Z. Wahrscheinlichkeitstheorie verw. Gebiete
68, 9–28 (1984)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On the Chacon-Jamison Theorem

John B. Walsh

Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada

The purpose of this article is to generalize a remarkable result on Markov processes due to R. Chacon and B. Jamison, and to examine some of its consequences. Our treatment is somewhat simpler as well as more general than the original - we can eliminate their hypothesis of quasi-left continuity, for instance - but it is based on the main idea of [3], and a good part of the simplification comes from the fact that a delicate measure-theoretical point has already been generalized in [6]; with this point out of the way the proof naturally becomes more transparent. Accordingly, this should properly be considered as an expository article, and we have written it in that style.

We will need to recall a number of definitions in order to state the theorem rigorously, but we can give an informal explanation now.

Let $X=\{X_t,\ t\geqq 0\}$ be a strong Markov process. Suppose an observer is watching a film of its evolution. Unbeknownst to him, however, the projector is running erratically, so that he sees, not X_t but $X_{g(t)}$, where g is a continuous increasing function with $g(0)=0$. According to the Chacon-Jamison theorem, he can, after observing a single sample path, determine the function g and so readjust the projector to run correctly.

Let us illustrate this with two examples. Suppose first that X is a standard Brownian motion. The quadratic variation of X_t up to time t_0 is exactly t_0, so that the quadratic variation of $X_{g(t)}$ up to time t_0 is exactly $g(t_0)$. Thus the observer can determine g by calculating the quadratic variation of the observed process.

Next, suppose that X is a process of stationary independent increments with an infinite Lévy measure ν. Let $N_\varepsilon(t)$ be the number of jumps of X of magnitude at least ε which occur before t. Then $N_\varepsilon(t)$ is Poisson with parameter $t\,\nu\{x\colon |x|\geqq\varepsilon\}$, so that if $\varepsilon_n\to 0$ quickly enough, $(\nu\{x\colon |x|\geqq\varepsilon_n\})^{-1}N_{\varepsilon_n}(t)\to t$ for all t and, in particular, $(\nu\{x\colon |x|\geqq\varepsilon_n\})^{-1}N_{\varepsilon_n}(g(t))\to g(t)$. Thus the observer can again recover g, this time by counting the jumps of $X_{g(t)}$.

The above argument breaks down when the Lévy measure is finite, and if X has holding points, the observer cannot recover $g(t)$. Indeed, if X is the simplest of all processes, the constant process, then $X_{g(t)}\equiv X_0$, so there is no hope of recovering g from observations of $X_{g(t)}$. Thus we will need to make one assumption: X has no traps or exponential holding points. Apart from minimal regularity assumptions, this is the only restriction we need to place on X.

We will prove the basic theorem in section one. We sharpen it and show how it is connected with additive functionals and time-changes in section two. The sharpest version is given there, in Theorem 2.1. We will give some applications in section three, and in particular, we will show how the Blumenthal-Getoor-McKean theorem follows.

For a different point of view see [9], where Y. Le Jan has recast the theorem entirely in terms of additive functionals. His proof, based on the theory of potentials of additive functionals, is considerably shorter than ours, but does not give quite as sharp a result.

1. The Basic Theorem

Let E be a Lusin space [10], that is, E is a Borel subset of a compact separable metric space $\bar{E}$. Adjoin a point δ (the cemetary) to $\bar{E}$ as an isolated point. Let $d(x,y)$ be the metric on $\bar{E}\cup\delta$, and suppose for convenience that $d(\delta,\bar{E})\geqq 1$. Let $\bar{\Omega}$ be the set of all functions from $[0,\infty)$ to $\bar{E}\cup\delta$ which are right continuous, admit left limits at all $t<\infty$ and have a (possibly infinite) lifetime ζ. Let $\Omega\subset\bar{\Omega}$ be the set of $\omega\in\bar{\Omega}$ such that $\omega(t)\in E\cup\delta$ for all t. Let $\bar{\mathscr{F}}$ be the σ-field on $\bar{\Omega}$ induced by the coordinate functions, and let $\mathscr{F}$ be the trace of $\bar{\mathscr{F}}$ on Ω.

Two functions ω and ω' are *equivalent* (denoted $\omega\sim\omega'$) if there are positive increasing right continuous functions f and g such that $\omega=\omega'\circ f$ and $\omega'=\omega\circ g$. Let $\bar{\mathscr{T}}$ be the σ-field of all $\Lambda\in\bar{\mathscr{F}}$ with the property that $\omega\sim\omega'\Rightarrow I_\Lambda(\omega)=I_\Lambda(\omega')$. The atoms of $\bar{\mathscr{T}}$ are the equivalence classes. Each equivalence class is called a *trajectory* and $\bar{\mathscr{T}}$ is called the σ-field of *spatial events*. We say that $\omega\in\bar{\Omega}$ is *nowhere constant* if it is not constant on any open sub-interval of $[0,\zeta(\omega))$. Of course ω is constant on $[\zeta(\omega),\omega)$, for it equals δ there.

Now $(\bar{\Omega},\bar{\mathscr{F}})$ is a Lusin space [10] and $\bar{\mathscr{T}}$ is a separable sub σ-field of $\bar{\mathscr{F}}$ [6] (this is half of the technical result we mentioned above) so by a theorem of Blackwell [1], if P is a probability measure on $\bar{\mathscr{F}}$, there exists a regular conditional probability $P_{\mathscr{T}}(\omega,\Lambda)$ on $\bar{\mathscr{F}}$, that is

(a) for $\omega\in\bar{\Omega}$, $P_{\mathscr{T}}(\omega,\cdot)$ is a probability measure on $\bar{\mathscr{F}}$;

(b) for each $\Lambda\in\bar{\mathscr{F}}$, $P_{\mathscr{T}}(\cdot,\Lambda)$ is a version of $P\{\Lambda|\mathscr{T}\}$.

We are really interested in E, not $\bar{E}$, so let $\mathscr{F}$ and $\mathscr{T}$ be the traces of $\bar{\mathscr{F}}$ and $\bar{\mathscr{T}}$ respectively on Ω. Note that $\omega\in\Omega$ and $\omega'\sim\omega\Rightarrow\omega'\in\Omega$, so that all atoms of $\mathscr{T}$ are also atoms of $\bar{\mathscr{T}}$. Now Ω is a universally measurable subset of $\bar{\Omega}$ – in fact it is the complement of an analytic set [10] – so that if $\Lambda\in\mathscr{F}$, $\Lambda\cap\Omega\in\bar{\mathscr{F}}^*$ where $\bar{\mathscr{F}}^*$ is the universal completion of $\bar{\mathscr{F}}$. Thus $P_{\mathscr{T}}(\omega,\Lambda\cap\Omega)$ makes sense and the restriction of $P_{\mathscr{T}}(\cdot,\Lambda\cap\Omega)$ to Ω is measurable with respect to the universal completion of $\mathscr{F}$.

We now turn our attention to E and Ω. $\bar{E}$ and $\bar{\Omega}$ will remain in the background for moral support, but their main function was to establish the existence of $P_{\mathscr{G}}(\omega, \Lambda)$ for $\omega \in \Omega$ and $\Lambda \in \mathscr{F}$. In particular, 'Borel measurable' below means measurable with respect to the topological Borel field $\mathscr{E}$ of E, and 'universally measurable' means measurable with respect to the universal completion $\mathscr{E}^*$ of $\mathscr{E}$.

Let X be a strong Markov process with state space $E \cup \delta$ whose sample paths are right continuous, have left limits, and admit a lifetime ζ. We assume, as we may, that X is canonically defined on Ω: $X_t(\omega) = \omega(t)$. Let $\mathscr{F}_t = \sigma\{X_s, s \leqq t\}$ be the natural filtration on Ω and let $\mathscr{F}_t^*$ and $\mathscr{F}^*$ be the usual right-continuous completions of $\mathscr{F}_t$ and $\mathscr{F}$ [2, p. 27].

We can summarize our hypotheses on X as follows:

(H1) X is canonically defined on Ω;

(H2) for each $x \in E \cup \delta$ there is a probability distribution P^x on $(\Omega, \mathscr{F}^*)$ such that if $\Lambda \in \mathscr{F}$, $x \to P^x\{\Lambda\}$ is universally measurable;

(H3) for each $(\mathscr{F}_t^*)$-stopping time T, $x \in E$, and $\Lambda \in \mathscr{F}$:

$$P^x\{\theta_T^{-1} \Lambda | \mathscr{F}_T^*\} = P^{X_T}\{\Lambda\} \; P^x \text{ a.s.},$$

where θ_t is the usual translation operator on Ω.

As usual, P^μ denotes the measure $\int P^x(\cdot)\,\mu(dx)$ on $(\Omega, \mathscr{F}^*)$. We can now state the Chacon-Jamison Theorem.

Theorem 1.1. *Let μ be a probability measure on E. Suppose X has no traps or holding points. Then for P^μa.e. ω, the regular conditional probability $P_{\mathscr{G}}^\mu(\omega, \cdot)$ is a point mass which sits on $\{\omega' : \omega' \sim \omega\}$.*

Remark. This may appear far removed from the informal explanation in the introduction, but consider the observer who sees a sample path of $X_{g(t)}$, say $X_{g(t)} = \omega(t)$. Now ω is necessarily equivalent to the true, non-time-changed path ω_0, for $\omega(t) = \omega_0(g(t))$. To determine g, he calculates $P_{\mathscr{G}}(\omega, \cdot)$ which, by the theorem, must put its mass on a single path, namely ω_0. Once he knows both ω and ω_0, he determines g uniquely from the equation $\omega = \omega_0 \circ g$; g is uniquely determined because ω_0 is nowhere constant.

