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Two Coupled Processors: The Reduction to a Riemann-Hilbert Problem

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Résumé. Beaucoup de problèmes liés au couplage de processeurs conduisent à des équations fonctionnelles. En général, les fonctions inconnues représentent les fonctions génératrices d'un processus stationaire. Nous étudions ici un problème particulier, mais la méthode proposée est applicable à des cas très généraux de marches aléatoires à deux dimensions.

Summary. Many problems arising from the coupling of processors require the solution of functional equations. Generally, the unknown functions are the generating functions for a stationary distribution of the studied process. In this paper, a particular problem is addressed but results lead to a computationally reasonable solution which applies to very general two dimensional random walks.

Introduction

Many problems arising from the coupling of processors [or, dualistically speaking, from the sharing of one resource by several classes of customers] require the solution of functional equations. Generally, the unknown functions are the generating functions for a stationary distribution of the studied process. In this paper we considera particular problem but the results offer a computationally reasonable solution which applies to very general two-dimensional random walks.

In the first five sections we describe the problem of coupling and show that the generating function $F(x,y)$ [for the joint distribution of the Markov process associated with the number of jobs in both queues] can be continued as a meromorphic function to the whole complex plane. We do not need a uniformizing parameter as in Flatto and MacKean [2] or in Malyshev [5].

Section VI is devoted to a processor – sharing strategy where only the top jobs in each queue share the processor. This strategy was introduced without analysis by Coffman and Mitrani [1]; it has been shown to be "complete" in the sense that every achievable vector of average response times can be obtained by such processor sharing (Mitrani and Hine [8]). We reduce the problem to a

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Dirichlet problem and obtain closed formulas which include elliptic functions of the third kind.

Section VII treats a more general case, equivalent to a Riemann-Hilbert problem. Again, closed formulas are obtained.

In Sect. VIII, we give conditions under which the preceding procedures are efficient.

I. Problem Formulation and Assumptions

Let us consider two parallel $M/M/1$ queues with infinite capacities under the following assumptions.

a) The arrivals form two independent Poisson processes with parameters λ_1, λ_2.

b) The service times are distributed exponentially with instantaneous service rates S_1 and S_2 depending on the system state in the following manner:

i) $S_1=\mu_1$, $S_2=\mu_2$ if both queues are busy,

ii) $S_1=\mu_1^*$ if queue 2 is empty,

iii) $S_2=\mu_2^*$ if queue 1 is empty.

c) The service discipline is FIFO (first in-first out) in each queue.

Let $p_t(m,n)$ be the probability that, at time t, there are m jobs in queue 1 and n jobs in queue 2.

$p(m,n)=\lim_{t\to\infty} p_t(m,n)$ will be referred to as the stationary probability of the state (m,n). We study the behaviour of the system at the steady state by means of the generating function $F(x,y)$ (see below).

From now on the terms "stability" or "ergodicity" will be used to mean "there exists a stationary distribution".

The Kolmogeroff forward equations for the $p(m,n)$ are the following

$$(\lambda_1+\lambda_2+\mu_1+\mu_2)\,p(m,n)=\lambda_1\,p(m-1,n)+\lambda_2\,p(m,n-1)$$
$$+\mu_1\,p(m+1,n)+\mu_2\,p(m,n+1), \qquad m,n>0, \tag{1.1a}$$

$$(\lambda_1+\lambda_2+\mu_2^*)\,p(0,n)=\lambda_2\,p(0,n-1)$$
$$+\mu_1\,p(1,n)+\mu_2^*\,p(0,n+1), \qquad m=0,n>0, \tag{1.1b}$$

$$(\lambda_1+\lambda_2+\mu_1^*)\,p(m,0)=\lambda_1\,p(m-1,0)$$
$$+\mu_1\,p(m+1,0)+\mu_2\,p(m,1), \qquad m>0,n=0, \tag{1.1c}$$

$$(\lambda_1+\lambda_2)\,p(0,0)=\mu_1^*\,p(1,0)+\mu_2^*\,p(0,1), \qquad m=n=0. \tag{1.1d}$$

We introduce the generating functions.

$$F(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p(m,n)\,x^m y^n$$

which are analytic with respect to x and y whenever $|x|,|y|<1$.

A straightforward but tedious computation yields

$$\boxed{T(x,y)\,F(x,y)=F(0,y)\,a(x,y)+F(x,0)\,b(x,y)+F(0,0)\,c(x,y)} \tag{1.2}$$

where:

$$a(x,y) \overset{\text{def}}{=} \mu_1\left(1-\frac{1}{x}\right)+q\left(1-\frac{1}{y}\right),$$

$$b(x,y) \overset{\text{def}}{=} \mu_2\left(1-\frac{1}{y}\right)+p\left(1-\frac{1}{x}\right),$$

$$c(x,y) \overset{\text{def}}{=} p\left(\frac{1}{x}-1\right)+q\left(\frac{1}{y}-1\right),$$

$$T(x,y) \overset{\text{def}}{=} \lambda_1(1-x)+\mu_1\left(1-\frac{1}{x}\right)+\lambda_2(1-y)+\mu_2\left(1-\frac{1}{y}\right),$$

$$p=\mu_1-\mu_1^*,$$

$$q=\mu_2-\mu_2.$$

Lemma 1.1. *The existence of $F(x,y)$ satisfying the functional equation* (1.2) *with $\sum_{m,n=0}^{\infty} |p(m,n)|<\infty$ (space L_1) is equivalent to stability. Moreover if $F(x,y)$ exists, it is unique up to a constant multiplier which can be suitably chosen so that all coefficients in the power series expansion are positive and sum to* 1. *In that case $F(x,y)$ is the generating function for a stationary distribution.*

Proof. See Malyshev [5].

A glance at relation (1.2) does not give much information concerning $F(x,y)$. However, a further investigation shows that the right side vanishes whenever

$$T(x,y)=0, \quad \text{provided } |x|, \quad |y|\leqq 1.$$

To obtain additional relations between $F(x,0)$ and $F(0,y)$, it is necessary to examine carefully the algebraic curve C defined by

$$R(x,y)=x\,y\cdot T(x,y)=0 \tag{1.3}$$

in the whole complex plane which we do in the next section.

II. $R(x,y)=0$

$R(x,y)$ is a polynomial of third degree w.r.t. x and y and of second degree w.r.t. each variable x or y.

The curve C has genus 1 and can be identified with the Riemann surface C over either of the extended x or y planes (more precisely, the algebraic extension of the field of rational functions of x, as defined by $R(x,y)=0$).

These assertions are well known (see for example Fuchs [3]) and will not be discussed further.

For curves of genus 1, the introduction of a uniformizing parameter requires elliptic functions and reduces the tractability of the computation if we try to construct the analytic continuation of the functions $F(x,0)$ and $F(0,y)$ (for an example concerning a curve of genus 0 (see Flatto and Mac Kean [2]).

Solving $R(x,y)=0$ for y, we have

$$y(x)=\frac{\lambda_2+\mu_2+A(x)\mp\sqrt{(\lambda_2+\mu_2+A(x))^2-4\lambda_2\mu_2}}{2\lambda_2} \tag{2.1}$$

where

$$A(x)=\lambda_1(1-x)+\mu_1\left(1-\frac{1}{x}\right). \tag{2.2}$$

We obtain two distinct branches which give a two sheeted covering over the x plane.

Lemma 2.1. *The algebraic function $y(x)$ defined by $R(x,y)$ has four real branch points x_1, x_2, x_3, x_4 with $0<x_1<x_2<1<x_3<x_4$.*

Proof. From (2.1), it follows that branch points are the zero's of the discriminant

$$\Delta(x)=[\lambda_2+\mu_2+A(x)]^2-4\lambda_2\mu_2$$

This can be written

$$A(x)+(\sqrt{\lambda_2}\mp\sqrt{\mu_2})^2=0$$

or

$$\lambda_1 x^2+\mu_1=(\Sigma+k)x \tag{2.3}$$

where

$$\Sigma=\lambda_1+\lambda_2+\mu_1+\mu_2,$$

$$k=\pm 2\sqrt{\mu_2\lambda_2}.$$

Obviously, the roots of (2.3) are real and positive. Moreover $\lambda_1+\mu_1\leqq\Sigma+k$. It follows that 1 lies between these roots. As k takes two values $+2\sqrt{\mu_2\lambda_2}$ and $-2\sqrt{\mu_2\lambda_2}$, $\Delta(x)$ vanishes for $x=x_1,x_2,x_3,x_4$ with $x_1<x_2<1<x_3<x_4$.

