Z. Wahrscheinlichkeitstheorie verw. Gebiete
47, 289–297 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Singular Random Measure Generated by Splitting [0, 1]

Jacques Peyrière

IBM T.J. Watson Research Center, POBox 218, Yorktown Heights, NY 10598, USA

1. Introduction, Definitions, and Summary

1.1. Definitions and Notations

Let $\{\varepsilon_j\}_{j \geqq 1}$ be a sequence of independent random variables such that ε_j assumes values $1, 2, \ldots, j$ with equal probabilities. $\{W_j\}_{j \geqq 1}$ is a sequence of random variables with values in $]0, 1[$.

We define recursively an increasing sequence, $\{F_n\}_{n \geqq 1}$, of finite random subsets of $[0, 1]$: $F_1 = \{0, 1\}$ and, if $x_1, x_2, \ldots, x_{n+1}$ are the elements of F_n enumerated in increasing order, we get F_{n+1} by adding up to F_n the point

$$x_{\varepsilon_n} + (x_{\varepsilon_n + 1} - x_{\varepsilon_n}) W_n.$$

The complement of F_n in $[0, 1]$ consists in n intervals, $\{I_{n,j}\}_{1 \leqq j \leqq n}$, numbered from left to right.

Let μ_n denote the measure on $[0, 1]$, the density of which with respect to the Lebesgue measure is the random function

$$n^{-1} \sum_{1 \leqq j \leqq n} |I_{n,j}|^{-1} 1_{I_{n,j}}.$$

(If I is a Borel set in $[0, 1]$, $|I|$ denotes its Lebesgue measure and 1_I its indicator function.) If x belongs to $[0, 1[$, $I_n(x)$ is the interval among $\{I_{n,j}\}_{j \geqq 1}$ that contains x. (To make this definition precise, we have to replace each $I_{n,j}$ by the corresponding interval semi-closed to the left.)

1.2. Results

Proposition 1. *Almost surely μ_n converges to a probability μ in the weak-star sense. Furthermore, almost surely μ is continuous and its support is the adherence in $[0, 1]$ of the set of nonisolated points of* $\bigcup_{n \geqq 1} F_n$.

0044-3719/79/0047/0289/$01.80

Proposition 2. *Almost surely we have* $\log n\,\mu(I_n(x))/\log\log n = 0(1)$ *for* μ – *almost every* x.

Proposition 3. *If* W_j*'s are mutually independent, independent of* ε*'s and equidistributed with a random variable* W*, then, almost surely for* μ*-almost every* x,

$$\lim_{n\to\infty} [\log |I_n(x)|]/\log n = E[\log W(1-W)].$$

Theorem. *Hypotheses are the same as in Proposition 3. Almost surely there exists a Borel set carrying* μ*, the Hausdorff dimension of which is* $D = -1/E[\log W(1-W)]$*. On the other hand, every Borel set of dimension* $< D$ *is almost surely of* μ *– measure* 0.

1.3. Historical Background

The construction discussed in this paper is due to Voss [11], who introduced it as a variant of the stochastic model of turbulent intermittency due to Mandelbrot [8, 9]. The original motivation for this variant was that it is much easier to study on the computer. Simulations had suggested that the distribution of mass in the present model tends very rapidly to a limit involving singular measures. Mandelbrot brought the problem to my attention, together with a wealth of properties and conjectures suggested by simulation, intuition, or physics, and we had a number of stimulating discussions about it.

Various other random sets and measures considered elsewhere in the literature bear various degrees of resemblance to the present one. One example is the above-mentioned model by Mandelbrot [8, 9] which has been studied further by Kahane and Peyrière [4]. A construction in Dubins and Freedman [2] and Kinney and Pitcher [6] also converges to almost surely singular measures. On the other hand, the construction in Kakutani [5], while bearing a superficial resemblance to the present one, leads to a very different result, namely equidistributed points on [0, 1]; see also Adler and Flatto [1], Lootgieter [7] and Van Zwet [10].