Before proving the theorem, let us define a family $(\tau_{\mathbf{j}}^n)$ of stopping times, where $n = 1, 2, \ldots$ and $\mathbf{j} = j_1 \ldots j_n$ is a multi-index. Let d be the metric on E inherited from $\bar{E}$ and define:

$$\tau_0^1 \equiv 0, \quad \tau_{j+1}^1 = \inf\{t > \tau_j^1 : d(X_t, X_{\tau_j^1}) > 1/2\} \wedge \zeta, \quad j = 0, 1, 2, \ldots$$

and for $n = 2$ and $i \geqq 0$:

$$\tau_{i0}^2 \equiv \tau_i^1, \quad \tau_{ij+1}^2 = \inf\{t > \tau_{ij}^2 : d(X_t, X_{\tau_{ij}^2}) > 1/4\} \wedge \tau_{i+1}^1, \quad j = 0, 1, 2, \ldots$$

and, in general, for $n \geqq 1$ and $i_1, \ldots, i_n$ positive integers:

$$\begin{aligned} &\tau_{i_1 \ldots i_n 0}^{n+1} = \tau_{i_1 \ldots i_n}^n, \\ \text{and} \quad &\tau_{i_1 \ldots i_n j+1}^{n+1} = \inf\{t > \tau_{i_1 \ldots i_n j}^{n+1} : d(X_t, X_{\tau_{i_1 \ldots i_n j}^{n+1}}) > 2^{-n-1}\} \wedge \tau_{i_1 \ldots i_n + 1}^n. \end{aligned} \tag{1.1}$$

Notice that at the $n+1^{\text{st}}$ stage we interpolate a sequence of stopping times between each pair of successive times from the n^{th} stage.

Let us introduce some notation for multi-indices. If $\mathbf{j}=j_1 \dots j_{n-1} j_n$, define $\mathbf{j}+1=j_1 \dots j_{n-1}(j_n+1)$. We also define $\mathbf{j}-1$: if $j_n \geqq 1$, $\mathbf{j}-1=j_1 \dots j_{n-1}(j_n-1)$, and if $j_n=0$, we set $\mathbf{j}-1=\mathbf{j}$. For an integer k, let $\mathbf{j}k=j_1 \dots j_n k$. We let $|\mathbf{j}|$ denote the length of $\mathbf{j}$. Thus, if $\mathbf{j}=257$ and $k=9$, then $|\mathbf{j}|=3$, $\mathbf{j}\,k=2579$, $\mathbf{j}-1=256$ and $\mathbf{j}+1=258$.

We have constructed the $\tau_{\mathbf{j}}^n$ so that for each k

$$\tau_{\mathbf{j}}^n \leqq \tau_{\mathbf{j}k}^{n+1} \leqq \tau_{\mathbf{j}+1}^n. \tag{1.2}$$

Note that the upper index n is redundant: it always equals the length of $\mathbf{j}$. We can drop it from our notation in the following, writing $\tau_{\mathbf{j}}$ instead of $\tau_{\mathbf{j}}^n$.

There is a natural order, namely lexicographic order, on the multi-indices such that if $\mathbf{j} \leqq \mathbf{k}$ in this order, then $\tau_{\mathbf{j}} \leqq \tau_{\mathbf{k}}$. This can be seen by induction from (1.2), but a more intuitive way to see it is to compare the ordering of the stopping times with the decimal fractions: these can be constructed by first marking off the points $0.1<0.2<0.3<\dots$, then further dividing each interval by points, say $0.2=0.20<0.21<\dots<0.29<0.3$ and so on. In our case we divide the interval $[\tau_2, \tau_3]$ by infinitely many (not necessarily distinct) points $\tau_2=\tau_{20} \leqq \tau_{21} \leqq \dots \leqq \tau_3$, but the principle is the same. Thus, informally, to see if $\mathbf{j} \leqq \mathbf{k}$, with $\mathbf{j}$ and $\mathbf{k}$ as "decimals": $\mathbf{j} \leqq \mathbf{k}$ iff $0.j_1 \dots j_n \leqq 0.k_1 \dots k_n$.

For example, τ_{257} is less than or equal to both τ_{2631} and τ_3, while it dominates both τ_{1732} and τ_2. We should point out that if $|\mathbf{j}|=m$ and if $n>m$, we can write $\tau_{\mathbf{j}}=\tau_{\mathbf{k}}$, where $|\mathbf{k}|=n$. Indeed, $\tau_{25}=\tau_{250}=\tau_{2500}=\dots$.

A stopping time T is *intrinsic* if $\omega \sim \omega'$ and $\omega=\omega' \circ f$ imply $T(\omega') \leqq f(T(\omega))$. These times were defined in [6] and most of their basic properties were given in Prop. 1.1 there. These are stopping times which can be defined in terms of the trajectory, rather than the path. Hitting times are primary examples. So are the $\tau_{\mathbf{j}}$ above. Two properties we shall need are these: if T is intrinsic and $\omega' \sim \omega$, then $\omega(T(\omega))=\omega'(T(\omega'))$. Consequently, if T is $\mathscr{F}$-measurable, $\omega(T(\omega))$ is $\mathscr{T}$-measurable by Blackwell's Theorem. Then we have:

Lemma 1.2. *Under the hypotheses of Theorem* 1.1

(i) *there is a total order on the multi-indices such that if* $\mathbf{i} \leqq \mathbf{j}$ *in this order, then* $\tau_{\mathbf{i}}(\omega) \leqq \tau_{\mathbf{j}}(\omega)$ *for all* $\omega \in \Omega$;

(ii) *the* $\tau_{\mathbf{j}}$ *are* $\mathscr{F}$*-measurable intrinsic times, and for* P^μ*-a.e.* ω *the family* $\{\tau_{\mathbf{j}}(\omega)\}$ *is dense in* $[0, \zeta(\omega))$.

(iii) *Let* $\mathscr{T}_n=\sigma\{X_{\tau_{\mathbf{j}}}: |\mathbf{j}|=n\}$. *Then* $\mathscr{T}_n \subset \mathscr{T}_{n+1}$ *and* $\mathscr{T}=\bigvee_n \mathscr{T}_n$ *up to* P^μ*-null-sets.*

Proof. (i) follows from (1.2). To see (ii), note that the $\tau_{\mathbf{j}}$ are successive hitting times of open sets, hence $\mathscr{F}$-measurable, and they are intrinsic by [6]. Now $d(X_{\tau_{j+1}^1}, X_{\tau_j^1}) \geqq 1/2$ on $\{X_{\tau_{j+1}^1}<\zeta\}$, so that $\tau_j^1 \to \zeta$ as $j \to \infty$, for if not the path would fail to have a left limit at $\lim \tau_j^1$. For the same reason, there are only finitely many k in (1.2) for which there is strict inequality. The density of the $\tau_{\mathbf{j}}$ then follows from the fact that the paths are P^μ a.s. nowhere constant (an immediate consequence of the hypothesis that there are no holding points). (iii) Note that $\{\tau_{\mathbf{j}}: |\mathbf{j}|=n\} \subset \{\tau_{\mathbf{j}}: |\mathbf{j}|=n+1\}$ so $\mathscr{T}_n \subset \mathscr{T}_{n+1}$. It then follows from Theorem 2.1 of [6] and Remark 3° following it that $\mathscr{T}=\bigvee_n \mathscr{T}_n$. (The auxiliary variables Q_{nj} of Theorem 2.1 are P^{μ}-a.s. zero since the paths of X are nowhere constant. This is the second half of the technical result we spoke of in the introduction.) QED

Proof of Theorem 1.1. Let $\Delta\tau_{\mathbf{j}}=\tau_{\mathbf{j}}-\tau_{\mathbf{j}-1}$. Then for each $\mathbf{j}\in J$

$$\tau_{\mathbf{j}}=\sum_{\substack{\mathbf{i}\leqq\mathbf{j}\\|\mathbf{i}|=|\mathbf{j}|}}\Delta\tau_{\mathbf{i}}. \tag{1.3}$$

Two remarks are in order here. First, the sum is only over those $\mathbf{i}$ for which $|\mathbf{i}|=|\mathbf{j}|$, not over $|\mathbf{i}|\leqq|\mathbf{j}|$, since the lower order indices are in fact included in the higher order ones: if $|\mathbf{i}|<|\mathbf{j}|$, then $\mathbf{k}=\mathbf{i}0\ldots0$ is an index for which $|\mathbf{k}|=|\mathbf{j}|$ and $\tau_{\mathbf{k}}=\tau_{\mathbf{i}}$. Next, suppose $\mathbf{j}=j_1\ldots j_n0$. Then $\mathbf{j}-1=\mathbf{j}$ by definition, so that $\Delta\tau_{\mathbf{j}}=0$. This is the correct value, for although $\mathbf{j}$ has no immediate predecessor, there exists some m such that $\tau_{j_1\ldots j_n0}(\omega)=\tau_{j_1\ldots(j_n-1)m}(\omega)$, so that the increment preceeding τ_j is already in the sum.

Let $|\mathbf{j}|=1$. By the strong Markov property, the process $\{X_{(\tau_{\mathbf{j}}+t)\wedge\tau_{\mathbf{j}+1}}, t\geqq0\}$ is conditionally independent of the processes $\{X_{t\wedge\tau_{\mathbf{j}}}, t\geqq0\}$ and $\{X_{\tau_{\mathbf{j}+1}+t}, t\geqq0\}$ given $X_{\tau_{\mathbf{j}}}$ and $X_{\tau_{\mathbf{j}+1}}$. (Informally: what happens on $(\tau_{\mathbf{j}}, \tau_{\mathbf{j}+1})$ is independent of what happens on the complement given $X_{\tau_{\mathbf{j}}}$ and $X_{\tau_{\mathbf{j}+1}}$.) Thus it is still conditionally independent given the larger σ-field $\mathscr{T}_1$. If $|\mathbf{j}|=n$, the definition of $\tau_{\mathbf{j}+1}$ depends on information in $\mathscr{T}_n$, but the strong Markov property again shows that the two processes are conditionally independent given $\mathscr{T}_n$. It follows that if $\mathbf{j}_1,\ldots,\mathbf{j}_r$ are multi-indices of length n,

$$E\left\{\exp\left(-\sum_{q=1}^{r}\lambda_q\Delta\tau_{\mathbf{j}_q}\right)\Big|\mathscr{T}_n\right\}=\prod_{q=1}^{r}E\{\exp(-\lambda_q\Delta\tau_{\mathbf{j}_q})|\mathscr{T}_n\}.$$