Verifying that

$$x(S)=\frac{\lambda_1+\mu_1+S-\sqrt{\lambda_1+\mu_1+S)^2-4\lambda_1\mu_1}}{2\lambda_1}$$

is a decreasing function of S for $S\geqq 0$, yields

$$x_1=\frac{\lambda_1+\mu_1+(\sqrt{\lambda_2}+\sqrt{\mu_2})^2-\sqrt{(\lambda_1+\mu_1+(\sqrt{\lambda_2}+\sqrt{\mu_2})^2)^2-4\lambda_1\mu_1}}{2\lambda_1}$$

$$x_2 = \frac{\lambda_1+\mu_1+(\sqrt{\lambda_2}-\sqrt{\mu_2})^2-\sqrt{(\lambda_1+\mu_1+(\sqrt{\lambda_2}-\sqrt{\mu_2})^2)^2-4\lambda_1\mu_1}}{2\lambda_1}$$
$$x_3 = \frac{\mu_1}{\lambda_1 x_2} \tag{2.4}$$
$$x_4 = \frac{\mu_1}{\lambda_1 x_1}.$$

The Lemma 2.1 is obviously valid for the branch points y_1, y_2, y_3, y_4 of the function $x(y)$.

Lemma 2.2. *The equation $R(x, y)=0$ has one root $y(x)=h(x)$ which is an analytic algebraic function of x in the whole complex plane cut along the two segments $[x_1 x_2]$ and $[x_3 x_4]$.*

Moreover $|h(x)| \leqq 1$ if $|x|=1$

$$|h(x)| \leqq \sqrt{\frac{\mu_2}{\lambda_2}} \;\forall x.$$

Similar propositions apply to $x(y)$: i.e. there exists $k(y)$ such that

$$(k(y), y)=0,$$
$$|k(y)| \leqq 1 \quad \text{if } |y|=1,$$
$$|k(y)| \leqq \sqrt{\frac{\mu_1}{\lambda_1}} \;\forall y.$$

Proof. The first part of the lemma results from the general theory of polynomials of two complex variables (Fuchs [3]).

The second assertion is proved by using Rouché's theorem as follows:

$R(x, y)=0$ is equivalent to

$$y x\left[\lambda_2+\mu_2+\lambda_1(1-x)+\mu_1\left(1-\frac{1}{x}\right)\right]=[\lambda_2 y^2+\mu_2]\, x.$$

When

$$|x|=1, \quad |\lambda_2+\mu_2+A(x)|>\lambda_2+\mu_2,$$
$$x \neq 1,$$

$A(x)$ given by (2.2).

Hence on the circle $|y|=1$ we have

$$|x y|\,|\lambda_2+\mu_2+A(x)|=|\lambda_2+\mu_2+A(x)|>\lambda_2+\mu_2 \geqq |x|\,|\lambda_2 y^2+\mu_2|.$$

We deduce that $R(x, y)=0$ with $|x|=1$, $x \neq 1$ has exactly one root $h(x)$ inside the unit circle.

$x=1$ yields $\left(\lambda_2-\frac{\mu_2}{y}\right)(1-y)=0$

and $h(1)=\min\left(1, \frac{\mu_2}{\lambda_2}\right)$.

Observing that $\alpha(y)=\frac{\mu_2}{\lambda_2 y}$ is a conformal map (automorphism) of the Riemann surface onto itself $[R(x,\alpha(y))=R(x,y)]$, we deduce that the second root of $R(x,y)=0$ is $\frac{\mu_2}{\lambda_2 h(x)}$. Moreover, when $x\in[x_1 x_2]\cup[x_3 x_4]$, $h(x)$ and $\frac{\mu_2}{\lambda_2 h(x)}$ are complex conjugate (from Lemma 2.1) with a common modules equal to $\sqrt{\frac{\mu_2}{\lambda_2}}$.

The curve $H=\{h(x), |x|=1\}$ is simple and closed.

The region inside the circle $|x|=1$ cut along the real axis from x_1 to x_2 is mapped by $h(x)$ onto the ring shaped region between the curve H and the circle $|y|=\sqrt{\frac{\mu_2}{\lambda_2}}$.

The last assertion of the lemma is derived from the "maximum modulus principle" (Fuchs [3], pp. 201–203) since

i) $h(x)=h\left(\frac{\mu_1}{\lambda_1 x}\right)=0$ if $x=\infty$,

ii) $h(x)$ is analytic of x in the whole complex plane cut along $[x_1 x_2]\cup[x_3 x_4]$.

Thus, the maximum modulus of $h(x)$ can be reached only on the boundary $[x_1 x_2]\cup[x_3 x_4]$.

But $|h(x)|=\sqrt{\frac{\mu_2}{\lambda_2}}$ if $x\in[x_1 x_2]\cup[x_3 x_4]$.

Remark. Upon setting $y=\sqrt{\frac{\mu_2}{\lambda_2}}\,z$, we get $\lambda_2(1-y)+\mu_2\left(1-\frac{1}{y}\right)=\lambda_2+\mu_2-\sqrt{\lambda_2\mu_2}\left(z+\frac{1}{z}\right)$.

Replacing z by $\frac{1}{z}$, this expression is unaffected.

Therefore $h(x)$ is the root corresponding to $|z|\leqq 1$ and it follows at once $|h(x)|\leqq\sqrt{\frac{\mu_2}{\lambda_2}}$.

The proof of Lemma 2.2 is thus terminated.

Lemma 2.3.

$$k(h(x))=\begin{cases} x & \text{if } |x|\leqq\sqrt{\frac{\mu_1}{\lambda_1}} \\ \frac{\mu_1}{\lambda_1 x} & \text{if } |x|>\sqrt{\frac{\mu_1}{\lambda_1}} \end{cases}$$
$$h(k(y))=\begin{cases} y & \text{if } |y|\leqq\sqrt{\frac{\mu_2}{\lambda_2}} \\ \frac{\mu_2}{\lambda_2 y} & \text{if } |y|>\sqrt{\frac{\mu_2}{\lambda_2}} \end{cases}. \tag{2.5}$$

The proof is easy using the arguments of Lemma 2.2.

It is important to note the strict inequalities in (2.5): for example $k(h(x)) = \frac{\mu_1}{\lambda_1 x}$ if $|x| > \sqrt{\frac{\mu_1}{\lambda_1}}$ because on the circle of radius $|x| = \sqrt{\frac{\mu_1}{\lambda_1}}$ there are two conjugate roots k and $\bar{k} = \frac{\mu_1}{\lambda_1 k}$, but k is reached from inside.

Thereby proof is concluded.

III. The Analytic Continuation of $F(x,0)$ and $F(0,y)$

Convention. To avoid repetitions, we introduce the notations:

$$C(R) = \{z \mid \text{modulus}(z) = R\},$$
$$B(R) = \{z \mid \text{modulus}(z) < R\},$$
$$\overline{B(R)} = \{z \mid \text{modulus}(z) > R\}.$$

In Sect. II, we have defined two curves

$$H = \{h(z)/|z| = 1\},$$
$$K = \{k(z)/|z| = 1\}.$$

which are simple, closed and lie inside $C(1)$. Similarly, let H' and K' be the curves obtained from H and K under the mappings (automorphisms) $z \to \frac{\mu_2}{\lambda_2 z}$ and $z \to \frac{\mu_1}{\lambda_1 z}$ respectively, i.e.:

$$H' = \frac{\mu_2}{\lambda_2 H},$$
$$K' = \frac{\mu_1}{\lambda_1 K}.$$

H' and K' are also simple, closed and lie outside $C(1)$. [The point $z=1$ on the positive real axis belongs to H' $\left((\text{resp. } K') \text{ iff } \frac{\mu_2}{\lambda_2} < 1\right)$ $\left(\text{resp. } \frac{\mu_1}{\lambda_1} < 1\right)$ as shown in Sect. II.]

We shall use throughout this paper the functions $h(x)$ and $k(y)$ introduced in II.

Theorem 3.1. 1) $F(x,0)$ [resp. $F(0,y)$] *can be continued as a meromorphic function to the whole complex plane cut along the real axis from* x_3 to x_4 [resp. *from* y_3 *to* y_4].

2) *The poles of* $F(x,0)$ [*resp.* $F(0,y)$] *are the zeros (if any) of* $\mu_2\left(1-\frac{1}{h(x)}\right)+p\left(1-\frac{1}{x}\right)$ $\left[\textit{resp. } \mu_1\left(1-\frac{1}{k(y)}\right)+q\left(1-\frac{1}{y}\right)\right]$ *outside* $C(1)$.

Proof. All the information known about $F(x,0)$ and $F(0,y)$ is that they are analytic for $|x|$, $|y|\leqq 1$.

We define two regions R_x and R_y as follows: R_y is the region which lies between the curves $C(1)$ and H. R_x is

$$\text{the region}\begin{cases}\text{between the curves } C(1) \text{ and } K & \text{if } \frac{\mu_1}{\lambda_1}>1\\ B(1) & \text{if } \frac{\mu_1}{\lambda_1}\leqq 1\end{cases}$$

First of all we establish the

Lemma 3.1. *All the couples* (x,y) *which are solutions of the system*

$$\begin{aligned}&R(x,y)=0\\ &|x|\leqq 1\\ &|y|\leqq 1\end{aligned}\qquad (3.1)$$

have the form $(x,h(x))$ *or* $(k(y),y)$ *where* $x\in R_x$ *and* $y\in R_y$.