1.4. This paper is organized as follows. Proposition 1 is proved in Section 2, propositions 2 and 3 in Section 3, and the Theorem in Section 4. Section 5 contains some comments.

I wish to thank the referee for his careful criticism of the original paper and suggestions for its improvement.

2. Convergence of the Sequence $\{\mu_n\}_{n>1}$

2.1. **Lemma.** *Let* a *and* b *be two integers such that* $0 \leqq b \leqq a$*, and let* F *be a subset of* $\{1, 2, \ldots, a\}$ *with cardinality* b*. Then, almost surely, as* n *tends to infinity,* $\mu_n(\bigcup_{j\in F} I_{a,j})$*, tends to a limit* $Z_{(a,F)}$*. Furthermore,* $Z_{(a,F)}$ *is independent of* $\{\varepsilon_j\}_{1\leqq j<a}$

and its law has $\frac{\Gamma(a)}{\Gamma(b)\,\Gamma(a-b)} t^{b-1}(1-t)^{a-b-1}$ *as density with respect to Lebesgue's measure on* $[0,1]$.

This lemma follows easily from the results on Polya urn scheme [3].

2.2. *Consequence:* Almost surely $\lim_{n\to\infty} \sup_{1\leqq j\leqq n} \lim_{m\to\infty} \mu_m(I_{n,j})=0$.

2.3. *Proof of Proposition 1.* Lemma 2.1 allows us almost surely to define an increasing function, g, on $\bigcup_{n\geqq 1} F_n$: $g(x)=\lim_{m\to\infty} \mu_m([0,x[)$. By 2.2, almost surely, $\inf(g(x); x>0)=0$, $\sup(g(x); x<1)=1$ and, for every y in $]0,1[$, $\sup(g(x); x<y) =\inf(g(x); x>y)$. So g has a continuous non-decreasing extension to $[0,1]$ whose derivative is easily shown to be the weak-star limit of the sequence $\{\mu_n\}_{n\geqq 1}$.

If some assumptions are made on the sequence $\{W_j\}_{j\geqq 1}$, precisions on the support of μ are obtained. The following lemma gives conditions insuring that the support is $[0,1]$.

2.4. **Lemma.** *Hypotheses are the same as in Proposition 3. Set*

$$\sigma=\sup_{t>1}\{[1-E(W^t+(1-W)^t)]/t\}.$$

Then almost surely $\sup_{n\geqq 1} n^{\sigma} \sup_{1\leqq j\leqq n} |I_{n,j}|$ *is finite.*

2.5. *Proof.* Let t be a real number such that $E[W^t+(1-W)^t]$ is finite. Set $\varphi(t) =1-E[W^t+(1-W)^t]$. First it will be shown that $n^{\varphi(t)} \sum_{1\leqq j\leqq n} |I_{n,j}|^t$ has an a.s. finite limit as n tends to infinity. We have

$$\sum_{1\leqq j\leqq n+1} |I_{n+1,j}|^t=\sum_{1\leqq j\leqq n} |I_{n,j}|^t-|I_{n,\varepsilon_n}|^t+|I_{n+1,\varepsilon_n}|^t+|I_{n+1,\varepsilon_n+1}|^t$$

so

$$\begin{aligned} E\Big(&\sum_{1\leqq j\leqq n+1} |I_{n+1,j}|^t \,|\, \varepsilon_1,\dots,\varepsilon_{n-1}, W_1,\dots,W_{n-1}\Big)\\ &=\sum_{1\leqq j\leqq n} |I_{n,j}|^t-E[|I_{n,\varepsilon_n}|^t(1-W_n^t-(1-W_n)^t)\,|\,\varepsilon_1,\dots,\varepsilon_{n-1}, W_1,\dots,W_{n-1}]\\ &=\Big[\sum_{1\leqq j\leqq n} |I_{n,j}|^t\Big](1-\varphi(t)/n). \end{aligned}$$

Therefore $\big(\sum_{1\leqq j\leqq n} |I_{n,j}|^t\big) \prod_{1\leqq j<n} (1-\varphi(t)/j)^{-1}$ is a positive martingale. On the other hand,

$$n^{\varphi(t)} \prod_{1\leqq j<n} (1-\varphi(t)/j)=\prod_{1\leqq j<n} (1+(1/j))^{\varphi(t)}\,(1-(\varphi(t)/j))$$

tends to a finite non-zero limit. This proves the above assertion.