If $p>n$ and $|\mathbf{j}|=n$

$$\Delta\tau_{\mathbf{j}}=\sum_{\substack{\mathbf{j}-1<\mathbf{i}\leqq\mathbf{j}\\|\mathbf{i}|=p}}\Delta\tau_{\mathbf{i}}. \tag{1.4}$$

The sums in (1.4) are disjoint for different $\mathbf{j}$ so that the $\Delta\tau_{\mathbf{j}}$, $|\mathbf{j}|=n$, are independent given $\mathscr{T}_p$ for each $p\geqq n$. If we put $Y_q=\exp(-\lambda_q\Delta\tau_{\mathbf{j}_q})$ and $Y=Y_1\ldots Y_r$, then

$$E\{Y|\mathscr{T}_p\}=\prod_{q=1}^{r}E\{Y_q|\mathscr{T}_p\}.$$

Let $p\to\infty$ and use Lemma 1.2(iii):

$$E\{Y|\mathscr{T}\}=\prod_{q=1}^{r}E\{Y_q|\mathscr{T}\},$$

or, if $P^{\mu}_{\mathscr{T}}$ is the regular conditional probability,

$$P^{\mu}_{\mathscr{T}}(\omega, Y)=\prod_{q=1}^{r}P^{\mu}_{\mathscr{T}}(\omega, Y_q) \tag{1.5}$$

for P^{μ}-a.e. ω. This is true simultaneously for all rational $\lambda_q>0$ so that for a.e. ω the $\Delta\tau_{\mathbf{j}_q}$ are independent – not just conditionally independent – under the measure $P^{\mu}_{\mathscr{T}}(\omega, \cdot)$. But if $p > |\mathbf{j}|$, then by (1.3) and (1.4)

$$\tau_{\mathbf{j}} = \sum_{\substack{\mathbf{i} \leqq \mathbf{j} \\ |\mathbf{i}| = p}} \Delta \tau_{\mathbf{i}}. \tag{1.6}$$

Now the paths are a.s. nowhere constant, so $\sup\{\Delta\tau_{\mathbf{i}}: \mathbf{i} \leqq \mathbf{j},\ |\mathbf{i}| = p\} \to 0$ as $p \to \infty$. By Proposition A of the appendix, $\tau_{\mathbf{j}}$ is a.e. constant, i.e. for P^{μ}-a.e. ω there exists a constant $t_{\mathbf{j}}$ such that $P^{\mu}_{\mathscr{T}}(\omega, \cdot)$ puts its mass on the set $\{\omega': \tau_{\mathbf{j}}(\omega') = t_{\mathbf{j}}\}$. On the other hand, $\omega' \to \omega'(\tau_{\mathbf{j}}(\omega'))$ is $\mathscr{T}$-measurable [6] so it is easily seen that $P^{\mu}_{\mathscr{T}}(\omega, \cdot)$ also puts its mass on $\{\omega': \omega'(\tau_{\mathbf{j}}(\omega')) = \omega(\tau_{\mathbf{j}}(\omega)),\ \forall \mathbf{j}\}$ (which is the set $\{\omega': \omega' \sim \omega\}$ by Theorem 2.1 of [6]).

Putting these two together, we see that there exist $t_{\mathbf{j}} \geqq 0$ and $x_{\mathbf{j}} \in E \cup \delta$ such that $P^{\mu}_{\mathscr{T}}$ puts all its mass on

$$\{\omega': \tau_{\mathbf{j}}(\omega') = t_{\mathbf{j}},\ \omega'(t_{\mathbf{j}}) = x_{\mathbf{j}},\ \forall \mathbf{j}\}.$$

But the $t_{\mathbf{j}}$ are dense so this set is evidently a singleton, hence $P^{\mu}_{\mathscr{T}}(\omega, \cdot)$ is a point mass which sits on $\{\omega': \omega' \sim \omega\}$. QED

2. Additive Functionals and Time Changes

Theorem 1.1 in its present form is not sufficiently sharp for many applications. One problem is that the regular conditional probabilities $P^{\mu}_{\mathscr{T}}$ are only determined up to a null set, and this null set may depend on μ. This can be gotten around as in [4] by defining a measurable version $P^{\mu}_{\mathscr{T}}$ simultaneously for all $x \in E$, but it still leaves a somewhat subtler problem: finding a version of $P^{\mu}_{\mathscr{T}}(\omega, \cdot)$ which works not only for ω, but for $\theta_t \omega$, all $t \geqq 0$. This is exactly analogous to the problem of finding a perfect version of an additive functional. We will show in this section that it is possible to construct a 'perfect' version of $P_{\mathscr{T}}$. In the process of doing this, we will make the connection between Theorem 1.1, additive functionals, and time changes explicit. The key step is to construct a perfect version of a certain additive functional. We retain the hypotheses of Theorem 1.1.

Theorem 2.1. *There exists a $\mathscr{T}^*$-measurable map $p: \Omega \to \Omega$ such that for each probability measure μ on E*

(i) $P^{\mu}\{\omega: \omega = p(\omega)\} = 1$;

(ii) $P^{\mu}_{\mathscr{T}}(\omega, \{p(\omega)\}) = 1$ *for* P^{μ}-*a.e.* ω.

Furthermore, there exists a set $\Gamma \in \mathscr{T}^*$, an $(\mathscr{F}_t^*)$-adapted continuous additive functional A_t defined on Γ, and its right continuous inverse T_t such that

(iii) $P^{\mu}\{\Gamma\} = 1$ *for all* μ *and* $\theta_t \Gamma \subset \Gamma$ *for all* t;

(iv) $p(\omega)(t) = \omega(T_t \omega)$ *and* $p(\omega) \sim \omega$ *for* $\omega \in \Gamma$;

(v) A *is perfect, i.e. if* $\omega \in \Gamma$ *and* $s, t \geqq 0$, *then*

$$A_{s+t}(\omega) = A_s(\omega) + A_t(\theta_s \omega). \tag{2.1}$$

Remarks. This theorem is our translation of the main theorem of [4]. We should point out that although the paths of X are nowhere-constant, there will be equivalent paths which do have flat spots. This complicates the proof: T_t is in general only right continuous, for instance, rather than continuous. The reader will find that if he simply assumes that all the paths are nowhere-constant, so that the functions T_t, $\hat{T}_t$, A_t and $\hat{A}_t$ appearing below are continuous and strictly increasing, he can eliminate a good third (or a bad third) of the proof.

Let us give another expression for the stopping times $\tau_{\mathbf{j}}$ of (1.1). Suppose $|\mathbf{j}| = n$. For each $p<n$ there is an index $\mathbf{i}_p$ of length p such that $\mathbf{i}_p \leqq \mathbf{j} \leqq \mathbf{i}_p + 1$. Let $\mathbf{i}_n = \mathbf{j}$. Then

$$\tau_{\mathbf{j}+1} = \inf\{t>\tau_{\mathbf{j}}\colon\ d(X_t, X_{\tau_{\mathbf{i}_p}}) > 2^{-p},\ \text{some } p=1,\dots,n\}.$$

Define a stopping time by

$$\sigma_n(x_1,\dots,x_n) = \inf\{t>\tau_{\mathbf{j}}\colon\ d(X_t, x_p) > 2^{-p}, \text{some } p=1,\dots,n\}$$

and let, for $\varepsilon>0$,

$$f_n^\varepsilon(x,y;x_1,\dots,x_{n-1}) = E^x\{\sigma_n(x_1,\dots,x_{n-1},x)\wedge\varepsilon \,|\, X_{\sigma_n(x_1,\dots,x_{n-1},x)} = y\}.$$

The conditional distribution P^x of X is universally measurable so that, by an argument of Doob, one can choose a version of the conditional expectation which is universally measurable in $(x,y,x_1,\dots,x_{n-1})$.

For the index $\mathbf{j}$ above, set

$$F_{\mathbf{j}}^\varepsilon = f_n^\varepsilon(X_{\tau_{\mathbf{j}-1}}, X_{\tau_{\mathbf{j}}}; X_{\tau_{\mathbf{i}_1}},\dots,X_{\tau_{\mathbf{i}_{n-1}}}). \tag{2.2}$$

Now as in the proof of Theorem 1.1, $\Delta\tau_{\mathbf{j}}$ is conditionally independent of $\mathscr{T}_n$ given $X_{\tau_{\mathbf{j}-1}}$, $X_{\tau_{\mathbf{j}}}$, and $X_{\tau_{\mathbf{i}_1}},\dots,X_{\tau_{\mathbf{i}_{n-1}}}$, so

$$E^\mu\{\Delta\tau_{\mathbf{j}}\wedge\varepsilon\,|\,\mathscr{T}_n\} = F_{\mathbf{j}}^\varepsilon.$$

We also saw that the $\Delta\tau_{\mathbf{j}}$, $|\mathbf{j}|=n$ were conditionally independent given $\mathscr{T}_n$, so from (1.3), (1.4) and Proposition B of the appendix, we have P^μ a.s. for all $\mathbf{j}\in J$ that

$$\tau_{\mathbf{j}} = \inf_m\{\liminf_{n\to\infty} \sum_{\substack{|\mathbf{i}|=n-1\\ \mathbf{i}k\leqq\mathbf{j}}} F_{1/m}(\mathbf{i}\,k)\}. \tag{2.3}$$

Accordingly, define for each $\omega\in\Omega$:

$$\sigma_{\mathbf{j}}(\omega) = \inf_m\{\liminf_{n\to\infty} \sum_{\substack{|\mathbf{i}|=n-1\\ \mathbf{i}k\leqq\mathbf{j}}} F_{1/m}(\mathbf{i}\,k)\}.$$

If we trace through the measurability, we can see that $\sigma_{\mathbf{j}}$ is $\mathscr{F}^*$-measurable. If T is intrinsic and $\omega\sim\omega'$, then $X_T(\omega)=X_T(\omega')$ [6]. The $\tau_{\mathbf{j}}$ are intrinsic, so that from (2.2),

$$f_{\mathbf{j}}^\varepsilon(\omega') = f_{\mathbf{j}}^\varepsilon(\omega), \qquad \text{hence } \sigma_{\mathbf{j}}(\omega') = \sigma_{\mathbf{j}}(\omega).$$

Thus $\sigma_{\mathbf{j}}$ is $\mathscr{T}^*$-measurable. In fact, notice that $F_{\mathbf{j}}^{\varepsilon}$ only depends on $X_{\tau_{\mathbf{i}}}$ for $\mathbf{i} \leqq \mathbf{j}$ so that, if $\mathscr{T}_{\tau_{\mathbf{j}}}^*$ is the σ-field generated by events of the form $\{X_{\tau_{\mathbf{i}}} \in A\}$ where $\mathbf{i} \leqq \mathbf{j}$, and $A \subset E$ is universally measurable, we have

a) $\sigma_{\mathbf{j}} \in \mathscr{T}_{\tau_{\mathbf{j}}}^* \subset \mathscr{F}_{\tau_{\mathbf{j}}}^*$;
b) $\mathbf{i} \leqq \mathbf{j} \Rightarrow \sigma_{\mathbf{i}} \leqq \sigma_{\mathbf{j}}$;
c) $P^{\mu}\{\sigma_{\mathbf{j}} = \tau_{\mathbf{j}}\} = 1, \ \forall \mu$.