The demonstration follows at once from Sect. II $\left(\text{assuming } \frac{\mu_1}{\lambda_1}>1\right)$. When $\frac{\mu_2}{\lambda_2}>1$ there is a bijection between R_x and R_y. More precisely,

$$R_x \xrightarrow{h} R_y \text{ and } R_y \xrightarrow{k} R_x.$$

When $\frac{\mu_2}{\lambda_2}<1$ there is a bijection between D_x and D_y,

$$\text{where}\begin{cases}D_x \text{ is the domain between the curves } C(1) \text{ and } K\\ D_y \text{ is the domain between the curves } C\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right) \text{ and } H.\end{cases}$$

Remark. The case $\frac{\mu_1}{\lambda_1}<1$, $\frac{\mu_2}{\lambda_2}<1$ is rejected: otherwise, as soon as both queues are busy, the queue lengths remain unbounded with positive probabilities $\left(1-\frac{\mu_1}{\lambda_1}\right.$ and $\left.1-\frac{\mu_2}{\lambda_2} \text{ respectively}\right)$ (Cohen [4]).

We are in a position to demonstrate Theorem 3.1.

Applying the notation of Sect. I, system (3.1) entails

$$\boxed{F(0,y)\,a(x,y)+F(x,0)\,b(x,y)+F(0,0)\,c(x,y)=0}\qquad (3.2)$$

In (3.2) either $x=k(y)$ or $y=h(x)$.

We continue the function $F(x,0)$ to $\overline{B(1)}$.

Two cases must be distinguished.

i) $\frac{\mu_2}{\lambda_2}<1$.

Then $|h(x)|\leqq\sqrt{\frac{\mu_2}{\lambda_2}}\leqq 1\ \forall\ x$.

$F(0,h(x), a(x,h(x))$ and $c(x,h(x))$ are analytic in $\overline{B(1)}$ cut along $[x_3\,x_4]$.

The relation (3.2) entails the same property for the product $F(x,0)\cdot b(x,h(x))$. It follows that $F(x,0)$ can be continued as a meromorphic function to $\overline{B(1)}-[x_3\,x_4]$, the poles of which are the roots (if any) of $b(x,h(x))=0$ in $\overline{B(1)}-[x_3\,x_4]$.

ii) $\frac{\mu_2}{\lambda_2}>1$.

Let $\mathscr{K}'$ denote the region inside the curve K', that is

$$|x|\geqq 1,$$
$$|h(x)|\geqq 1.$$

Note that x_3 and x_4 are interior points of $\mathscr{K}'$.

In $\overline{B(1)}-\mathscr{K}'$, $|h(x)|\leqq 1$: the same argument as in i) holds and the product $F(x,0)\cdot b(x,h(x))$ is analytic.

The analytic continuation of $F(x,0)$ to $\mathscr{K}'-[x_3\,x_4]$ is made according to the following procedure:

α) In (3.2), we use the fact that $F(x,0)$ is analytic in $B(1)$ to continue $F(0,h(x))$ to $C(1)-[x_1\,x_2]$ as in the foregoing lines.

β) On account of the automorphism $z\rightarrow\frac{\mu_1}{\lambda_1 z}$, we know that

$$h\left(\frac{\mu_1}{\lambda_1 x}\right)=h(x),$$

$$[x_3\,x_4]=\frac{\lambda_1}{\mu_1[x_2\,x_1]}.$$

Hence, $F(0,h(x))$ is continued as a meromorphic function to $\mathscr{K}'-[x_3\,x_4]$.

γ) We use (3.2) again and β to obtain the analytic continuation of $F(x,0)$: the product $F(x,0)\cdot b(x,h(x))$ is analytic everywhere in the complex plane cut along $[x_3\,x_4]$.

In $\overline{B(1)}$, the poles of $F(x,0)$ coincide with the zeros of $b(x,h(x))$.

Same conclusions can be drawn for $F(0,y)$ and the proof of theorem 3.1 is terminated.

IV. A Simplification of the Relation (3.2)

We try to reduce (3.2) to an homogeneous equation (i.e. without right member).

An elementary computation shows that when $pq-\mu_1\mu_2 \neq 0$, the transformation (translation)

$$\begin{aligned} F(0,y) &= \frac{p(q-\mu_2)}{pq-\mu_1\mu_2} F(0,0)+H(y) \\ F(x,0) &= \frac{q(p-\mu_1)}{pq-\mu_1\mu_2} F(0,0)+G(x) \end{aligned} \tag{4.0}$$

yields the system

$$\begin{aligned} &R(x,y)=0, \quad |x|, |y| \leqq 1 \\ &H(y)\, a(x,y)+G(x)\, b(x,y)=0. \end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned} H(0) &= \frac{\mu_2 \mu_1^* F(0,0)}{\mu_1\mu_2-pq} \\ G(0) &= \frac{\mu_1 \mu_2^* F(0,0)}{\mu_1\mu_2-pq}. \end{aligned} \tag{4.2}$$

Section III gives the analytic continuations of $G(x)$ and $H(y)$.

When $pq=\mu_1\mu_2$, the preceding transformation is not valid. However, remembering that:

$$p=\mu_1-\mu_1^* \text{ and } q=\mu_2-\mu_2^*,$$

$pq-\mu_1\mu_2=0$ can be rewritten as

$$\frac{\mu_1}{\mu_1^*}+\frac{\mu_2}{\mu_2^*}=1,$$

or

$$\begin{aligned} \mu_1 &= \xi \mu_1^* \\ \mu_2 &= (1-\xi)\mu_2^* \end{aligned} \quad 0 \leqq \xi \leqq 1, \text{ provided that } \mu_1^*, \mu_2^* \neq 0. \tag{4.3}$$

If $\mu_1^*=\mu_2^*=0$, (4.1) yields the system

$$\begin{aligned} &R(x,y)=0, \quad |x|, |y| \leqq 1, \\ &F(0,x)+F(y,0)=F(0,0), \quad (x,y) \neq (1,1), \end{aligned}$$

which clearly has no admissible solution due to the fact that $F(x,0)+F(0,y)>F(0,0)$.

Using (4.3), (3.2) takes the following form:

$$\begin{aligned} &R(x,y)=0, \quad |x| \leqq 1, \quad |y| \leqq 1 \\ &\left[\mu_1^*\left(1-\frac{1}{x}\right)-\mu_2^*\left(1-\frac{1}{y}\right)\right][\xi F(0,y)-(1-\xi)F(x,0)] \\ &\quad =\left[\mu_1^*\left(\frac{1}{x}-1\right)(1-\xi)+\mu_2^*\left(\frac{1}{y}-1\right)\xi\right]F(0,0). \end{aligned} \tag{4.4}$$

V. The Roots of $b^1(x)=p\left(1-\frac{1}{x}\right)+\mu_2\left(1-\frac{1}{h(x)}\right)=0$ inside $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$

Following the remark in Sect. III, *it will be assumed* from now on $\mu_1>\lambda_1$.

Section III reveals that a stepping stone towards the solutions of the original problem is the study of the roots of the equation $b^1(x)=0$. More precisely, we require only the roots inside $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$ for reasons given in Sects. VI and VII. It seems convenient to introduce the following notations (valid until the end of the paper),

$$\begin{aligned} b^1(x)&=p\left(1-\frac{1}{x}\right)+\mu_2\left(1-\frac{1}{h(x)}\right)\\ a^1(x)&=q\left(1-\frac{1}{h(x)}\right)+\mu_1\left(1-\frac{1}{x}\right)\\ b^2(y)&=p\left(1-\frac{1}{k(y)}\right)+\mu_2\left(1-\frac{1}{y}\right)\\ a^2(y)&=q\left(1-\frac{1}{y}\right)+\mu_1\left(1-\frac{1}{k(y)}\right). \end{aligned} \tag{5.1}$$

Up to a change of the parameters, the conclusions drawn for $b^1(x)=0$ will hold for $a^2(y)=0$.

The term "root" or "zero" will always design a number different from 1.

Lemma 5.1. *Excluding the trivial root* $x=1$, $b^1(x)=0$ *has at most two roots inside* $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$. *Moreover, these roots (if any) are real, positive and belong to the set* $[0,x_1]\cup\left[x_2,\sqrt{\frac{\mu_1}{\lambda_1}}\right]$.

Proof. i) $b^1(x)=0$ together (1.4) entails

$$\varphi(x)\overset{\text{def}}{=}\lambda_1(\mu_2+p)x^2-x[\mu_1^*\mu_2+(\lambda_1+\lambda_2+\mu_1^*)p]+p\mu_1^*=0. \tag{5.2}$$

The first part of the lemma says that $\varphi(x)=0$ has two real roots. This is proved in Appendix A.