If $t>0$, we get $\sup_{n\geqq 1}(n^{(\varphi(t)/t)} \sup_{1\leqq j\leqq n} |I_{n,j}|)<\infty$ a.s. The lemma then follows from the fact that $\varphi(t)/t$ actually assumes the value σ on $]1,+\infty[$ ($E(W^t+(1-W)^t)$ is a decreasing function of t).

Similarly, almost surely, $\inf_{n\geqq 1}(n^{\sigma'} \inf_{1\leqq j\leqq n} |I_{n,j}|)>0$, where $\sigma'=\inf_{t<0}(\varphi(t)/t)$.

3. Study of the Limit Measure

3.1. Let $(\Omega, \mathscr{A}, P)$ be the probability space on which are defined variables W_j and ε_j $(j \geqq 1)$. We endow $\Omega \times [0,1]$ with the σ-field $\mathscr{F}$, product of $\mathscr{A}$ by the Borel sets of $[0,1]$ and we define on it a probability Q in the following way.

$$Q(A) = E_P(\int 1_A \, d\mu).$$

So Q-almost surely exactly means P-almost surely μ-almost everywhere.

3.2. *Proof of Proposition 2.* Set $M_n(x) = \mu(I_n(x))$. The sequence, $\{M_n\}_{n \geqq 1}$, is clearly non-increasing. Using Lemma 2.1, one gets

$$Q(2^n M_{2^n} > 2 \operatorname{Log} n) = 2^n(2^n - 1) \int_{2^{1-n} \operatorname{Log} n}^{1} t(1-t)^{2^n - 2} \, dt$$
$$= (1 + (2^n - 1)\, 2^{1-n} \operatorname{Log} n)(1 - 2^{1-n} \operatorname{Log} n)^{2^n - 1} = O(n^{-2} \operatorname{Log} n).$$

Borel-Cantelli lemma and the decrease of M_n show that Q-a.s., $\limsup_{n \to \infty} \operatorname{Log} n M_n / \operatorname{Log} \operatorname{Log} \operatorname{Log} n \leqq 1$.

Similarly, if $\alpha > 1/2$, one has

$$Q(2^n M_{2^n} < n^{-\alpha}) = 2^n(2^n - 1) \int_0^{2^{-n} n^{-\alpha}} t(1-t)^{2^n - 2} \, dt$$
$$= 1 - (1 + (2^n - 1)\, 2^{-n} n^{-\alpha})(1 - 2^{-n} n^{-\alpha})^{2^n - 1} = O(n^{-2\alpha}).$$

Borel-Cantelli lemma and the decrease of M_n show that Q-a.s. $\liminf_{n \to \infty} \operatorname{Log} n M_n / \operatorname{Log} \operatorname{Log} n \geqq -\alpha$.

3.3. *Proof of Proposition 3.* Set $L_n = \sum_{1 \leqq j \leqq n} |I_{n,j}| \, 1_{I_{n,j}}$.

3.3.1. **Lemma.** *The random variables* $\{L_{n+1}/L_n\}_{n \geqq 1}$ *(defined on* $(\Omega \times [0,1], \mathscr{F}, Q)$ *) are idependent. Moreover,* $Q(L_{n+1}/L_n \leqq t) = [P(W \leqq t) + P(1 - W \leqq t)]/(n+1)$ *when* $0 \leqq t < 1$.