Definition. $\hat{T}_t(\omega) = \inf\{\tau_{\mathbf{j}}(\omega) \colon \ \sigma_{\mathbf{j}}(\omega) > t\}$;

$$\hat{A}_t(\omega) = \inf\{s \colon \ \hat{T}_s(\omega) > t\}.$$

Notice that $\hat{A}_t$ is $\mathscr{F}_t^*$-measurable since

$$\{\hat{A}_t < s\} = \bigcup_n \{\hat{T}_s < t - 1/n\} = \bigcup_{n, \mathbf{j}} \{\omega \colon \ \sigma_{\mathbf{j}}(\omega) > s; \ \tau_{\mathbf{j}}(\omega) < t - 1/n\}$$

which is in $\mathscr{F}_t^*$.

Definition. $\Omega_0 = \{\omega \in \Omega \colon \ t \to \omega(t)$ is nowhere constant$\}$.

$$\Gamma_0 = \{\omega \in \Omega_0 \colon \ \sigma_{\mathbf{j}}(\omega) = \tau_{\mathbf{j}}(\omega), \ \forall \mathbf{j}\}.$$

$$\Gamma_1 = \{\omega \in \Omega \colon \ \exists \, \omega' \in \Gamma_0 \ni \omega' \sim \omega\}.$$

Lemma 2.2. *Let μ be an initial distribution. Then*

(i) *$\Omega_0 \in \mathscr{F}$, $\Gamma_0 \in \mathscr{F}^*$, and $\Gamma_1 \in \mathscr{T}^*$;*
(ii) *$P^{\mu}(\Gamma_0) = P^{\mu}(\Gamma_1) = 1$;*
(iii) *if $\omega \in \Gamma_0$, $\hat{T}_t(\omega) = \hat{A}(\omega) = t$, all $t < \zeta(\omega)$.*

Now suppose $\omega \in \Gamma_1$. Then

(iv) *if $\omega \sim \omega'$, then $\omega(\hat{T}_t(\omega)) = \omega'(\hat{T}_t(\omega'))$, all t;*

(v) *if $\omega''(t) = \omega(\hat{T}_t(\omega))$, $t \geqq 0$, then $\omega'' \in \Gamma_0$ and $\omega(t) = \omega''(\hat{A}_t(\omega))$. In particular, $\omega'' \sim \omega$. Moreover, ω'' is the unique element in Γ_0 which is equivalent to ω.*

(vi) *$t \to \hat{A}_t(\omega)$ is continuous, $t \to \hat{T}_t(\omega)$ is strictly increasing, and $\sup_t \hat{T}_t(\omega) = \zeta(\omega)$.*

(vii) *if $s < t$, then $\hat{A}_s(\omega) = \hat{A}_t(\omega)$ iff $u \to \omega(u)$ is constant on $(s, t]$.*

Proof. (i) Clearly $\Omega_0 \in \mathscr{F}$, so $\Gamma_0 \in \mathscr{F}^*$ because $\tau_{\mathbf{j}}$ and $\sigma_{\mathbf{j}}$ are $\mathscr{F}^*$-measurable. It then follows that $\Gamma_1 \in \mathscr{T}^*$. (ii) $P^{\mu}(\Gamma_0) = 1$ by (c) and $\Gamma_1 \supset \Gamma_0$. (iii) is clear. To see (iv) note that

$$\omega(\hat{T}_t(\omega)) = \lim_{\sigma_{\mathbf{j}}(\omega) \downarrow t} \omega(\tau_{\mathbf{j}}(\omega)).$$

Now $\sigma_{\mathbf{j}}(\omega) = \sigma_{\mathbf{j}}(\omega')$ by (a) and $\omega(\tau_{\mathbf{j}}(\omega)) = \omega'(\tau_{\mathbf{j}} \, \omega'))$ by [6] so

$$= \lim_{\sigma_{\mathbf{j}}(\omega') \downarrow t} \omega'(\tau_{\mathbf{j}}(\omega'))$$

$$= \omega'(\hat{T}_t(\omega)).$$

Now there exists $\omega' \in \Gamma_0$ such that $\omega' \sim \omega$. $\hat{T}_t(\omega') = t$ by (iii), so $\omega(\hat{T}_t(\omega)) = \omega'(t)$. By equivalence there is an increasing g such that $\omega(t) = \omega'(g(t))$, hence g must be the inverse of $\hat{T}_t$, i.e. $g(t) = \hat{A}_t(\omega)$. To see uniqueness, note that if $\omega', \omega'' \in \Omega_0$, by (iii) and (iv)

$$\omega'(t) = \omega'(\hat{T}_t(\omega')) = \omega''(\hat{T}_t(\omega'')) = \omega''(t), \qquad \text{i.e. } \omega' = \omega''.$$

If $\omega \in \Gamma_1$, then $\omega \sim \omega'$ for some $\omega' \in \Gamma_0$. Since ω' is nowhere constant and $\omega'(t) = \omega(\hat{T}_t)$, $\hat{T}_t(\omega)$ must be strictly increasing and its inverse $\hat{A}_t$ must be continuous. If, now, $t < \zeta(\omega)$, then there exists $\mathbf{j}$ such that $t < \tau_{\mathbf{j}}(\omega) < \zeta(\omega)$ (see Remark 2.5 below). If $\sigma_{\mathbf{j}}(\omega) = s$, $\hat{T}_s(\omega) \geqq t$. Then (vii) follows from (vi) and the equation $\omega(u) = \omega'(\hat{A}_u(\omega))$. QED

We are next going to show that $\hat{A}$ is an additive functional, at least at certain random times. For this we will introduce a class of intrinsic times which are a bit easier to use than the $\tau_{\mathbf{j}}$.

For each n, let $B_{n1}, \ldots, B_{nk_n}$ be a finite partition of E into Borel sets of diameter at most $1/n$. We can do this since E is a subset of the compact space $\bar{E}$. Let D_{nk} be the closed $1/n$ - neighborhood of B_{nk}. Thus $B_{nk} \subset D_{nk}$. We will consider the 'upcrossings' of $D_{nk} - B_{nk}$. Fix n and k and define a sequence of stopping times by

$$V_0^{nk} = 0, \qquad U_1^{nk} = \inf\{t > 0 \colon X_t \in B_{nk}\}$$
$$V_1^{nk} = \inf\{t > U_1^{nk} \colon X_t \notin D_{nk}\}$$

and, for $m \geqq 2$, set

$$U_m^{nk} = \inf\{t > V_{m-1}^{nk} \colon X_t \in B_{nk}\}$$
$$V_m^{nk} = \inf\{t > U_m^{nk} \colon X_t \notin D_{nk}\}.$$

The V_m^{nk}, $m = 1, 2, \ldots$ are the successive upcrossing times of $D_{nk} - B_{nk}$. Note that $d(X_{U_m^{nk}}, X_{V_m^{nk}}) \geqq 1/n$ if $V_m^{nk} < \zeta$, so $V_m^{nk} \to \infty$ as $m \to \infty$; otherwise the path would have oscillatory discontinuities.

Let

$$\mathscr{V} = \{V_m^{nk} \colon \text{ all } n, k, m \geqq 0\}$$
$$\mathscr{V}_+ = \{V_m^{nk} \colon \text{ all } n, k \geqq 0 \text{ and } m \geqq 1\}.$$

Note that if $V \in \mathscr{V}_+$, $V > 0$.

Lemma 2.3. (i) *If $V \in \mathscr{V}$, V is an intrinsic ($\mathscr{F}_t^*$)-stopping time;*

(ii) *$\sigma_{\mathbf{j}}(\theta_V \omega) = \tau_{\mathbf{j}}(\theta_V \omega)$ P^μ-a.s. for all $V \in \mathscr{V}$, $\mathbf{j} \in J$ and μ;*

(iii) *for each ω*

$$\{s \colon s = t + V(\theta_t \omega) \text{ some } t \geqq 0,\ V \in \mathscr{V}_+\} = \{s \colon s = V(\omega), \text{ some } V \in \mathscr{V}_+\}. \tag{2.4}$$

Proof. (i) is immediate since the V are successive hitting times of Borel sets, and (ii) follows from (c) and the strong Markov property. For (iii), fix ω and let $W(\omega) = t + V_k^{nj}(\theta_t \omega)$. Then $W(\omega)$ is the time of the k^{th} upcrossing of $D_{nj} - B_{nj}$ after t. If there have been p previous upcrossings, then $W(\omega)$ equals either $V_{k+p}^{nj}(\omega)$ or $V_{k+p+1}^{nj}(\omega)$. QED

If $\Lambda \subset \Omega$, let us define a set Λ^+ by

$$\Lambda^+ = \{\omega: \theta_V \omega \in \Lambda, \text{all } V \in \mathscr{V}_+\} = \bigcap_{V \in \mathscr{V}_+} \theta_V^{-1}(\Lambda).$$

If $\Lambda \in \mathscr{T}^*$, $\Lambda^+ \in \mathscr{T}^*$, since there are only countably many $V \in \mathscr{V}_+$, and each is intrinsic. If Λ has the property that $P^\mu\{\Lambda\}=1$ for all initial measures μ, so does Λ^+ by the strong Markov property applied to each $V \in \mathscr{V}_+$.

