) when $\eta \downarrow 0$. □

Remarks 1. Let $X_1, X_2, \ldots$ be independent rv with zero means and finite variances. Define $S_n = X_1 + \ldots + X_n$, $s_n^2 = E(S_n^2)$ and $t_n = (2 \log\log s_n^2)^{1/2}$; assume $s_n \to \infty$. Since $S_n/(s_n t_n) \to 0$ in probability by Čebyšev's inequality, (2.5) holds with $a_n = \delta s_n t_n$, for any $\delta > 0$, by the Degenerate Convergence Criterion ([13], p. 217). If (2.4) and (2.6) hold with $\beta = 1$, $B_n = s_n$ and $b_n = t_n$, then $T_- \leqq \limsup_{n \to \infty} S_n/(s_n t_n) \leqq T_+ \leqq 1$ a.s. by Theorem 2.1.

Under these circumstances, it might be expected that the function $T_n(\varepsilon)$ could be replaced by something simpler. In fact, defining $H_n(\varepsilon) = s_n^{-2} \sum_{i=1}^{n} E(X_i^2 I(|X_i| \leqq \varepsilon s_i t_i^{-1}))$, it is not hard to show that T_- and T_+ may be respectively replaced by H_- and H_+, defined using H_n in lieu of T_n in (2.3), provided $s_n^{-2} \sum_{i=1}^{n} \{E(|X_i| I(|X_i| \geqq \varepsilon s_i t_i^{-1}))\}^2 \to 0$. In the case where $H_- = H_+ = 1$, this modifed result yields a theorem of Teicher [16]. This result also shows that $H_- \leqq \limsup_{n \to \infty} S_n/(s_n t_n) \leqq H_+$ a.s. under the sole condition that $\sum_{n=1}^{\infty} P[|X_n| \geqq \varepsilon s_n t_n^{-1}] < \infty$ for every $\varepsilon > 0$; this theorem is due to Tomkins [21].

Notice the direct relationship between $H_n(\varepsilon)$ and the Lindeberg function $L_n(\varepsilon)=s_n^{-2}\sum_{i=1}^{n} E(X_i^2 I(|X_i|>\varepsilon s_i))$. While the connection between the Lindeberg functions and the Central Limit Theorem has been known for more than half a century, only recently has the relationship between these functions and the law of the iterated logarithm been studied by several authors, including Egorov [3–6], Teicher [16, 17] and Tomkins [19, 21].

2. Suppose $Y_1, Y_2, \ldots$ are i.i.d. rv with $E(Y_1)=0$. Let $\{\sigma_n, n\geqq 1\}$ be non-negative numbers; define $A_n=\sum_{i=1}^{n}\sigma_i$ and assume that $A_n\to\infty$ and $n\sigma_n/A_n=0((\log\log A_n)^\beta)$ for some $\beta\geqq 0$. Theorem 3.1 of Fernholz and Teicher [8] shows that $\sum_{j=1}^{n}\sigma_j Y_j/(A_n(\log_2 A_n)^\beta)\to 0$ a.s. To see that this result follows readily from Theorem 2.1, let $a_n=A_n(\log\log A_n)^\beta$, $X_n=\sigma_n Y_n$ and define B_n according to the equation $B_n b_n=A_n(\log\log A_n)^\beta$. Then (2.4) holds because $E(Y_1)=0$, (2.5) follows easily from the Toeplitz lemma, while (2.6) holds with $\beta=1$ (cf. p. 769 of [8]). Moreover, $EX_n^2 I(|X_n|\leqq\varepsilon B_n/b_n)\leqq(\varepsilon\sigma_n B_n/b_n)E|Y_1|$ so that $T_n(\varepsilon)\leqq\varepsilon E|Y_1| A_n B_n/b_n$. It is now easy to show that $\limsup_{n\to\infty} T_n(\varepsilon)=O(\varepsilon)$ and, hence, that $T_+=0$. Theorem 2.1 now yields the desired result.

3. Corollary 2.6 of [8] follows quite readily from Theorem 3.2. However, Theorem 2.5 of [8] does not seem to be a consequence of any of the theorems in this article, in spite of some obvious similarities.

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