To show that these roots are positive, we substract $b^1(x)=0$ from (1.4) and obtain

$$\lambda_1(1-x)+\mu_1^*\left(1-\frac{1}{x}\right)+\lambda_2(1-h(x))=0:$$

$x\leqq 0$ would lead to $h(x)>1$: this is impossible. Hence, any root is positive.

From Sect. II it is obvious that a root x cannot belong to the cut $[x_1 x_2]$ by checking $h(x)$ is a complex number and $b^1(x)$ does not vanish in this case.

The proof of Lemma 5.1 is concluded.

We make now the assumption (valid in the whole Sect. V) $|x|\leqq\sqrt{\frac{\mu_1}{\lambda_1}}$ when the range of variation is not explicitly stated.

The following system of notations will also be used:

$$\frac{db^1(x)}{dx}=\frac{p}{x^2}+\frac{\mu_2 h'(x)}{h^2(x)}$$

$$\frac{dh(x)}{dx}=h'(x)=\frac{\lambda_1-\frac{\mu_1}{x^2}}{\frac{\mu_2}{h^2(x)}-\lambda_2} \tag{5.3}$$

$$\Psi(x)=\frac{\lambda_1 x-\mu_1}{\mu_2-\lambda_2 h(x)}; \qquad \frac{d\Psi(x)}{dx}=\frac{\lambda_1(\mu_2-\lambda_2 h(x))+\lambda_2(\lambda_1 x-\mu_1)h'(x)}{[\mu_2-\lambda_2 h(x)]^2}.$$

Thus, $b^1(x)$ can be rewritten at our convenience

$$b^1(x)=\left(1-\frac{1}{x}\right)[p+\mu_2 \Psi(x)]. \tag{5.4}$$

Two cases are now investigated

1. $p<0$

yields

$$\frac{db^1(x)}{dx}<0 \quad \text{for } x\in[0,x_1]\cup\left[x_2,\sqrt{\frac{\mu_1}{\lambda_1}}\right] \tag{5.3}$$

1.a $\mu_2\geqq\lambda_2$

Then $\Psi(x)<0$.

Hence $p+\mu_2\Psi(x)<0$ and there is no root in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$

1.b $\mu_2<\lambda_2$

Then $b_1(1)=\mu_2-\lambda_2<0$.

This implies:

* there is a root (and only one) in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$ if $b^1(x_2)>0$,

Moreover, this root belongs to $[x_2,1]$

* there is no root in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$ if $b^1(x_2)<0$.

2. $p>0$

2.a $\mu_2\geqq\lambda_2$

yields

$$\frac{d\Psi(x)}{dx}>0 \quad \text{for } x\in[0,x_1]\cup\left[x_2,\sqrt{\frac{\mu_1}{\lambda_1}}\right] \tag{5.3}$$

i) If $b^1(x_2)>0$ then $\begin{cases} \text{one root in } \left[x_2, \sqrt{\frac{\mu_1}{\lambda_1}}\right] & \text{if } b^1\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)>0 \\ \text{no root in } \left[x_2, \sqrt{\frac{\mu_1}{\lambda_1}}\right] & \text{if } b^1\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)<0 \\ \text{one root in } [0,x_1] & \text{if } b^1(x_1)<0 \\ \text{no root in } [0,x_1] & \text{if } b^1(x_1)>0. \end{cases}$

ii) If $b^1(x_2)<0$ then $b^1(x_1)<0$ (see Appendix B): we deduce there is only one root in $[0,x_1]$ and no other root in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$.

2.b $\mu_2<\lambda_2$

As in 2.a, $\frac{d\Psi(x)}{dx}>0$ for $x\in[0,x_1]$. Thus, there is one root in $[0,x_1]$ iff $b^1(x_1)<0$. yields

$$h'(x)<0 \quad \text{for } x\in\left[x_2, \sqrt{\frac{\mu_1}{\lambda_1}}\right]. \tag{5.3}$$

When x increases from x_2 to $\sqrt{\frac{\mu_1}{\lambda_1}}$, $h(x)$ decreases from $\sqrt{\frac{\mu_2}{\lambda_2}}$ to y_2 [y_2 is the second branch point of $k(y)$]

It follows: $\begin{cases} \Psi(x)>0 & \text{for } x\in[x_2,1] \\ \frac{d\Psi(x)}{dx}>0 & \text{for } x\in\left[1,\sqrt{\frac{\mu_1}{\lambda_1}}\right]. \end{cases}$

This implies:

* There is one root in $\left[1,\sqrt{\frac{\mu_1}{\lambda_1}}\right]$ if $b^1\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)>0$.
* There is no root in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)-[0,x_1]$ if $b^1\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)<0$.

Similar conclusions can be drawn for $a^2(y)$ as follows:

- When $\mu_2\geqq\lambda_2$, it suffices to change the name of the parameters.
- When $\mu_2<\lambda_2$, let us set

$$a^2(y)=\left(1-\frac{1}{y}\right)[q+\mu_1\,\Phi(y)], \qquad \text{where } \Phi(y) \overset{\text{def}}{=} \frac{\lambda_2 y-\mu_2}{\mu_1-\lambda_1 k(y)},$$

$\Phi(y)$ increases on $\left[\sqrt{\frac{\mu_2}{\lambda_2}},1\right]$.

i) $q<0$: $a^2(y)$ decreases on $\left[y_2, \sqrt{\frac{\mu_2}{\lambda_2}}\right]$.

There is no root on $\left[\frac{\mu_2}{\lambda_2}, \sqrt{\frac{\mu_2}{\lambda_2}}\right]$ if $q+\mu_1 \Phi(1)<0$.

If $q+\mu_1 \Phi(1)>0$, then

$$\text{there is one root} \begin{cases} \text{in } \left[\sqrt{\frac{\mu_2}{\lambda_2}}, 1\right] & \text{if } a^2\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)>0 \\ \text{in } \left[\frac{\mu_2}{\lambda_2}, \sqrt{\frac{\mu_2}{\lambda_2}}\right] & \text{if } a^2\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)<0 \end{cases}$$

ii) $q>0$:

* no root in $\left[\frac{\mu_2}{\lambda_2}, 1\right]$,
* one root in $\left[y_2, \frac{\mu_2}{\lambda_2}\right]$ iff $a^2(y_2)>0$,
* one root in $[0, y_1]$ iff $a^2(y_1)<0$.

VI. The Case $pq=\mu_1 \mu_2$: Determination of $F(x,0)$ and $F(0,y)$ by Solving a Dirichlet Problem for a Circle

From now on, let $\mathscr{I}_m(z)$ and $R_e(z)$ denote respectively the imaginary and the real part of the complex number z.

From Sect. I, we know that

$\mathscr{I}_m F(0,y)=0$ for $y\in[y_1\, y_2]$, since the power series expansion of $F(0,y)$ has positive coefficients.

Hence, by using system (4.4) and Sect. III, it follows:

$$\mathscr{I}_m(1-\xi)F(x,0)=\mathscr{I}_m \frac{\mu_2^*\left(\frac{1}{h(x)}-1\right)F(0,0)}{-\mu_1^*\left(1-\frac{1}{x}\right)+\mu_2^*\left(1-\frac{1}{h(x)}\right)} \quad \text{for } |x|=\sqrt{\frac{\mu_1}{\lambda_1}}. \tag{6.1}$$

Here, $b^1(x)=-\mu_1^*\left(1-\frac{1}{x}\right)+\mu_2^*\left(1-\frac{1}{h(x)}\right)$.

– If $b^1(x)$ has no zero in $B\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$, we reduce the problem to that of finding a function $F(x,0)$ analytic in $B\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$, continuous on $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$ and satisfying the boundary condition (6.1). This is a particular of a Dirichlet problem for a circle.

– If $b^1(x)$ has a zero in $B\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$, say x_0, we have still a Dirichlet problem for the function $(x-x_0)F(x,0)$.

Provided that $b^1(x)$ has no root in $B\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$; $F(x, C)$ is determined from (6.1), in $B\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$, up to a constant, by Schwarz's formula (Muskhelishvili [6], Sect. 41)

$$F\left(\sqrt{\frac{\mu_1}{\lambda_1}}\, z, 0\right) = \frac{i}{2\pi} \int_{-\pi}^{\pi} u(\rho) \frac{e^{i\rho}+z}{e^{i\rho}-z}\, d\rho + D, \qquad |z|<1 \tag{6.2}$$

where D is a real constant and

$$u(\rho) = \frac{1}{1-\xi} \mathscr{I}_m \frac{\mu_2^* \left(1 - \frac{1}{h(x)}\right) F(0,0)}{\mu_1^* \left(1 - \frac{1}{x}\right) - \mu_2^* \left(1 - \frac{1}{h(x)}\right)}.$$

With $x = \sqrt{\frac{\mu_1}{\lambda_1}}\, e^{i\rho}$,

$$u(\rho) = \frac{-\lambda_1 \sin\rho\, H(\rho)}{[\rho_2^*(\mu_1^* - \mu_2^*) H^2(\rho) + (\mu_2^* - \mu_1^* + \lambda_1 + \lambda_2) H(\rho) - \mu_2^*](1-\xi)} \tag{6.3}$$

where

$$H(\rho) = h\left(\sqrt{\frac{\mu_1}{\lambda_1}}\, e^{i\rho}\right)$$
$$= \frac{\lambda_2 + \mu_2 + \alpha - \sqrt{[(\sqrt{\lambda_2} + \sqrt{\mu_2})^2 + \alpha][(\sqrt{\lambda_2} - \sqrt{\mu_2})^2 + \alpha]}}{2\lambda_2}$$

and $\alpha = \lambda_1 + \mu_1 - 2\sqrt{\lambda_1 \mu_1} \cos\rho$.