Lemma 2.4. *Let* $\Gamma_2 = \{\omega \in \Omega: \exists\, \omega' \in \Gamma_0 \cap \Gamma_0^+ \ni \omega' \sim \omega\}$. *Then*

(i) $\Gamma_2 \in \mathscr{T}^*$ *and* $P^\mu\{\Gamma_2\}=1$ *for all initial* μ.

(ii) *If* $\omega \in \Gamma_2$, $V \in \mathscr{V}$ *and* $t \geqq 0$, *then*

$$\hat{A}_{V+t}(\omega) = \hat{A}_V(\omega) + \hat{A}_t(\theta_V \omega). \tag{2.5}$$

Proof. (i) follows from the preceeding remarks. Consider $\hat{T}_t$ and put $\omega'(t) = \omega(\hat{T}_t(\omega))$. Then $\omega' \in \Gamma_0$ (Lemma 2.2(v)); and, since ω' is unique, evidently $\omega' \in \Gamma_0^+$, so $\theta_V \omega' \in \Gamma_0$. V is intrinsic, so $\omega \sim \omega' \Rightarrow \theta_V \omega \sim \theta_V \omega'$. By Lemma 2.2(iv) and (ii),

$$\theta_V \omega'(t) = \theta_V \omega(\hat{T}_t(\theta_V \omega)),$$

hence

$$\omega'(V(\omega')+t) = \omega(V(\omega) + \hat{T}_t(\theta_V \omega)).$$

Now ω' is nowhere constant so the time-change from ω to ω' is unique, and we must have $V(\omega)+\hat{T}_t(\theta_V \omega) = \hat{T}_{V(\omega')+t}(\omega)$. By Lemma 2.2(v), $V(\omega') = \hat{A}_V(\omega)$, giving us

$$V(\omega) + \hat{T}_t(\theta_V \omega) = \hat{T}_{\hat{A}_V(\omega)+t}(\omega). \tag{2.6}$$

We get (2.5) by inverting (2.6), using Lemma 2.2(vii) to handle the situation in which ω is constant on some interval. QED

We must modify $\hat{A}$ in order to get a perfect additive functional. Let $T_0(\omega) = \inf\{V(\omega): V \in \mathscr{V}_+\}$. Then we define

$$A_t(\omega) = \begin{cases} \liminf\limits_{\substack{V \downarrow T_0(\omega) \\ V \in \mathscr{V}_+}} \hat{A}_{t-V(\omega)}(\theta_V \omega) & \text{if } t > T_0(\omega) \\ 0 & \text{if } t \leqq T_0(\omega); \end{cases} \tag{2.7}$$

$$T_t(\omega) = \inf\{s: A_s(\omega) > t\}; \tag{2.8}$$

$$\Gamma = \Gamma_2^+ \cap \{\omega \in \Omega: t \to A_t \text{ is continuous}\}. \tag{2.9}$$

Remark 2.5. Let N be the set of $\omega \in \Omega$ which have the property that $\exists s < t \ni \omega$ is constant on (s,t) but discontinuous at t. By [6], $N \in \mathscr{J}$, and, since $N \cap \Omega_0 = \phi$, $N \cap \Gamma_2 = \phi$, so we can ignore N in what follows. We can also ignore the set

$$M = \{\omega: \zeta(\omega) = \infty \text{ and } \exists s \ni \omega \text{ is constant on } (s, \infty)\}$$

for the same reason. Note that if $\omega \notin N \cup M$, each $V \in \mathscr{V}_+$ is a limit point of other $U \in \mathscr{V}_+$ and in particular, $\exists\, U_n$ and $V_n \in \mathscr{V}_+ \ni U_n(\omega) \downarrow T_0(\omega)$ and $V_n(\omega) \uparrow \zeta(\omega)$.

In addition, if $\omega \sim \omega'$, then $\zeta(\omega') < \infty$ iff $\zeta(\omega) < \infty$ (for ζ is intrinsic [6]). We have ignored the time-change on (ζ, ∞), but it is easily handled: if $t > \zeta(\omega)$, set $T_t(\omega) = T_\zeta(\omega) + t - \zeta(\omega)$ and $A_t(\omega) = A_\zeta(\omega) + t - \zeta(\omega)$.

Proof of Theorem 2.1. We must verify that the quantities A, T and Γ defined above satisfy the conclusions of the theorem U, V and W will represent elements of $\mathscr{V}_+$.

If $U(\omega) < V(\omega) < t$, $\exists W \in \mathscr{V}_+ \ni V(\omega) = U(\omega) + W(\theta_U \omega)$ by (2.4), so if $\omega \in \Gamma$, we apply (2.5):

$$\hat{A}_{t-U(\omega)}(\theta_U \omega) = \hat{A}_W(\theta_U \omega) + \hat{A}_{t-V(\omega)}(\theta_V \omega) \geqq \hat{A}_{t-V(\omega)}(\theta_V \omega). \tag{2.10}$$

Thus the lim inf as $V(\omega) \downarrow T_0(\omega)$ in (2.7) is actually an increasing limit. The limit is finite by definition of Γ.

We now verify (2.1). There are four cases, although if ω is nowhere constant, only case 4 occurs.

Case 1. If $T_0(\omega) \geqq s + t$, both sides of (2.1) vanish.

Case 2. $0 \leqq s \leqq T_0(\omega) < s + t$. Then $T_0(\omega) = s + T_0(\theta_s \omega)$, and $\hat{A}_s(\omega) = 0$ while

$$A_{t+s}(\omega) = \lim_V \hat{A}_{t+s-V(\omega)}(\theta_V \omega).$$

Choose U such that $V(\omega) = s + U(\theta_s \omega)$ (by (2.4) again):

$$= \lim_U \hat{A}_{t-U(\theta_s \omega)}(\theta_U \theta_s \omega) = A_t(\theta_s \omega).$$

Case 3. $T_0(\omega) < s$, $T_0(\theta_s \omega) \geqq t$.

Let $U(\omega) < s$. Since $T_0(\theta_s \omega) \geqq t$, ω is constant on $[s, t]$, hence $\theta_U \omega$ is constant on $[s - U, s + t - U]$, so by Lemma 2.2 (vii), $\hat{A}(\theta_U \omega)$ is constant on the same interval, and

$$\begin{aligned} A_{t+s}(\omega) &= \lim_U \hat{A}_{t+s-U(\omega)}(\theta_u \omega) \\ &= \lim_U \hat{A}_{s-U(\omega)}(\theta_U \omega) = A_s(\omega). \end{aligned}$$

Since $T_0(\theta_s \omega) \geqq t$, $A_t(\theta_s \omega) = 0$ and (2.1) follows.

Case 4. $T_0(\omega) < s$, $T_0(\theta_s \omega) < t$.

Choose $U(\omega) < s < V(\omega) < s + t$, and choose $W, W' \ni V(\omega) = U(\omega) + W(\theta_U \omega) = s + W'(\theta_s \omega)$. Then

$$\begin{aligned} A_{s+t}(\omega) &= \lim_U \hat{A}_{s+t-U(\omega)}(\theta_U \omega) \\ &= \lim_U \hat{A}_{V(\omega)-U(\omega)}(\theta_U \omega) + \hat{A}_{s+t-V(\omega)}(\theta_W \theta_U \omega) \end{aligned}$$

where we have used the fact that $\theta_U \omega \in \Gamma_2$ to apply (2.5). Now $\theta_\omega \theta_U \omega = \theta_{W'} \theta_s \omega$, so

$$= A_V(\omega) + \hat{A}_{t-W'(\theta_s \omega)}(\theta_{W'} \theta_s \omega).$$

Let $W'(\theta_s\omega)\downarrow T_0(\theta_s\omega)$. Then $A_V \to A_s$ so:

$$=A_s(\omega)+A_t(\theta_s\omega),$$

proving (v).

Define $p\colon \Omega\to\Omega$ by

$$p(\omega)(t)=\begin{cases}\omega(T_t(\omega)) & \text{if } \omega\in\Gamma\\ \delta & \text{otherwise.}\end{cases}$$

If $\omega\in\Gamma_2$, $A_t(\omega)=\hat{A}_t(\omega)$ by (2.5) and (2.6). Inverting, $T_t(\omega)=\hat{T}_t(\omega)$. Now let $\omega\sim\omega'\in\Gamma$ and $V\in\mathscr{V}_+$. Then $\theta_V\omega\sim\theta_V\omega'\in\Gamma_2$ so by Lemma 2.2

$$\begin{aligned}\theta_V\omega'(\hat{T}_t(\theta_V\omega'))&=\theta_V\omega(\hat{T}_t(\theta_V\omega))\\ &=\omega(V(\omega)+T_t(\theta_V\omega))\\ &=\omega(T_{A_V(\omega)+t}(\omega))\end{aligned}$$

which follows since (2.6) holds for T as well as $\hat{T}$ – one gets it by inverting (2.1). Doing the same for ω', we see that $\omega'(T_{A_V(\omega)+t}(\omega))=\omega(T_{A_V(\omega)+t}(\omega))$. Let V decrease so that $A_V\to 0$. By right continuity, $\omega'(T_t(\omega'))=\omega(T_t(\omega))$, i.e. $p(\omega)=p(\omega')$. But p is clearly $\mathscr{F}^*$ measurable, and constant on the atoms of $\mathscr{T}^*$, hence it is $\mathscr{T}^*$-measurable.

Returning to (2.10), notice that if $s<t$ and $T_0(\omega)<V(\omega)<s$, then $A_t(\omega)-A_s(\omega)=A_{t-V(\omega)}(\theta_V\omega)-A_{s-V(\omega)}(\theta_V\omega)$. It follows from Lemma 2.2(vii) that $A_s(\omega)=A_t(\omega)$ iff ω is constant on $(s,t]$. It is not hard to see that the same is true if $T_0(\omega)\geqq s$, too.