Proof. Let $\mathscr{F}_n$ be the smallest sub σ-field of $\mathscr{F}$ with respect to which the following X's are measurable:

$$X = \sum_{1 \leqq j \leqq n} X_j 1_{I_{n,j}},$$

where each X_j is measurable with respect to $\sigma(\varepsilon_1, \ldots, \varepsilon_{n-1}, W_1, \ldots, W_{n-1})$.

L_n is measurable with respect to $\mathscr{F}_n$.

Let us compute $E_Q(1_A(L_{n+1}/L_n) | \mathscr{F}_n)$ when A is a Borel subset of $[0,1[$. So if X is a bounded $\mathscr{F}_n$-measurable function, we have

$$E_Q[X 1_A(L_{n+1}/L_n)] = E_P(X_{\varepsilon_n} \mu(I_{n+1,\varepsilon_n}) 1_A(W_n) + X_{\varepsilon_n} \mu(I_{n+1,\varepsilon_n+1}) 1_A(1 - W_n)).$$

Using Lemma 2.1, we get

$$E_Q[X 1_A(L_{n+1}/L_n)] = (1/(n+1)) \, E[1_A(W) + 1_A(1-W)] \, E_P(X_{\varepsilon_n})$$
$$= (1/(n+1)) \, E[1_A(W) + 1_A(1-W)] \, E_Q(X).$$

Therefore,

$$E_Q[1_A(L_{n+1}/L_n)|\mathscr{F}_n]=E[1_A(W)+1_A(1-W)]/(n+1)$$

and

$$E_Q[1_A(L_{n+1}/L_n)|L_1,\dots,L_n]=E[1_A(W)+1_A(1-W)]/(n+1).$$

This proves the lemma.

3.3.2. **Lemma.** $\sum_{n\geqq 1} Q(L_{n+1}/L_n<1/n)\leqq 2\log 2-E[\log W(1-W)]$.

Proof. If $0<t\leqq 1$, we have, by 3.3.1,

$$Q(L_{n+1}/L_n<t)=(1/(n+1))\,E[1_{[0,t[}(W)+1_{[0,t[}(1-W)],$$

$$\begin{aligned}\sum_{n\geqq 1} Q(L_{n+1}/L_n<1/n)&\\ &=E[\sum_{n\geqq 1}(1/(n+1))\,1_{[0,(1/n)[}(W)+\sum_{n\geqq 1}(1/(n+1))\,1_{[0,(1/n)[}(1-W)]\\ &\leqq E[\log((1/W)+1)+\log((1/(1-W))+1)]\\ &\leqq 2\log 2+E[\log(1/W(1-W))].\end{aligned}$$

3.3.3. **Lemma.**

$$\sum_{n\geqq 1} E_Q[(\log\sup(L_{n+1}/L_n,1/n))^2]/[\log(n+1)]^2\leqq 2E[\log(1/W(1-W))].$$

Proof. If $0<t\leqq 1$, by Lemma 3.3.1, we have

$$\begin{aligned}&E_Q[\log^2\sup(L_{n+1}/L_n,1/n)]\\ &\quad=(1/(n+1))\,E[\log^2\sup(W,1/n)]+(1/(n+1))\,E[\log^2\sup(1-W,1/n)].\end{aligned}$$

But

$$\begin{aligned}&\sum_{n\geqq 1}(1/(n+1)\log^2(n+1))\log^2\sup(W,1/n)\\ &\quad=\sum_{n>(1/W)}(\log^2 W/(n+1)\log^2(n+1))\\ &\qquad+\sum_{1\leqq n\leqq(1/W)}(\log^2 n/(n+1)\log^2(n+1))\leqq 2\log(1/W),\end{aligned}$$

the second term is handled in the same way.