Note that $H(\rho)$ is real. From (6.3), we deduce easily that $u(\rho)$ is an odd function of ρ. This implies

$$\boxed{F\left(\sqrt{\frac{\mu_1}{\lambda_1}}\, z, 0\right) = \frac{1}{\pi} \int_0^{\pi} \frac{z \sin\rho\, u(\rho)\, d\rho}{1 + z^2 - 2z\cos\rho} + F(0,0)} \qquad |z|<1 \tag{6.4}$$

The expression of $F\left(\sqrt{\frac{\mu_1}{\lambda_1}}\, z, 0\right)$ in terms of elliptic functions of the third kind is given in Appendix C.

1) If $\mu_2 \geqq \lambda_2$, we have seen in Sect. V that $b^1(x)$ has no zeros in $B\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$. In that case, $a^2(y)$ has no zeros in $B\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)$ and

$$\boxed{F\left(0, \sqrt{\frac{\mu_2}{\lambda_2}}\, z\right) = \frac{1}{\pi} \int_0^{\pi} \frac{z \sin\theta\, v(\theta)\, d\theta}{1 + z^2 - 2z\cos\theta} + F(0,0)\, |z|<1} \tag{6.5}$$

$$v(\theta) = \frac{-\lambda_2 \sin\theta\, K(\theta)}{[\rho_1^*(\mu_2^* - \mu_1^*) K^2(\theta) + (\mu_1^* - \mu_2^* + \lambda_1 + \lambda_2) K(\theta) - \mu_1^*]\, \xi} \tag{6.6}$$

where

$$K(\theta)=k\left(\sqrt{\frac{\mu_1}{\lambda_1}}\,e^{i\theta}\right)$$

$$=\frac{\lambda_1+\mu_1+\beta-\sqrt{[(\sqrt{\lambda_2}+\sqrt{\mu_2})^2+\beta][(\sqrt{\lambda_2}-\sqrt{\mu_2})^2+\beta]}}{2\lambda_1}$$

and $\beta=\lambda_2+\mu_2-2\sqrt{\lambda_2\mu_2}\,\mathrm{Cos}\,\theta$.

Note that $K(\theta)$ is real.

2) If $\mu_2<\lambda_2$, $b^1(x)$ may have a zero on $[x_2,1]$ which is equivalent to

"$a^2(y)$ may have a zero on $\left[\frac{\mu_2}{\lambda_2},\sqrt{\frac{\mu_2}{\lambda_2}}\right]$".

But Eq. (4.4) and Sect. III entail that the system is ergodic iff

$$a^2(y)\neq 0 \quad \text{in } C(1). \tag{6.7}$$

Indeed, (6.5) determines $F(0,y)$ in $B\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)$ and the same formula (6.5) holds for $\frac{\mu_2}{\lambda_2}\leqq y\leqq 1$ iff (6.7) is satisfied.

The Ergodicity Condition (6.7)

From Sect. V we deduce immediately thar (6.7) is satisfied iff $q+\mu_1\Phi(1)<0$, which yields:

$$q+\mu_1\frac{\lambda_2-\mu_2}{\mu_1-\lambda_1}<0 \quad \text{or, using } \mu_1=\xi\mu_1^* \text{ and } \mu_2=(1-\xi)\mu_2^*,$$

$$\boxed{1-\rho_1^*-\rho_2^*>0}, \quad \text{where } \rho_1^*\overset{\text{def}}{=}\frac{\lambda_1}{\mu_1^*} \text{ and } \rho_2^*\overset{\text{def}}{=}\frac{\lambda_2}{\mu_2^*}. \tag{6.8}$$

$F(0,0)$ is easily obtained from (1.2): it is sufficient to couple x and y by the relation $\mu_1^*\left(1-\frac{1}{x}\right)=\mu_2^*\left(1-\frac{1}{y}\right)$ and to write $F(1,1)=1$.

It follows $F(0,0)=1-\rho_1^*-\rho_2^*$ and (6.8) yields $F(0,0)>0$.

Using Little's formula, the mean waiting times W_1 and W_2 in queue 1 and queue 2 can be readily obtained from (6.4), (6.5) and (1.2).

$$W_1\overset{\text{def}}{=}\frac{1}{\lambda_1}\frac{d}{dx}F(x,y)_{|x=y=1},$$

$$W_2\overset{\text{def}}{=}\frac{1}{\lambda_2}\frac{d}{dy}F(x,y)_{|x=y=1}.$$

It is also possible to verify that Kleinrock's conservation law (Kleinrock [9]) is satisfied, that is:

$$\rho_1^* W_1 + \rho_2^* W_2 = \frac{1}{1-\rho_1^*-\rho_2^*}\left(\frac{\lambda_1}{\mu_1^{*2}}+\frac{\lambda_2}{\mu_2^{*2}}\right).$$

This tedious computation will not be done there.

VII. The Case $pq \neq \mu_1 \mu_2$: Determination of $F(x,0)$ and $F(0,y)$ by Solving an Homogeneous Riemann-Hilbert Problem for a Circle

In that section we use intensively the procedure given in Muskhelishvili [6], Chapter 5, Sects. 39–40.

Theorem 7.1. *Our problem is a particular case of a famous general problem due to Riemann and first studied by Hilbert.*

This problem is as follows:

Let S^+ be a finite or infinite region, bounded by a single simple contour L. It is required: to find a function $\Omega(z)=u+iv$, holomorphic in S^+ and continuous in $S^+ + L$, satisfying the boundary condition

$$\mathscr{R}_e[U(z)\cdot\Omega(z)] = V(z) \quad on \ L, \tag{7.1}$$

where $U(z)$, $V(z)$ are continuous functions given on L. Then homogeneous problem is obtained by putting

$$V(z)\equiv 0 \quad in \ (7.1).$$

Demonstration.

Lemma 7.1.

$$G(z)=\frac{[z-k(\alpha_2)]^{i_1}[z-k(\beta_2)]^{i_2}\tilde{G}(z)}{[z-\gamma_1]^{i_3}}, \quad |z|<\sqrt{\frac{\mu_1}{\lambda_1}} \tag{7.2}$$

where i_j, $j=1,2,3$ takes the values 0 or 1 and

i) *γ_1 is the eventual zero of $b^1(x)$ in $\left[1,\sqrt{\frac{\mu_1}{\lambda_1}}\right]$;*

ii) *α_2,β_2 are the eventual zeros of $a^2(y)$ in $[0,y_1]\cup[y_2,1]$;*

iii) *$\tilde{G}(z)$ is analytic in $B\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$.*

Proof of Lemma 7.1. *Proof of* i). Applying the notations of Sect. V, the second Eq. of (4.1) becomes

$$H(h(x))\,a^1(x)+b^1(x)\,G(x)=0, \tag{7.3}$$

where $G(x)$ is meromorphic for $x\in\left[1,\sqrt{\frac{\mu_1}{\lambda_1}}\right]$.

Section V shows that, in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)-C(1)$; $b^1(x)$ has at most one zero $(\neq 1)$ denoted here by γ_1.

Moreover it can be seen that $a^1(x)$ and $b^1(x)$ have no common zero $(\neq 1)$. Hence, γ_1 is a pole of $G(x)$ if $H(h(\gamma_1))\neq 0$. But, $h(\gamma_1)\in[y_2,1]$. It suffices to show that $H(y)\neq 0$ for $y\in[y_2,1]$, which is a consequence of equation (4.2) as follows:

* if $\mu_1\mu_2>pq$, then $H(0)>0$: $H(y), y\in[0,1]$ has a power series expansion with real positive coefficients and this, in turn, implies $H(y)>0$ for $y\in[0,1]$.

* if $\mu_1\mu_2<pq$, then $p<0$ and $q<0$: in that case $b^1(x)\neq 0$, $x\in\left[1,\sqrt{\frac{\mu_1}{\lambda_1}}\right]$ and γ_1 does not exist.

The proof of i) is terminated.

Proof of ii). We use again system (4.1) to derive

$$H(y)\,a^2(y)+G(k(y))\,b^2(y)=0 \tag{7.4}$$

$H(y)$ must be analytic in $B(1)$: thus the zeros of $a^2(y)$ in $C(1)$ are zeros of $G(k(y))$.