To prove (iv), let $\omega'=p(\omega)$, so $\omega'(t)=\omega(T_t(\omega))$. We claim that $\omega(t)=\omega'(A_t(\omega))$. If $T_0(\omega)=0$ and T_t is continuous, this is clear, for T_t is then onto and A is its inverse. This remains true when T is discontinuous, for if $T_{t-}(\omega)<T_t(\omega)$, then A is constant on the interval $[T_{t-}(\omega),T_t(\omega)]$, hence so is ω by the above remarks. Thus $\omega\sim\omega'$, proving (iv).

To prove (iii), note that by (2.1), if $\omega\in\Gamma$, $t\to A_t(\theta_s\omega)$ will be continuous if $t\to A_t(\omega)$ is. Moreover, it follows from (2.4) that for any s and any $V\in\mathscr{V}_+$, there is a $W\in\mathscr{V}_+$ such that $\theta_V(\theta_s\omega)=\theta_W\omega$, so $\theta_V\Gamma_2^+\subset\Gamma_2^+\Rightarrow\theta_s\Gamma_2^+\subset\Gamma_2^+$. (iii) now follows from Lemma 2.4 and the remarks preceeding it.

Next, note that $P^\mu_{\mathscr{T}}(\omega,\cdot)$ sits on $\Gamma_0\cap\{\omega'\sim\omega\}$ by Lemma 2.2(ii). By Lemma 2.2(v), this last set is the singleton $\{p(\omega)\}$. Thus $P^\mu_{\mathscr{T}}(\omega,p(\omega))=1$ P^μ-a.e., proving (ii). Then (i) follows immediately. QED

§ 3. Applications to Time Changes

Any process having right continuous paths with left limits in $E\cup\delta$ and admitting a lifetime can be defined canonically on the space $(\Omega,\mathscr{F})$ of § 1. Let X be a strong Markov process satisfying (H1)–(H3), and suppose X is defined canonically on $(\Omega,\mathscr{F})$. Let A_t and T_t be the additive functional and its inverse described in Theorem 2.1. Then T_t is a universal time-change for canonically defined processes in the following sense.

Theorem 3.1. *Let Y be a process defined canonically on $(\Omega, \mathscr{F})$. Suppose X and Y have distributions P and Q respectively. If $Q|_{\mathscr{T}} = P|_{\mathscr{T}}$, then X is a time-change of Y in the sense that $\{Y_{T_t}, t \geqq 0\}$ has the same distribution as $\{X_t, t \geqq 0\}$.*

Proof. Let p: $\Omega \to \Omega$ map $\omega(t) \to \omega(T_t(\omega))$. Define

$$\begin{aligned}\hat{X}_t(\omega) &= Y_{T_t}(\omega)\\ &= \omega(T_t(\omega)) = p(\omega)(t).\end{aligned}$$

Now if $\Lambda \in \mathscr{F}$, Λ is a set in path space, so $\{\omega\colon \hat{X}_{\cdot}(\omega) \in \Lambda\} = p^{-1}(\Lambda)$ makes sense. Thus

$$\begin{aligned}Q(X_{\cdot} \in \Lambda\} &= Q(p^{-1}(\Lambda))\\ &= P(p^{-1}(\Lambda)\}\end{aligned}$$

since $p^{-1}(\Lambda) \in \mathscr{T}^*$ and $Q = P$ on $\mathscr{T}$, hence on $\mathscr{T}^*$. But P sits on $\{\omega\colon p(\omega) = \omega\}$, so this is

$$\begin{aligned}&= P\{p^{-1}(\Lambda) \cap \{p(\omega) = \omega\}\}\\ &= P\{\Lambda\}\\ &= P\{X_{\cdot} \in \Lambda\}. \quad \text{QED}\end{aligned}$$

Let us define stopping times S_t, $t \geqq 0$ by

$$S_t(\omega) = \inf\{s > 0\colon d(X_s, X_0) > t\}.$$

Define times T_{nj} in terms of the S_t: if $n \geqq 1$ is an integer, set

$$T_{n0} = 0, \qquad T_{n1} = S_{2^{-n}}, \quad \text{and} \quad T_{nj+1} = T_{nj} + S_{2^{-n}} \circ \theta_{T_{nj}}.$$

Here is a useful criterion for deciding whether two measures agree on $\mathscr{T}$.

Proposition 3.2. *Let P and Q be probability measures on $(\Omega, \mathscr{F})$ such that $P\{\Omega_0\} = Q\{\Omega_0\} = 1$. A necessary and sufficient condition that $P|_{\mathscr{T}} = Q|_{\mathscr{T}}$ is that for each large enough n, each K, and each collection $A_1, \dots, A_K$ of open subsets of $E \cup \delta$,*

$$P\{X_{T_{nk}} \in A_k,\ k = 1, \dots, K\} = Q\{X_{T_{nk}} \in A_k,\ k = 1, \dots, K\}. \tag{3.2}$$

Proof. Let us abuse notation and use two different letters to denote the canonical process on Ω: X is the process whose distribution is P and Y the process whose distribution is Q. Write $X_{nk} = X_{T_{nk}}$ and $Y_{nk} = Y_{T_{nk}}$. By Theorem 2.1 of [6], $\mathscr{T}$ is generated - up to null sets - by the X_{nk}. (The auxiliary variables Q_{nk} vanish a.s. under both P and Q by our hypothesis - see Remark (2.5).) Now (3.2) is clearly necessary. To see it is sufficient, we must show that for any M, the families $\{X_{T_{nk}}, n \geqq 0, k \geqq 0\}$ and $\{Y_{_{nk}}, n \geqq 0, k \geqq 0\}$ have the same distribution. What we know from (3.2) is that for each *fixed* n, $\{X_{T_{nk}}, k = 0, 1, 2, \dots\}$ and $\{Y_{T_{nk}}, k = 0, 1, 2, \dots\}$ have the same distribution.

For each n, let

$$S_t^n(\omega) = \inf_j \{T_{nj}(\omega)\colon d(X_{T_{nj}}, X_0) > t\}.$$

Let us establish some of the elementary properties of the S_t^n.

1. Note that $X_{S_t^n}$ is a function of the $\{X_{T_{nj}}, j=0,1,2,\ldots\}$, for if $N=\inf\{j: d(X_{T_{nj}}, X_0)>t\}$, then $X_{S_t^n}=X_{T_{nN}}$.

2. $S_t \leqq S_t^n$ for all $t \geqq 0$ and $S_t^n \leqq S_{t+\varepsilon}$ if $t \geqq 0$ and $2^{-n}<\varepsilon$. Indeed, the first statement is clear. For the second, note that $d(X_v, X_{T_{nj}}) \leqq 2^{-n}<\varepsilon$ if $T_{nj} \leqq v < T_{n+1,j}$, so that if $S_t^n \geqq u$, $d(X_v, X_0) \leqq t+\varepsilon$, hence $S_{t+\varepsilon} \geqq u$ as well.

3. Let (a_n) be a sequence of positive reals converging to zero and let $0<\varepsilon<t$. Then for all large enough n,

$$S_{t-\varepsilon}(\omega) \leqq a_n + S_t(\theta_{a_n}\omega) \leqq S_{t+\varepsilon}(\omega).$$

To see the first inequality, note that $d(X_{a_n}, X_0) \to 0$ by right continuity of X_t. If $S_{t-\varepsilon}(\omega)>u$, then for all $v \leqq u$, $d(X_v, X_0) \leqq t-\varepsilon$. Thus $d(X_v, X_{a_n}) \leqq d(X_v, X_0) + d(X_0, X_{a_n})$. For large n, this is smaller than t, hence $S_t(\theta_{a_n}\omega) \geqq u - a_n$. This proves the first inequality, and the second follows by a similar argument.

Fix N. We now approximate the times T_{Nj} by iterates of the S_t^n for $n>N$. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of strictly positive reals, and let $t=2^{-N}$. Then define

$$T^n(\varepsilon_1)=S^n_{t+\varepsilon_1}, \qquad T^n(\varepsilon_1,\varepsilon_2)=S^n_{t+\varepsilon_1}+S^n_{t+\varepsilon_2}\circ\theta_{S^n_{t+\varepsilon_1}},$$

and, by induction,

$$T^n(\varepsilon_1,\ldots,\varepsilon_{j+1})=T^n(\varepsilon_1,\ldots,\varepsilon_j)+S^n_{t+\varepsilon_{j+1}}\circ\theta_{T^n(\varepsilon_1,\ldots,\varepsilon_j)}.$$

4. (i) $\lim\limits_{\varepsilon_{j-1}\downarrow 0} \ldots \lim\limits_{\varepsilon_1\downarrow 0} \lim\limits_{n\to\infty} T^n(\varepsilon_1,\ldots,\varepsilon_j) \geqq T_{Nj}$;

(ii) $\lim\limits_{\varepsilon_j\downarrow 0} \ldots \lim\limits_{\varepsilon_1\downarrow 0} \lim\limits_{n\to\infty} T^n(\varepsilon_1,\ldots,\varepsilon_j) = T_{Nj}$.

We will verify (i) and (ii) in the following form.

(iii) There exist strictly positive functions n_j and $d_{j1},\ldots,d_{jj}$ such that if $\delta>0$ and if $n \geqq n_j(\varepsilon_1,\ldots,\varepsilon_j)$, $0<\varepsilon_i \leqq d_{ji}(\varepsilon_{i+1})$ for $i=1,\ldots,j-1$, and if $0<\varepsilon_j \leqq d_{jj}(\delta)$, then

$$T_{Nj}(\omega) \leqq T^n(\varepsilon_1,\ldots,\varepsilon_j;\omega) \leqq T_{Nj}(\omega)+\delta.$$

(The functions n_j and d_{ji} depend on ω, but we have suppressed this in the notation.)

We will prove (iii) by induction. The case $j=1$ follows from 2 and the right continuity of $t \to S_t$. Suppose (iii) holds for $j=1,\ldots,k$.