3.3.4. *End of the Proof of Proposition 3.* Suppose first $E(\log W(1-W))>-\infty$. By Lemmas 3.3.1 and 3.3.3 and the law of large numbers,

$$(1/\log(n+1))\sum_{1\leqq j\leqq n}\{\log\sup(L_{j+1}/L_j,1/j)-E_Q[\log\sup(L_{j+1}/L_j,1/j)]\}$$

tends to zero Q-a.s. By lemma 3.3.2, Q-a.s. beyond a certain rank, $L_{n+1}/L_n\geqq 1/n$, and by Lemma 3.3.1,

$$\begin{aligned}&E_Q[\log\sup(L_{n+1}/L_n,1/n)]\\ &\quad=(1/(n+1))\,E[\log\sup(W,1/n)+\log\sup(1-W,1/n)].\end{aligned}$$

Proposition 3 follows easily from these facts.

Suppose now $E(\log W(1-W))=-\infty$. By the law of large numbers and Lemma 3.3.1, we have, Q-a.s.,

$$\lim_{n\to\infty}\Big[\sum_{1\leqq k\leqq l}\sum_{1\leqq j\leqq n} 1_{]-\infty,-k]}(\log L_{j+1}-\log L_j)\Big]/\log n = \sum_{1\leqq k\leqq l}(P(W\leqq e^{-k})+P(1-W\leqq e^{-k})).$$

But

$$\sum_{1\leqq k\leqq l}[P(W\leqq e^{-k})+P(1-W\leqq e^{-k})] = E\{\inf(l,[\log 1/W])+\inf(l,[\log(1/(1-W)])\}$$

tends to $+\infty$ when l tends to $+\infty$. On the other hand,

$$-\sum_{1\leqq j\leqq n}(\log L_{j+1}-\log L_j)\geqq \sum_{1\leqq k\leqq l}\sum_{1\leqq j\leqq n} 1_{]-\infty,-k]}(\log L_{j+1}-\log L_j),$$

so $\log L_n/\log n$ tends to $-\infty$, Q-a.s.

4. Proof of Theorem

4.1. **Lemma.** *Set* $u_{n,j}=\operatorname{card}\{I_{m,k};\,1\leqq m<n,\ 1\leqq k\leqq m,\ I_{n,j}\subset I_{m,k},\ I_{n,j}\neq I_{m,k}\}$ *and* $u_n(x)=u_{n,j}$ *if* $x\in I_{n,j}$. *Then almost surely for* μ*-almost every* x, *we have* $\lim_{n\to\infty} u_n(x)/\log n=2$.

Proof. Set $U_n=\sum_{1\leqq j\leqq n} u_{n,j}1_{I_{n,j}}$ and let X be a $\mathscr{F}_n$-measurable bounded function. We have

$$E_Q[X(U_{n+1}-U_n)]=E_P[X_{\varepsilon_n}\mu(I_{n,\varepsilon_n})]=(2/(n+1))\,E_P(X_{\varepsilon_n})=(2/(n+1))\,E_Q(X)$$

and

$$E_Q[(U_{n+1}-U_n)^2]=E_P(\mu(I_{n,\varepsilon_n}))=2/(n+1).$$

Therefore $\sum(1/\log(n+1))\,(U_{n+1}-U_n-(2/(n+1)))$ converges Q-almost surely and the lemma is proved.

4.2. We have, almost surely for μ-almost every x,

$$\lim_{n\to\infty}\log\mu(I_n(x)|/\log|I_n(x)|=-1/E(\log W(1-W))$$

and

$$\lim_{n\to\infty}\log|I_n(x)|/u_n(x)=E(\log W(1-W))/2.$$

Then the Theorem derives from the following result, part of which is implicit in [6]. (Anticipating notations of the following paragraph, the theorem is obtained by setting $\mathscr{G}_n=\{I_{m,j};\,u_{m,j}=n\}$ and applying Lemma 4.3.2.)