Observing that $a^2(y)$ has no negative roots (see Sect. V), point ii) follows readily.

Finally, using i) and ii), formula (7.2) holds and the proof of Lemma 7.1 is concluded. Now, we are in a position to demonstrate Theorem 7.1 as in Sect. VI, $\mathscr{I}_m H(y)=0$ for $y\in[y_1,y_2]$, since $H(y)$ is analytic in $B(1)$ and its power series expansion has positive coefficients.

Hence, upon setting

$$U(z)=\frac{b^1(z)}{a^1(z)}\cdot\frac{[z-k(\alpha_2)]^{i_1}[z-k(\beta_2)]^{i_2}}{(z-\gamma_1)^{i_3}} \tag{7.5}$$

we obtain

$$\mathscr{R}_e\, i\, U(z)\,\tilde{G}(z)=0 \qquad \text{for } |z|=\sqrt{\frac{\mu_1}{\lambda_1}}. \tag{7.6}$$

The boundary condition (7.6) is a particular case of (7.1), putting $V(z)\equiv 0$.

This terminates the proof of Theorem 7.1.

From [6] formula (40–10) – Sect. 40, $\tilde{G}(z)$ is given by:

$$\tilde{G}(z)=D\,e^{\Gamma(z)} \qquad \text{for } |z|<\sqrt{\frac{\mu_1}{\lambda_1}} \tag{7.7}$$

where D is a constant, non zero, and:

$$\Gamma(z)=\frac{1}{2i\pi}\int_{C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)}\frac{\log[t^{-\chi}J(t)]\,dt}{t-z} \tag{7.8}$$

where

$$J(t)=\frac{\overline{i\,U(t)}}{i\,U(t)}$$

and

$$\chi=\frac{1}{2i\pi}\log\left[\frac{\bar{U}(t)}{U(t)}\right]_{C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)}$$

or

$$\chi=-\frac{1}{\pi}[\arg U(t)]_{C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)}, \tag{7.9}$$

denoting by arg(z) the function "argument of z".

Lemma 7.2.

$$\chi=2[N_p\cdot\operatorname{sgn}(\mu_2-\lambda_2)-N_z]+\frac{1}{\pi}\arg\frac{b^1(x)}{a^1(x)}_{\overleftarrow{\overrightarrow{x_1x_2}}} \tag{7.10}$$

[*denoting by* sgn(x) *the function* "sign *of* x" *with* sgn(0)=0]
where

* N_p *is the number of zeros of* $a^2(y)$ *on* $\left[1,\sqrt{\frac{\mu_2}{\lambda_2}}\right]$,
* N_z *is the number of zeros of* $b^1(x)$ *on* $[0,x_1]\cup[x_2,1]$,
* $\arg\left[\frac{b^1(x)}{a^1(x)}\right]_{\overleftarrow{\overrightarrow{x_1x_2}}}$ *is the variation of the argument of* $\frac{b^1(x)}{a^1(x)}$ *along the "contour"* $[x_1,x_2]$, *starting from* x_1 *above* $[x_1x_2]$, *going to* x_2 *and coming back to* x_1 *below* $[x_1x_2]$.

Demonstration. Using formulas (7.5), (7.9) and the principle of the argument (Fuchs [3]), we obtain:

$$\chi=-\frac{1}{\pi}\arg[U]_{C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)}=2(l-N_z)+\frac{1}{\pi}\arg[U]_{\overleftarrow{\overrightarrow{x_1x_2}}}$$

[the positive direction on $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$ is counter clockwise]
where

i) N_z = number of zeros of $b^1(x)$ in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$ − number of zeros of $b^1(x)$ in $\left[1,\sqrt{\frac{\mu_1}{\lambda_1}}\right]$, that is

N_z = number of zeros of $b^1(x)$ in [0, 1].

ii) $l=l_1-l_2$, with

l_1 = number of zeros of $a^1(x)$ in $C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)$,

l_2 = number of zeros of $a^2(y)$ in [0, 1] [l_2 is due to α_2 and β_2 in formula (7.5)].

Using the mapping $h(x)$, $\left[-\sqrt{\frac{\mu_1}{\lambda_1}},\sqrt{\frac{\mu_1}{\lambda_1}}\right]\xrightarrow{h(x)}\left[-\sqrt{\frac{\mu_2}{\lambda_2}},\sqrt{\frac{\mu_2}{\lambda_2}}\right]$ [there is a bijection between the two intervals], we can express l_1 as the number of zeros of $a^2(y)$ in $\left[-\sqrt{\frac{\mu_2}{\lambda_2}},\sqrt{\frac{\mu_2}{\lambda_2}}\right]$. From Sect. V we deduce

$$l=\begin{cases} N_p & \text{if } \mu_2>\lambda_2 \\ -N_p & \text{if } \mu_2<\lambda_2 \\ 0 & \text{if } \mu_2=\lambda_2. \end{cases}$$

In 7.5, $k(\alpha_2)$, $k(\beta_2)$ and γ_1 are real numbers outside $[x_1, x_2]$: Hence

$$\arg\left[\frac{[z-k(\alpha_2)]^{i_1}[z-k(\beta_2)]^{i_2}}{[z-\gamma_1]^{i_3}}\right]_{\overleftarrow{\overrightarrow{x_1x_2}}}=0$$

and we have $\frac{1}{\pi}\arg[U]_{\overleftarrow{\overrightarrow{x_1x_2}}}=\frac{1}{\pi}\arg\left[\frac{b^1(x)}{a^1(x)}\right]_{\overleftarrow{\overrightarrow{x_1x_2}}}$.

The proof of Lemma 7.2 is concluded

Lemma 7.3.

$$\frac{1}{\pi}\arg\frac{b^1(x)}{a^1(x)}_{\overleftarrow{\overrightarrow{x_1x_2}}}=\operatorname{Sgn}(\mu_1\mu_2-pq)\left[\operatorname{sgn}\frac{b^1(x_1)}{a^1(x_1)}-\operatorname{sgn}\frac{b^1(x_2)}{a^1(x_2)}\right]. \tag{7.11}$$

Demonstration. $h(x_1)=-\sqrt{\frac{\mu_2}{\lambda_2}}$ and $h(x_2)=\sqrt{\frac{\mu_2}{\lambda_2}}$ are real. When x reaches $]x_1, x_2[$ from above, $h(x)$ has a negative imaginary part [referring to the mapping $x\to h(x)$, $[x_1, x_2]\to C\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)$ and the upper part of $[x_1, x_2]$ corresponds to the inferior half circle of radius $\sqrt{\frac{\mu_2}{\lambda_2}}$. Then it is easy to derive $\operatorname{sgn}\mathscr{I}_m\frac{b^1(x)}{a^1(x)}=\operatorname{sgn}[\mu_1\mu_2-pq]$ for $x\in[x_1, x_2]$, which yields (7.11). The Lemma 7.3 is proved.

$$\frac{1}{\pi}\left[\arg\frac{b^1(x)}{a^1(x)}\right]_{\overleftarrow{\overrightarrow{x_1x_2}}}=\operatorname{sgn}(pq-\mu_1\mu_2)\left[\operatorname{sgn}\frac{b^1(x_2)}{a^2\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)}\right]+\operatorname{sgn}b^1(x_1)\Bigg] \tag{7.12}$$

remarking that $a^1(x_1)=a^2\left(-\sqrt{\frac{\mu_2}{\lambda_2}}\right)<0$.

Lemma 7.4. 1) *For $\chi\leqq -2$, the homogeneous Riemann-Hilbert problem has no solutions different from zero.*

2) *For $\chi\geqq 0$, the homogeneous Riemann-Hilbert problem has exactly $\chi+1$ linearly independent solutions: the general solution is given by*

$$\theta(z)=\tilde{G}(z)(c_0 z^\chi+c_1 z^{\chi-1}\ldots+c_\chi)$$

where $c_0, c_1, \ldots, c_\chi$ are constants subject to $c_\chi=\bar{c}_{\chi-k}$, $k=\alpha, 1, \ldots, \chi$ but otherwise arbitrary

Demonstration. See Muskhelishvili [6], Sect. 40, p. 100.

Theorem 7.2. 1) *The homogeneous Hilbert problem satisfying the boundary condition* (7.6) *has, at most, one solution. In other words,* $\chi \leqq 0$.