Let $\omega'=\theta_{T_{Nk}}\omega$ and let $\delta>0$. By the right continuity of $s \to S_s$, there exists $d = d(\delta)>0$ such that if $0<\varepsilon<d$, then

$$S_{t+3\varepsilon}(\omega') \leqq S_t(\omega')+\delta.$$

Having chosen ε, we can apply 3 to choose $\delta'=\delta'(\varepsilon)>0$ such that if $0 \leqq a \leqq \delta'$, then

$$S_t(\omega') \leqq a + S_{t+\varepsilon}(\theta_a\omega') \leqq a + S_{t+2\varepsilon}(\theta_a\omega') \leqq S_{t+3\varepsilon}(\omega').$$

Now choose $n_0=n_0(\varepsilon)$ such that $2^{-n_0}<\varepsilon$. If $n \geqq n_0$, we have by 2 that

$$S_{t+\varepsilon}(\theta_a\omega') \leqq S^n_{t+\varepsilon}(\theta_a\omega') \leqq S_{t+2\varepsilon}(\theta_a\omega'),$$

so that by the previous inequalities

$$S_t(\omega') \leqq a + S^n_{t+\varepsilon}(\theta_a \omega') \leqq S_t(\omega') + \delta.$$

Now choose $\varepsilon_1 \leqq d_{k1}(\varepsilon_2), \ldots, \varepsilon_k \leqq d_{kk}(\delta'(\varepsilon))$ and $n \geqq n_k(\varepsilon_1, \ldots, \varepsilon_k)$. Then by (iii), which holds for $j=k$ by the induction hypothesis,

$$T_{Nk} \leqq T^n(\varepsilon_1, \ldots, \varepsilon_k) \leqq T_{Nk} + \delta'.$$

Set $\varepsilon_{k+1} = \varepsilon$ and $a = T^n(\varepsilon_1, \ldots, \varepsilon_k) - T_{Nk}$. Then

$$T^n(\varepsilon_1, \ldots, \varepsilon_{k+1}) = T_{Nk} + a + S^n_{t+\varepsilon_{k+1}}(\theta_a \omega').$$

Now $T_{Nk+1} = T_{Nk} + S_t(\omega')$. Make n larger if necessary, so that $n \geqq n_0(\varepsilon)$ to see that (iii) holds for $j=k+1$ with the functions $d_{k+1,k+1}(\delta) = d(\delta)$; $d_{k+1,k}(\varepsilon) = d_{kk}(\delta'(\varepsilon))$; $d_{k+1,i}(\varepsilon) = d_{ki}(\varepsilon)$, $1 \leqq i \leqq k-1$; and $n_{k+1}(\varepsilon_1, \ldots, \varepsilon_{k+1}) = \max(n_0(\varepsilon_{k+1}), n_k(\varepsilon_1, \ldots, \varepsilon_k))$.

But this implies the proposition, since by hypothesis $\{X_{T_{nj}}, j=0,1,\ldots\}$ and $\{Y_{T_{nj}}, j=0,1,\ldots\}$ have the same distributions, hence $X_{T^n(\varepsilon_1,\ldots,\varepsilon_j)}$ and $Y_{T^n(\varepsilon_1,\ldots,\varepsilon_j)}$ have the same distributions, being functions of the $X_{T_{nj}}$ and $Y_{T_{nj}}$ respectively (see 1). By 4 and the right continuity of X_t, $X_{T^n(\varepsilon_1,\ldots,\varepsilon_j)} \to X_{T_{Nj}}$ and $Y_{T^n(\varepsilon_1,\ldots,\varepsilon_j)} \to Y_{T_{Nj}}$ as $n \to \infty$ and the ε_i decrease to zero in the right order. This implies that for any bounded continuous f, $E\{f(X_{T_{Nj}})\}$ is the limit of $E\{f(X_{T^n(\varepsilon_1,\ldots,\varepsilon_j)})\}$. The same holds for Y, so it follows that $X_{T_{Nj}}$ and $Y_{T_{Nj}}$ have the same distribution. We can do this simultaneously for $N=1,\ldots,M$ and $j=0,\ldots,M$, so that we can conclude that $\{X_{T_{Nj}}: N, j=0,\ldots,M\}$ and $\{Y_{T_{Nj}}: N, j=0,\ldots,M\}$ have the same distribution. QED

Notice that the process Y of Theorem 3.1 need not be Markov and its paths need not be nowhere constant. We can apply it to a martingale to get the Dubins-Schwartz theorem as follows.

Theorem 3.3 (Dubins-Schwartz). *Let $\{M_t, \mathscr{G}_t, t \geqq 0\}$ be a continuous martingale whose paths are unbounded, such that $M_0 = 0$. Then M can be time-changed into a standard Brownian motion.*

Proof. Suppose M and a standard Brownian motion B are defined canonically on Ω with distributions Q and P respectively. The paths of M are unbounded so that if T_{nk} is the stopping time introduced above, $Q\{T_{nk} < \infty\} = 1$ for each n and k. Note that for each n, $M_{T_{n1}}, M_{T_{n2}}, \ldots$ is a symmetric random walk on $2^{-n}\mathbb{Z}$. Indeed, by continuity $M_{T_{nj+1}} - M_{T_{nj}} = \pm 2^{-n}$ and by the stopping theorem

$$Q\{M_{T_{nj+1}} - M_{T_{nj}} = 1 \mid \mathscr{F}_{T_{nj}}\} = Q\{M_{T_{nj+1}} - M_{T_{nj}} = -1 \mid \mathscr{F}_{T_{nj}}\} = 1/2,$$

so that the differences $M_{T_{nj+1}} - M_{T_{nj}}$, $j=1,2,\ldots$ are iid. The same is true of B, so that Lemma 3.2 implies that $Q=P$ on $\mathscr{T}$. The conclusion follows from Theorem 3.1. QED

Note. With hindsight one can see that the main idea in Dubins' and Schwartz's original proof was to prove that $Q=P$ on $\mathscr{T}$, using embedded random walks.

They then use the quadratic variation to construct the time-change. This remains perhaps the most interesting, though not the shortest, proof of this theorem [7].

Following [3], we apply Theorem 3.1 to the case in which Y is itself a Markov process. One result of this is a general version of the Blumenthal-Getoor-McKean theorem. Since this theorem has recently been generalized by J. Glover [8] to cover our approximate situation, we will only prove the special case in which the processes have no traps or holding points, which is a rather direct consequence of Theorem 3.1.

We should point out, however, that the general case can be reduced to this. We will indicate how after the proof.

Definition. If B is Borel in E, let $T_B(\omega)=\inf\{t\colon \omega(t)\in B\}$. For a strong Markov process X, define

$$\pi_B(x, A)=P^x\{X_{T_B}\in A\}.$$

Theorem 3.4. *Let X and Y be strong Markov processes on $E\cup\delta$ having no traps or holding points in E, and which satisfy* (H1)–(H3). *Suppose that they both have the same hitting probabilities π_B and the same initial distribution. Then there exists a perfect continuous additive functional A_t whose inverse T_t is also continuous, such that the process $\{Y_{T_t}, t\geqq 0\}$ has the same distribution as $\{X_t, t\geqq 0\}$.*

Proof. Suppose X and Y are canonically defined on $(\Omega, \mathscr{F})$ with distributions P and Q respectively. We need only show that $P=Q$ on $\mathscr{T}$ and then apply Theorem 3.1. Both X and Y have nowhere-constant paths, so the time change is necessarily continuous and strictly increasing, as is A_t (see e.g. Lemma 2.2 (v)).

To show $P=Q$ on $\mathscr{T}$, consider $T_{n1}<T_{n2}<\ldots<T_{nk}$ and let $D(x,r)=\{y\colon d(x,y)>r\}$. If $A_1, \ldots, A_k$ are Borel,

$$P^x\{X_{T_{nj}}\in A_j,\ j=1,\ldots,k\}=E^x\{X_{T_{nj}}\in A_j,\ j=1,\ldots,k-1;\ P\{X_{T_{nk}}\in A_k|\mathscr{F}_{T_{nk-1}}\}\}.$$

But T_{nk} is the first time after T_{nk-1} that X hits $D(X_{T_{nk-1}}, 2^{-n})$ so by the strong Markov property

$$=E^x\{X_{T_{nj}}\in A_j, j=1,\ldots,k-1, \pi_{D(X_{T_{nk-1}}, 2^{-n})}(A_k)\}$$
$$=\int\ldots\int \pi_{D(x_1, 2^{-n})}(x_1, dx_2)\ldots \pi_{D(x_{k-1}, 2^{-n})}(x_{k-1}, A_k).$$

Since X and Y have the same hitting probabilities:

$$=Q^x\{Y_{T_{nj}}\in A_j, j=1,\ldots,k\}.$$

Now apply Lemma 3.2 to see that $P=Q$ on $\mathscr{T}$. QED

Remark 1. Suppose X has holding points. If x is a holding point for X, it is a holding point for Y, since then $\Pi_{E-x}(x,\{x\})=0$. Now X holds at x for an exponential time with parameter, say, $\lambda(x)$; the holding time for Y is also exponential, with parameter, say, $\mu(x)$. To time-change Y to X, we must use an additive functional of the form $dA_t=\dfrac{\lambda(x)}{\mu(x)}dt$ when $Y_t=x$. Thus, make a preliminary time-change which affects only the holding points: $dA_t = \frac{\lambda(x)}{\mu(x)} dt$ if $Y_t = x$ and x is a holding point, and $dA_t = dt$ otherwise. One must check that A_t is finite, which is true through not immediate.

After this change, X and Y have the same holding points with the same parameters but are otherwise unchanged. Now let $a(t, X_t) = t - \sup\{s < t: X_s \neq X_t\}$ be the *age* of the state occupied by X_t, and consider the pair $(X_t, a(t, X_t))$ as a process on $E \cup \delta \times \mathbb{R}_+$. It is still strongly Markov, and it has the same hitting probabilities - in $E \cup \delta \times \mathbb{R}_+$ - as $(Y_t, a(t, Y_t))$. (For $a \equiv 0$ when the process is not at a holding point, and the processes are identical on the holding points thanks to the time-change.) Moreover they have nowhere-constant paths since $a(t, X_t)$ increases when the process is at a holding point, so that Theorem 3.4 now applies.