4.3. *A way to estimate Hausdorff dimensions.*

4.3.1. The notations are particular to this section. Let us consider a sequence $\{\mathcal{G}_n\}_{n\geqq 0}$ of partitions of $[0,1[$ in semi-open intervals such that every element I of $\mathcal{G}_{n+1}$ is strictly contained in one, $\tilde{I}$, of $\mathcal{G}_n$. We set $\mathcal{G}=\bigcup_{n\geqq 0}\mathcal{G}_n$, $\mathcal{G}'=\bigcup_{n\geqq 1}\mathcal{G}_n$,

$$g(I)=n \quad \text{if } I\in\mathcal{G}_n,$$
$$b(I)=\operatorname{card}\{J\in\mathcal{G}';\tilde{J}=\tilde{I}\},$$
$$J_n(x)=I \quad \text{if } x\in I\in\mathcal{G}_n.$$

4.3.2. **Lemma.** *Let μ be a positive measure on $[0,1]$.*

a) *The set E of points $x\in[0,1]$ such that* $\lim_{n\to\infty}|J_n(x)|=0$ *and*
$\limsup_{n\to\infty}\log\mu(J_n(x))/\log|J_n(x)|<\alpha$

is such that $\dim E\leqq\alpha$.

b) *If a set F is such that $\mu(F)>0$ and for every $x\in F$ we have*

(i) $\liminf\log\mu(J_n(x))/\log|J_n(x)|>\alpha$

(ii) $-\gamma<\liminf(1/n)\log|J_n(x)|\leqq\limsup(1/n)\log|J_n(x)|<-\beta<0$,

then we have $\dim F\geqq(\alpha\beta+\beta-\gamma)/\gamma$. *If, furthermore, we have* $\limsup(1/n)\log b(J_n(x))<\delta$, *then* $\dim F\geqq(\alpha\beta-\delta)/\gamma$.

4.3.3. *Proof of a).* Let us observe first that $E\cap\bigcup(I\in\mathcal{G};\ \mu(I)=0)$ is empty. Let t be a positive number. Set

$$F_t=\bigcap_{n\geqq 0}\bigcup\{I\in\mathcal{G};\ g(I)=n \text{ and } \mu(I)\geqq\inf(t,|I|^\alpha)\}.$$

F_t increases as t decreases and $\bigcup_{t>0}F_t$ contains E. We shall now show that for every t, $\dim F_t\cap E\leqq\alpha$.

Let ε be a positive number less than $t^{1/\alpha}$. For each $x\in F_t\cap E$ such that $\mu(\{x\})<\varepsilon^\alpha$, choose an element I_x of $\mathcal{G}$ such that $x\in I_x$ and $\mu(I_x)<\varepsilon^\alpha$. By definition of F_t, $|I_x|^\alpha\leqq\mu(I_x)$. Now let $x_1,\dots,x_v$ be the elements of $F_t\cap E$ such that $\mu(\{x_j\})\geqq\varepsilon^\alpha$. Choose, for each j, an interval I_{x_j} of $\mathcal{G}$ such that $x_j\in I_{x_j}$, $|I_{x_j}|<\varepsilon$ and $\sum_{j=1}^{v}|I_{x_j}|^\alpha<1$. Then if $\{J_\lambda\}_{\lambda\in\Lambda}$ is a sub-family of $\{I_x\}_{x\in F_t\cap E}$ covering $E\cap F_t$ and whose elements are mutually disjoint, one has

$$\sup_{\lambda\in\Lambda}|J_\lambda|<\varepsilon \quad\text{and}\quad \sum_{\lambda\in\Lambda}|J_\lambda|^\alpha\leqq\mu([0,1[)+1.$$

This proves the above claim.

4.3.4. *Proof of b).* Let us assume first hypotheses (i) and (ii). Let t be a positive integer. Set

$$F_t=\bigcup\{I\in\mathcal{G}_0;\ |I|<t\}\cup\bigcup\{I\in\mathcal{G}';\ |I|<t \text{ and } (\mu(I)\geqq|I|^\alpha \text{ or } |\tilde{I}|\geqq e^{-\beta g(I)} \text{ or } |I|\leqq e^{-\gamma g(I)})\}.$$

F_t decreases as t decreases. $x \in \bigcap_{t>0} F_t$ implies that for infinitely many n we have $\mu(J_n(x)) \geqq |J_n(x)|^\alpha$ or $|J_{n-1}(x)| \geqq e^{-\beta n}$ or $|J_n(x)| \leqq e^{-n\gamma}$, so $F \cap F_t$ decreases to ϕ when t decreases to zero. We fix t in order to have $\mu(F \cap F_t^c) > 0$.