2) *The system is ergodic iff* $\chi = 0$, *which is equivalent to*

$$\left.\begin{aligned} \frac{db^1(x)}{dx}_{|x=1} < 0 \Leftrightarrow \mu_1^* > \frac{\mu_2\lambda_1 - \mu_1\lambda_2}{\mu_2 - \lambda_2} \\ \frac{da^2(y)}{dy}_{|y=1} < 0 \Leftrightarrow \mu_2^* > \frac{\mu_1\lambda_2 - \mu_2\lambda_1}{\mu_1 - \lambda_1} \end{aligned}\right\} \quad \text{if } \mu_2 \geqq \lambda_2, \tag{7.13}$$

$$\frac{da^2(y)}{dy}_{|y=1} < 0 \Leftrightarrow \mu_2^* > \frac{\mu_1\lambda_2 - \mu_2\lambda_1}{\mu_1 - \lambda_1} \quad \text{if } \mu_2 \leqq \lambda_2. \tag{7.14}$$

Proof. We build a table for all possible values of χ using formulas (7.10), (7.12) and Sect. V. It is convenient to introduce the variables T_p, T_q, F_p, F_q where

$$T_p \equiv [p + \mu_2 \Psi(1) < 0] \qquad F_p \equiv [p + \mu_2 \Psi(1) > 0],$$
$$T_q \equiv [q + \mu_1 \Phi(1) < 0] \qquad F_q \equiv [q + \mu_1 \Phi(1) > 0],$$

$\Phi(y)$ and $\Psi(x)$ have been defined in Sect. V and, in the tables "if $\mathscr{P}$" means "if $\mathscr{P}$ is true".

i) $\mu_2 \leqq \lambda_2$

N_p \ N_z	0	1	2
0	$\chi = \begin{cases} -2 & \text{if } F_q \\ 0 & \text{if } T_q \end{cases}$	$\chi = \begin{cases} -2 & \text{if } (F_p \text{ or } F_q) \\ 0 & \text{if } T_p \cdot T_q \end{cases}$	$\chi = -2$
1	$\chi = 0$	$\chi = \begin{cases} 0 & \text{if } T_p \\ -2 & \text{if } F_p \end{cases}$	$\chi = -2$

ii) $\mu_2 \geqq \lambda_2$

N_p \ N_z	0	1
0	$\chi = \begin{cases} -2 & \text{if } F_q \\ 0 & \text{if } T_q \end{cases}$	
1	T_q and $\chi = -2$	

These tables show that

$$\chi = 0 \quad \text{iff} \begin{cases} (7.13) & \text{when } \mu_2 \geqq \lambda_2 \\ (7.14) & \text{when } \mu_2 < \lambda_2 \end{cases}. \qquad \text{Then } i_2 = 0 \text{ in (7.2).}$$

Remarks. a) To obtain Table i), the reader will notice that certain inequalities cannot hold simultaneously, as for example

$$\begin{matrix} a^2\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)>0 \\ b^1(x_2) \quad <0 \end{matrix} \qquad \text{or} \quad \begin{cases} p+\mu_2\,\Psi(1)>0 \\ q+\mu_1\,\Phi(1)>0 \end{cases}$$

which would yield $\mu_1^*\,\mu_2^*<0$.

b) In Table ii), the values of χ do not rely on the values of N_z. Moreover, the ergodicity condition (7.14) can be derived from (7.4) and is the same as condition (6.7) [Sect. VI].

This concludes the proof of Theorem 7.2.

Theorem 7.3. *Assuming* (7.13) *or* (7.14), $F(z,0)$ *is given by*

$$\frac{F(z,0)}{F(0,0)}=\frac{q\,\mu_1^*}{\mu_1\,\mu_2-p\,q}+\frac{\mu_1\,\mu_2^*}{\mu_1\,\mu_2-p\,q}\,\frac{G(z)}{G(0)}, \qquad |z|<\sqrt{\frac{\mu_1}{\lambda_1}}. \tag{7.15}$$

Where

* $F(0,0)=\left[\frac{\mu_2\,\lambda_1-\mu_1\,\lambda_2+\mu_2^*(\mu_1-\lambda_1)}{\mu_1\,\mu_2^*}\right]\cdot\frac{G(0)}{G(1)}$,
* $G(z)$ is derived from (7.2) and (7.7) using

$$\Gamma(z)=\frac{1}{2\pi}\int_{C\left(\sqrt{\frac{\mu_1}{\lambda_1}}\right)}\frac{\Delta(t)\,dt}{t-z} \quad \text{and} \quad \Delta(t)=\arg\left[\frac{-\bar{U}(t)}{U(t)}\right].$$

Proof. (7.15) is implied readily by (7.8) and Theorem 7.2. $\Gamma(z)$ can be expressed by means of elliptic functions of the third kind.

The reader may verify that the "elliptic part" of the integral (7.15) vanishes iff $p+q=0$. Then

$$\begin{aligned} F(x,0)&=\frac{F(0,0)}{1-\alpha x} \\ F(x,y)&=\frac{F(0,0)}{(1-\alpha x)(1-\beta y)}, \qquad \alpha,\beta \text{ being obtained from (7.13) or (7.14).} \end{aligned} \tag{7.16}$$

As can be seen, this is a product form solution. In fact, a necessary and sufficient condition for a product form solution in Eq. (1.2) is

$$p+q=0 \quad \text{i.e.,} \quad \boxed{\mu_1+\mu_2=\mu_1^*+\mu_2^*} \tag{7.17}$$

Hence, $\mu_1^*>\mu_1$ leads to $\mu_2^*<\mu_2$. The physical explanation is not obvious. We would like to call this a "power steal": one class of customers (or one processor) steals power from the other one. $\Gamma(z)$ in (7.15) will not be explicitly computed. The complexity is equivalent to that of the previous Sect. VI.

VIII. Application to General Two Dimensional Random Walks [t.d.r.w.]

The method used in Sect. VI and VII can be applied in a more general context. Let us assume that a t.d.r.w. has a stationary distribution with a generating function $F(x,y)=\sum_{i,j} p_{ij}x^i y^j$ satisfying

$\sum |p_{ij}|<\infty$ (space L_1) and the following functional equation

$$R(x,y)\cdot F(x,y)=A(x,y)\cdot F(x,0)+B(x,y)\cdot F(0,y)+C(x,y)$$

where $A(x,y)$, $B(x,y)$ and $C(x,y)$ are known and continuous. Then the determination of $F(x,y)$ reduces to the solution of a Riemann-Hilbert problem whenever the following four conditions are met (the notation is the same as before):

1) The continuous function $k(y)$ [resp. $h(x)$] is such that

$$|k(y)|\leqq 1 \text{ [resp. } |h(x)|\leqq 1] \quad \text{if } |y|=1 \text{ [resp. } |x|=1$$

2) $k(y)$ [resp. $h(x)$] has two branch points inside $C(1)$, y_1 and y_2 [resp. x_1 and x_2] which are the ends of a "cut" $[y_1, y_2]$ (resp. $[x_1\, x_2]$).

3) The cut $[y_1\, y_2]$ (resp. $[x_1\, x_2]$) is mapped onto a simple closed curve C_k[resp. C_h] under the mapping $y\to k(y)$ [resp. $x\to h(x)$].

4) Moreover, the regions inside $C(1)$ and C_k(resp. C_h) must have a non-empty intersection D_k(resp. D_h) such that

$$|k(y)|\leqq 1 \text{ [resp. } |h(x)|\leqq 1] \quad \text{for } y\in D_k \text{ [resp. } x\in D_h].$$

Up to a conformal transformation, condition 3) says that it is possible to reduce the general case to that of a circular region. Specifically, the four preceding conditions hold in the problem of t.d.r.w. studied by Malyshev [5]. It is worthwhile to note the possibility of solving the Hilbert problem for arcs by using the arc $[x_3\, x_4]$ and the analytic continuation of $F(x,0)$ to the whole complex plane (see Sect. III).

When the genus of $R(x,y)$ is greater than 1, some of the conditions 1 to 4 may be not satisfied. Then, the method is still valid, (an example will be given in a future paper) but the computations are more complex.

Appendix A

The equation

$$f(x) \stackrel{\text{def}}{=} \lambda_1(\mu_2+p)x^2 - x[\mu_1^*\mu_2+(\lambda_1+\lambda_2+\mu_1^*)p]+p\mu_1^*=0$$

has two real roots.

Proof. Let $\Delta(p)$ be the discriminant of $f(x)$. Then,

$$\Delta(p)=[(\lambda_1+\lambda_2+\mu_1^*)\,p+\mu_2\,\mu_1^*]^2-4\,\lambda_1\,\mu_1^*\,p(\mu_2+p),$$

so that

$$\Delta(p)=[(\lambda_1+\lambda_2+\mu_1^*)^2-4\,\lambda_1\,\mu_1^*]+2\,p\,\mu_1^*\,\mu_2(\mu_1^*+\lambda_2-\lambda_1)+(\mu_1^*\,\mu_2)^2.$$

We consider now that $\Delta(p)$ is a polynomial of second degree w.r.t. the variable H. Then, $\delta(p)\stackrel{\text{def}}{=}-16(\mu_1^*\,\mu_2)^2\,\lambda_1\,\lambda_2$ is the discriminant of $\Delta(p)$. Obviously $\delta(p)$ is negative. It follows, since p is a real variable, that $\Delta(p)$ is always positive. The proof is concluded.