Remark 2. It is not necessary to know the hitting probabilities of X and Y for all Borel sets. It is enough to have them equal for sets of the form $D(x, r)$ for small enough r, for these are all that enter the proof.

Y need not be strongly Markov. It is enough that its hitting distributions depend only on the state [5]. More exactly:

Corollary 3.5. *Let X be as in Theorem 3.4 with hitting probabilities π_B. Suppose Y is a process defined canonically on $(\Omega, \mathscr{F})$ with distribution Q, and that $Q\{\Omega_0\} = 1$. Suppose further that for each pair of Borel sets A and B, and each stopping time T, if $\tau = T + T_B \circ \theta_T$, then*

$$P\{Y_\tau \in A \mid \mathscr{F}_T\} = \pi_B(Y_T, A).$$

If X and Y have the same initial distribution, then there exists a perfect continuous additive functional whose inverse T_t is continuous, such that $\{Y_{T_t}, t \geq 0\}$ has the same distribution as $\{X_t, t \geq 0\}$.

Note that there is nothing new to prove here, for the proof of Theorem 3.4 is still valid. We might point out that this extension is non-trivial, for the Dubins-Schwartz theorem is a special case of Cor. 3.5, but not of Theorem 3.4.

Appendix

We will prove two related results which are needed in sections one and two. The first one is a part of the folklore of the subject. We were unable to find an exact reference so we have provided a short and elementary proof. The second may be new.

Proposition A. *Let X and Y_{nj}, $n = 1, 2, \ldots, j = 1, 2, \ldots$ be positive random variables such that*

(i) $\sup_j Y_{nj} \to 0$ *in probability as* $n \to \infty$;

(ii) *for each* n, $Y_{n1}, Y_{n2}, \ldots$ *are independent;*

(iii) *for each* n, $X = \sum_j Y_{nj}$.

Then X is a constant.

Proof. Fix $0<\varepsilon<1$ and set $Z_{nj}=\varepsilon\wedge Y_{nj}$ and $X_n=\sum_j Z_{nj}$. Then

$$E\{e^{-X_n}\}=\prod_j E\{e^{-Z_{nj}}\}.$$

Let $\alpha=1-\varepsilon/2$. $Z_{nj}\leqq\varepsilon$ so $e^{-Z_{nj}}\leqq 1-\alpha Z_{nj}$, and

$$0<E\{e^{-X_n}\}=\prod_j (1-\alpha E\{Z_{nj}\}).$$

It follows that $E\{X_n\}=\sum_j E\{Z_{nj}\}<\infty$. But

$$\log(\prod_j (1-\alpha E\{Z_{nj}\}))=\sum_j \log(1-\alpha E\{Z_{nj}\})\leqq -\alpha E\{X_n\}$$

so that

$$E\{e^{-X_n}\}\leqq e^{-\alpha E\{X_n\}}.$$

Now $X_n\leqq X$, and $X_n\to X$ as $n\to\infty$; indeed, $P\{X_n\neq X\}=P\{\sup_j Y_{nj}>\varepsilon\}\to 0$. Thus by bounded convergence and Fatou's lemma:

$$E\{e^{-X}\}\leqq e^{-\alpha E\{X\}}.$$

The left hand side is strictly positive so $E\{X\}<\infty$. Let $\varepsilon\to 0$, so that $\alpha\to 1$, and use Jensen's inequality:

$$e^{-E\{X\}}\leqq E\{e^{-X}\}\leqq e^{-E\{X\}}.$$

Thus there is equality, but this can happen only if $X=E\{X\}$ a.s. QED

We now extend Proposition A to a statement involving conditional probabilities.

Proposition B. *Let X and $Y_{n1}, Y_{n2}, \ldots$ be positive random variables, and let $\mathscr{F}_1\subset\mathscr{F}_2\subset\ldots$ be σ-fields such that*

(i) $\sup_j Y_{nj}\to 0$ *a.e. as* $n\to\infty$;

(ii) *for each n,* $Y_{n1}, Y_{n2}, \ldots$ *are conditionally independent given* $\mathscr{F}_n$;

(iii) *for each n,* $X=\sum_j Y_{nj}$.

Then X is measurable with respect to $\mathscr{F}_\infty=\bigvee_n \mathscr{F}_n$, *and if* $X_n^\varepsilon=\sum_j \varepsilon\wedge Y_{nj}$, *then*

$$X=\lim_{\varepsilon\to 0}[\lim_{n\to\infty} E\{X_n^\varepsilon|\mathscr{F}_n\}] \qquad (1)$$

where we let $\varepsilon\to 0$ *through any sequence.*

Note. We do not claim that either X_n or X is integrable, but that $E\{X_n^\varepsilon|\mathscr{F}_n\}$ is defined and finite.

Proof. Set $Z_{nj}=\varepsilon\wedge Y_{nj}$. Then $X_n^\varepsilon=\sum_j Z_{nj}$, and

$$E\{e^{-X_n^\varepsilon}|\mathscr{F}_n\}=\prod_j E\{e^{-Z_{nj}}|\mathscr{F}_n\}.$$

Let $\alpha = 1 - \varepsilon/2$. $0 \leqq Z_{nj} \leqq \varepsilon$ so that $e^{-Z_{nj}} \leqq 1 - \alpha Z_{nj}$. Thus

$$0 < E\{e^{-X_n^\varepsilon} | \mathscr{F}_n\} \leqq \prod_j (1 - \alpha E\{Z_{nj} | \mathscr{F}_n\}). \tag{2}$$

The product is strictly positive, so $E\{X_n^\varepsilon | \mathscr{F}_n\} = \sum_j E\{Z_{nj} | \mathscr{F}_n\}$ must be finite a.s. Now $\log(1 - \alpha E\{Z_{nj} | \mathscr{F}_n\}) \leqq -\alpha E\{Z_{nj} | \mathscr{F}_n\}$ so

$$\log \prod_j (1 - \alpha E\{Z_{nj} | \mathscr{F}_n\}) \leqq -\alpha E\{X_n^\varepsilon | \mathscr{F}_n\}$$

and (2) becomes

$$E\{e^{-X_n^\varepsilon} | \mathscr{F}_n\} \leqq e^{-\alpha E\{X_n^\varepsilon | \mathscr{F}_n\}}. \tag{3}$$

Let $n \to \infty$. $X_n^\varepsilon \to X$; indeed $P\{X_n^\varepsilon \neq X\} = P\{\sup_j Y_{nj} > \varepsilon\}$ which tends to zero. By Hunt's lemma

$$E\{e^{-X_n^\varepsilon} | \mathscr{F}_n\} \to E\{e^{-X} | \mathscr{F}_\infty\}.$$

Similarly, if $M > 0$

$$E\{X_n^\varepsilon | \mathscr{F}_n\} \geqq E\{M \wedge X_n^\varepsilon | \mathscr{F}_n\} \to E\{M \wedge X | \mathscr{F}_\infty\}.$$

Thus $\liminf_{n \to \infty} E\{X_n^\varepsilon | \mathscr{F}_n\} \geqq E\{X | \mathscr{F}_\infty\}$ so that (3) becomes

$$E\{e^{-X} | \mathscr{F}_\infty\} \leqq e^{-\alpha E\{X | \mathscr{F}_\infty\}}. \tag{4}$$

The left-hand side is strictly positive, so that $E\{X | \mathscr{F}_\infty\} < \infty$ a.s. Now let $\varepsilon \to 0$ so that $\alpha \to 1$, and apply Jensen's inequality:

$$e^{-E\{X | \mathscr{F}_\infty\}} \leqq E\{e^{-X} | \mathscr{F}_\infty\} \leqq e^{-E\{X | \mathscr{F}_\infty\}}. \tag{5}$$

Thus there is equality in (5). This can happen only if $X = E\{X | \mathscr{F}_\infty\}$, i.e. if X is $\mathscr{F}_\infty$-measurable.

To prove (1), go back to (3) and use (5):

$$e^{-E\{X | \mathscr{F}_\infty\}} \leqq e^{-\alpha \limsup E\{X_n^\varepsilon | \mathscr{F}_n\}}$$

or

$$\limsup_{n \to \infty} E\{X_n^\varepsilon | \mathscr{F}_n\} \leqq \frac{1}{\alpha} E\{X | \mathscr{F}_\infty\}.$$

Thus with probability one

$$\begin{aligned} E\{X | \mathscr{F}_\infty\} &\leqq \liminf_{n \to \infty} E\{X_n^\varepsilon | \mathscr{F}_n\} \\ &\leqq \limsup_{n \to \infty} E\{X_n^\varepsilon | \mathscr{F}_n\} \leqq \left(1 - \frac{\varepsilon}{2}\right)^{-1} E\{X | \mathscr{F}_\infty\}. \end{aligned}$$

Let $\varepsilon \to 0$ through any sequence to get (1). QED

References

1. Blackwell, D.: On a class of probability spaces. Proc. 3rd Berkeley Sympos. Math. Statist. Probability Univ. Calif. **2**, 1–6 (1956)
2. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968
3. Chacon, R.V., Jamison, B.: A fundamental property of Markov processes with an application to equivalence under time changes. Israel J. Math. **33**, 241–269 (1979)
4. Chacon, R.V., Jamison, B.: Sample path consistency for Markov processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **58**, 169–182 (1981)
5. Chacon, R.V., Jamison, B.: Processes with state-dependent hitting probabilities. Adv. in Math. **32**, 1–35 (1979)
6. Chacon, R., Le Jan, Y., Walsh, J.: Spatial trajectories. Seminaire de Probabilités 15. Lecture Notes in Math. **850**, pp. 290–306. Berlin-Heidelberg-New York: Springer 1982
7. Dubins, L., Schwartz, G.: On continuous martingales. Proc. Nat. Acad. Sci. **53**, 913–916 (1965)
8. Glover, J.: Markov processes with identical hitting probabilities. Trans. Amer. Math. Soc. **275**, 131–141 (1983)
9. Le Jan, Y.: Arc length associated to a Markov process. Adv. in Math. **42**, 136–142 (1981)
10. Meyer, P.-A., Dellacherie, C.: Probabilités et Potentiel. Chapitres 0 à IV. Hermann: Paris 1975

Received May 15, 1983