Set $\mu' = \mu.\, 1_{F \cap F_t^c}$ and $\alpha' = (\alpha\beta + \beta - \gamma)/\gamma$. Let x and y be such that $0 \leqq x \leqq y$, and consider the set $\{I_\lambda\}_{\lambda \in \Lambda}$ of maximal elements for inclusion of $\{I \in \mathscr{G};\, I \subset]x, y[\}$. The set $F \cap (]x, y[\setminus \bigcup_{\lambda \in \Lambda} I_\lambda)$ is empty because every element of F is by (ii) contained in arbitrarily small elements of $\mathscr{G}$, so $\mu'(]x, y[) = \sum_{\lambda \in \Lambda} \mu'(I_\lambda)$. We want to show that there exists C independent of x and y such that $\mu'(]x, y[) \leqq C(y-x)^{\alpha'}$ whenever $0 < y - x < t$. Observe first that $\mu'(I_\lambda) = 0$ if $I_\lambda \subset F_t$; this is the case in particular if $g(I_\lambda) = 0$. Set $n_0 = \inf\{g(I_\lambda);\, \lambda \in \Lambda \text{ and } g(I_\lambda) > 0\}$. Let n be an integer not less than n_0. It results from the definition of I_λ's that those I_λ's such that $g(I_\lambda) = n$ are contained in at most two intervals of $\mathscr{G}_{n-1}$. Consider an interval I_λ such that $g(I_\lambda) = n$ and $I_\lambda \not\subset F_t$. We have $|I_\lambda| > e^{-\gamma n}$ and $|\tilde{I}_\lambda| < e^{-\beta n}$. The number of such intervals is therefore less than $2e^{(\gamma-\beta)n}$. Thus we have

$$\mu'(]x, y[) \leqq \sum_{n \geqq n_0} 2e^{(\gamma-\beta)n} e^{-\beta\alpha n} \leqq C e^{-(\beta\alpha + \beta - \gamma) n_0}$$

provided $\beta + \alpha\beta - \gamma > 0$ (if this is not the case, we have nothing to prove). But $e^{-n_0\gamma} < y - x$, so $\mu'(]x, y[) \leqq C(y-x)^{(\alpha\beta+\beta-\gamma)/\gamma}$.

Now if $\{J_\lambda\}_{\lambda \in \Lambda}$ is a covering of $F \cap F_t^c$ by intervals (not necessarily in $\mathscr{G}$) such that $|J_\lambda| < t$, we have

$$0 < \mu(F \cap F_t^c) \leqq \sum_{\lambda \in \Lambda} \mu'(J_\lambda) \leqq \sum_{\lambda \in \Lambda} |J_\lambda|^{(\alpha\beta+\beta-\gamma)/\gamma}.$$

This proves that $\dim F \cap F_t^c \geqq (\alpha\beta + \beta - \gamma)/\gamma$.

The proof of the last assertion is similar. This time we just have to set

$$F_t = U\{I \in \mathscr{G}_0;\, |I| < t\} \cup \bigcup \{I \in \mathscr{G}';\, |I| < t \text{ and } (\mu(I) \geqq |I|^\alpha \text{ or } |I| \geqq e^{-\beta g(I)} \text{ or } |I| \leqq e^{-\gamma g(I)} \text{ or } b(I) \geqq e^{\delta g(I)})\}.$$