Appendix B

When $\lambda_1 \leqq \mu_1$ and $\lambda_2 \leqq \mu_2$,

$$b^1(x_2)<0 \text{ entails } b^1(x_1)\leqq 0.$$

The notation is the same as in Sect. V.

Proof.

$$b^1(x_2)=\left(1-\frac{1}{x_2}\right)[p+\mu_2\,\Psi(x_2)],$$

$$b^1(x_1)=\left(1-\frac{1}{x_1}\right)[p+\mu_2\,\Psi(x_1)].$$

It suffices to show that $\Psi(x_1)>\Psi(x_2)$.

Formula (2.4), Sect. II leads us to introduce the function

$$f(s)=\frac{\lambda_1-\mu_1+s^2-\sqrt{(\lambda_1+\mu_1+s^2)^2-4\,\lambda_1\,\mu_1}}{2\,s}$$

Upon setting $s_1=\sqrt{\mu_2}+\sqrt{\lambda_2}$ and $s_2=\sqrt{\mu_2}-\sqrt{\lambda_2}$, we get

$$\Psi(x_2)=\frac{1}{\sqrt{\mu_2}}f(s_2),$$

$$\Psi(x_1)=\frac{1}{\sqrt{\mu_2}}f(s_1),$$

$$\frac{df}{ds}=\frac{1}{2\,s^2}\,\frac{g(s)}{q(s)}$$

where $g(s)=s^2[q(s)-s^2]+(\mu_1-\lambda_1)\,q(s)+(\mu_1-\lambda_1)^2$ and

$$q(s)=\sqrt{(\lambda_1+\mu_1+s^2)^2-4\,\lambda_1\,\mu_1}\,.$$

Obviously $g(s)>0$.

Hence, $f(s)$ is an increasing function of s for $s>0$ and $f(s_1)>f(s_2)$. This terminates the proof.

Appendix C

Expression of the Integral in Formula (6.3)

Upon setting

$$t=\frac{z^2+1}{2z}, \qquad u=\cos\rho, \qquad \Sigma=\lambda_1+\lambda_2+\mu_1+\mu_2,$$

$$\alpha=\frac{\Sigma+2\sqrt{\mu_2\lambda_2}}{2\sqrt{\mu_1\lambda_1}}, \qquad \beta=\frac{\Sigma-2\sqrt{\mu_2\lambda_2}}{2\sqrt{\mu_1\lambda_1}}$$

it follows

$$I(z)=\frac{(1-\xi)\pi}{\lambda_1 F(0,0)}\cdot\left[F\left(\sqrt{\frac{\mu_1}{\lambda_1}}z,0\right)-F(0,0)\right]=\int_{-1}^{1}\frac{\sqrt{1-u^2}/du}{(t-u)[a+bu+cR]} \tag{A.1}$$

where

$$a=\frac{\Sigma}{2}\left[\frac{\xi}{1-\xi}-\frac{\mu_1^*}{\mu_2^*}\right]-2\mu_1^*,$$

$$b=\sqrt{\lambda_1\mu_1^*\xi}\left[\frac{\mu_1^*}{\mu_2^*}-\frac{2-\xi}{1-\xi}\right],$$

$$c=\frac{1}{2}\left[\frac{\mu_1^*}{\mu_2^*}+\frac{\xi}{1-\xi}\right],$$

$$R=2\sqrt{\mu_1\lambda_1}\cdot\sqrt{(\alpha-u)(\beta-u)}.$$

Formula (A.1) yields

$$I(z)=I_1(z)+I_2(z) \tag{A.2}$$

$I_1(z)$ [resp. $I_2(z)$] is the "rational" [resp. "elliptic"] part of $I(z)$. We proceed now to derive $I_1(z)$ and $I_2(z)$.

1) $$I_1(z)=\int_0^{\pi}\frac{[a+b\cos\rho]\sin^2\rho\,d\rho}{(t-\cos\rho)(\alpha_1-\cos\rho)(\alpha_2-\cos\rho)(b^2-4\lambda_1\mu_1c^2)}$$

where

$$\alpha_i=\frac{1+\delta_i^2}{2\delta_i}, \qquad |\alpha_i|>1, \qquad |\delta_i|<1, \quad \mathrm{i}=1,2$$

δ_i being real numbers. [This results from Sect. V.]

But,

$$\frac{a+b\cos\rho}{(t-\cos\rho)(\alpha_1-\cos\rho)(\alpha_2-\cos\rho)}=\frac{A}{t-\cos\rho}+\frac{B_1}{\alpha_1-\cos\rho}+\frac{B_2}{\alpha_2-\cos\rho}$$

where

$$A=\frac{a+bt}{(t-\alpha_1)(t-\alpha_2)},$$

$$B_1=\frac{a+b\alpha_1}{(t-\alpha_1)(\alpha_2-\alpha_1)},$$

$$B_2=\frac{a+b\alpha_2}{(t-\alpha_2)(\alpha_1-\alpha_2)}.$$

Using Gradshteyn [7] p. 366 formula 3.613-3, we get

$$I_1(z)=\frac{\pi}{(b^2-4\lambda_1\mu_1c^2)}[zA+\delta_1B_1+\delta_2B_2]$$

and, after a somewhat lengthy computation,

$$I_1(z)=\frac{2\delta_1\delta_2\pi}{b^2-4\lambda_1\mu_1c^2}\left[bz+\frac{2a(\delta_1+\delta_2)}{1-\delta_1\delta_2}\right]\cdot\frac{z}{(1-z\delta_1)(1-z\delta_2)}.$$

2) $$I_1(z)=\frac{2c\sqrt{\lambda_1\mu_1}}{b^2-4\lambda_1\mu_1c^2}\int_{-1}^{1}\frac{(1-u^2)(\beta-u)(\alpha-u)\,du}{(t-u)(u-\alpha_1)(u-\alpha_2)\sqrt{(1+u)(1-u)(\beta-u)(\alpha-u)}}.$$

Repeatedly applying Gradshteyn [7], formulas 3.147-3, 3.148-3, and 3.151-3, we obtain

$$I_2(z)=K\left[\frac{(t-\alpha)(t+1)}{(t-\alpha_1)(t-\alpha_2)}\cdot\prod\left(\frac{\pi}{2},\frac{2(t-\beta)}{(\beta+1)(t-1)},r\right)+\frac{L}{t-\alpha_1}+\frac{M}{t-\alpha_2}+N\right],$$

where

* $$K=\frac{4c\sqrt{\lambda_1\mu_1}(\beta-1)}{(b^2-4\lambda_1\mu_1c^2)\sqrt{(\alpha-1)(\beta+1)}},$$

* $\Pi(\rho,n,k)$ is the elliptic integral of the third kind:

$$\Pi(\rho,n,k)=\int_0^{\sin\rho}\frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

* $$r=\sqrt{\frac{2(\alpha-\beta)}{(\alpha-1)(\beta+1)}},$$

* $$L=\frac{(1+\alpha_1)(\alpha-\alpha_1)}{\alpha_1-\alpha_2}\Pi\left(\frac{\pi}{2},\frac{2(\alpha_1-\beta)}{(\beta+1)(\alpha_1-1)},r\right),$$

* $$M=\frac{(1+\alpha_2)(\alpha-\alpha_2)}{\alpha_2-\alpha_1}\Pi\left(\frac{\pi}{2},\frac{2(\alpha_2-\beta)}{(\beta+1)(\alpha_2-1)},r\right),$$

* $$N=-\Pi\left(\frac{\pi}{2},\frac{2}{\beta+1},r\right).$$

References

1. Coffman, E.G., Mitrani, I.: Selecting a Scheduling Rule that Meets Pre-Specified Response Time Demands. Proceedings, 5th Symposium on Operating Systems Principles, Austin 1975
2. Flatto, L., MacKean, H.: Two Parallel Queues with Equal Servicing Rates. IBM Research Center Watson, Report RC 5916, March 1976
3. Fuchs, B.A., Shabat, B.V.: Functions of a Complex Variable and Some of Their Applications, Volume I. Oxford, London, New York, Paris: Pergamon Press 1964
4. Cohen, J.W.: The Single-Server Queue. Amsterdam: North-Holland Publishing Co. 1969
5. Malyshev, V.A.: An Analytical Method in the Theory of Two-Dimensional Positive Ramdon Walks [translated from Sibirskii]. Mathematicheskii Zhurnal, Vol. **13**, No. 6, pp. 1314–1329, November-December 1972
6. Muskhelishvili, N.I.: Singular Integral Equations. Groningen, Holland-Moscow: P. Noordhoff 1946
7. Gradshteyn, I.S., Ryzhik, I.M.: Tables of Integral Series Products. New York and London: Academic Press 1965
8. Mitrani, I., Hine, J.H.: Complete Parameterized Families of Job Scheduling Strategies. Technical Report Series, University of Newcastle upon Tyre, Number 81, October 1975
9. Kleinrock, L.: Queueing systems, II. New York: Wiley 1976

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