5. Comments

5.1. In cases where assumption of equidistribution of W_j's is dropped, Lemma 3.3.1 needs a modification:

$$Q(L_{n+1}/L_n \leqq t) = 1/(n+1)\,[P(W_n \leqq t) + P(1 - W_n \leqq t)], \qquad \text{when } 0 \leqq t < 1.$$

One can restate Proposition 3 as follows. If

$$\sum_{n \geqq 1} E[(\log W_n)^2 + (\log(1-W_n))^2]/(n+1)\,[\log(n+1)]^2 < \infty,$$

then Q-almost surely $\{\log L_{n+1} - \sum_{1 \leqq j \leqq n} (1/(j+1))\; E[\log W_j(1-W_j)]\}/\log n$ tends to zero. Then Lemma 4.3.2 allows us to make some assertions about the Hausdorff dimension of sets of μ-measure > 0.

5.2. *Connection with the Construction of Dubins and Freedman.* Let $W = (W^1, W^2)$ be now a random variable with value in $[0,1[\times[0,1[$. We consider two sequences of r.v.'s $\{W_j\}_{j\geqq 1}$ and $\{\varepsilon_j\}_{j\geqq 1}$, satisfying hypotheses analogous to those of proposition 3. We define as previously an increasing sequence $\{F_n\}_{n\geqq 1}$, of finite random subsets of $[0,1]\times[0,1]$. Projections of F_n on both coordinates axes give rise to two families of intervals, $\{I^1_{n,j}\}_{1\leqq j\leqq n}$ and $\{I^2_{n,j}\}_{1\leqq j\leqq n}$. For almost every sequence $\{\varepsilon_j\}_{j\geqq 1}$ the set $\bigcap_{n\geqq 1} \bigcup_{1\leqq j\leqq n} I^1_{n,j}\times I^2_{j,n}$ is the graph of the random increasing function considered by Dubins and Freedman [2]. (The probability used to perform the construction is the law of W.)

We define the measures μ^k_n, $k=1,2$, the density of which, with respect to Lebesgue's measure, is $n^{-1}\sum_{1\leqq j\leqq n} |I^k_{n,j}|^{-1}\, 1_{I^k_{n,j}}$. Almost surely μ^k_n converges to μ^k. μ^2 is the image of μ^1 by the Dubins-Freedman function. μ^k is a random measure resulting from the construction considered in Section 1, performed with W^k $(k=1,2)$.

References

1. Adler, R.L., Flatto, L.: Uniform distribution of Kakutani's interval splitting procedure. Z. Wahrscheinlichkeitstheorie verw. Gebiete **38**, 253–259 (1977)
2. Dubins, L.E., Freedman, D.A.: Random distribution functions. Fifth Berkeley Symp. Math. Statist Probab. Univ. Calif. Vol. II, 183–214 (1967)
3. Feller, W.: An Introduction to Probability Theory and Its Applications. Vol. II. New York: J. Wiley 1971
4. Kahane, J.-P., Peyrière, J.: Sur certaines martingales de Benoit Mandelbrot. Advances in Math. **22**, 131–145 (1976)
5. Kakutani, S.: A problem of equidistribution on the unit interval [0, 1]. Proc. of the Oberwolfach Conference on Measure Theory (1975). Lecture Notes in Math **541.** Berlin-Heidelberg-New York: Springer 1976
6. Kinney, J.R., Pitcher, T.S.: The dimension of the support of a random distribution function. Bull. Amer. Math. Soc. **70**, 161–164 (1964)
7. Lootgieter, J.-C.: Sur la répartition des suites de Kakutani. C.R. Acad. Sci. Paris **285**, 403–406 (1977)
8. Mandelbrot, B.B.: Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire. C.R. Acad. Sci. Paris **278**, 289–292 (1974)
9. Mandelbrot, B.B.: Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire: quelques extensions. C.R. Acad. Sci. Paris **278**, 567–569 (1974)
10. Van Zwet, W.R.: A proof of a conjecture of Kakutani. To appear.
11. Voss, R.F.: Personal communication

Received February 4, 1978; in revised form September 12, 1978