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Asymptotic Properties of Stationary Point Processes with Generalized Clusters

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In Memoriam ROLLO DAVIDSON

1. Introduction

Lewis (1971) has extended the notion of a generalized branching Poisson process or cluster point process (Bartlett, 1963; Lewis, 1964) to a generalized branching renewal process and obtained some asymptotic results for these processes, suggesting that some of the asymptotic results of Lewis (1969) which extend to branching renewal processes should also extend to branching stationary point processes. The object of this paper is to establish some analogous asymptotic results for a further generalization which encompasses Bartlett-Lewis and Neyman-Scott cluster processes and also the process with one ancillary variable in Vere-Jones (1970) and the Discussion thereon.

It is hoped that the generality of the model and the methods used in the proofs indicate the pertinent features in the structure of the process that lead to the asymptotic properties established. Apart from assumptions concerning finiteness, the most important of these features are the ergodicity of the primary point process, the independence of different clusters, and their independence of the primary process.

2. Notation and Preliminary Results

Underlying our observed point process $N(.)$ is a *primary point process* $N^*(.)$ of which a typical sample realization consists of the points $\{t_j\}$ $(j=0, \pm 1, \ldots)$ say. Each point t_j is the origin of a *subsidiary point process* $n_j(.)$. The processes $\{n_j(.)\}$ are assumed to be mutually independent realizations of the generic subsidiary point process $n(.)$ defined on the Borel subsets of the real line R with $n(R)<\infty$ almost surely (a.s.), and

$$\begin{aligned} m_1(x)&=E\big(n(-\infty, x]\big), \qquad & m_2(x)&=E\big(n^2(-\infty, x]\big), \\ m_1&=m_1(\infty), & m_2&=m_2(\infty)<\infty. \end{aligned} \tag{1}$$

The complete point process $N(.)$ is then defined for any bounded Borel set A by

$$N(A)=\sum_{\text{all } j} n_j(A-t_j). \tag{2}$$

Perusal of the results below will show that they hold equally well if $n(.)$, instead of being a point process (i.e., a random integer-valued set function) is simply a non-negative random measure satisfying the finiteness conditions at (1).

Presumably it is also true that the non-negativity can also be weakened so that $n(.)$ and $N(.)$ may be signed random measures, but such a generalization involves further work which would be a digression in this paper.

The simplest stationarity assumption to make concerning $N^*(.)$ is that for bounded Borel sets $A_1, A_2, \ldots$, non-negative integers $i_1, i_2, \ldots$, positive integral r and real y, the probability measure $\mathrm{pr}\{.\}$ for $N^*(.)$ satisfies

$$\mathrm{pr}\{N^*(A_1)=i_1, \ldots, N^*(A_r)=i_r\}=\mathrm{pr}\{N^*(A_1+y)=i_1, \ldots, N^*(A_r+y)=i_r\}. \quad (3)$$

Then $N^*(.)$ is a *stationary point process*, and it follows easily that, assuming $N(A)$ is finite for bounded Borel sets A, $N(.)$ is stationary also. When

$$\mu=E\big(N^*(0,1]\big)<\infty, \quad (4)$$

it is known (e.g. Slivnyak, 1962, 1966) that there is associated with $\mathrm{pr}\{.\}$ a unique probability measure $\mathrm{pr}_0\{.\}$ for a point process $N_0^*(.)$ (in general, non-stationary) whose sample paths $\{t_j'\}$ satisfy $\cdots \leqq t'_{-1} \leqq t'_0 = 0 \leqq t'_1 \leqq \cdots$, and $\mathrm{pr}_0\{.\}$ is such that the sequence $\{T_j\}=\{t'_{j+1}-t'_j\}$ of non-negative random variables is a stationary random sequence, i.e., for any integers satisfying $-\infty<i_1<\cdots<i_r<\infty$, non-negative $x_1, x_2, \ldots$, positive integral r and $j<i_1$,

$$\mathrm{pr}_0\{t'_{i_1}-t'_j \leqq x_1, \ldots, t'_{i_r}-t'_j \leqq x_r\}=\mathrm{pr}_0\{t'_{i_1-j} \leqq x_1, \ldots, t'_{i_r-j} \leqq x_r\}. \quad (5)$$

It is equally natural to replace the stationary process $N^*(.)$ and $\{t_j\}$ in (2) by $N_0^*(.)$ and $\{t_j'\}$ to give

$$N_0(A)=\sum_{\text{all } j} n_j(A-t_j'). \quad (6)$$

(In more picturesque language, the process $N^*(.)$ is associated with arbitrary time sampling or asynchronous counting, while $N_0^*(.)$ arises from using arbitrary events or taking synchronous counts.)

In (2) and (6) it is envisaged that the primary point process concerned (i.e., $N^*(.)$ or $N_0^*(.)$) evolves over the whole time axis. Lewis (1969, 1971) also studies a *transient process* in which the primary point process is confined to the non-negative time axis; it may then be sensible on practical grounds to assume also that $n((-\infty,\infty))=n([0,\infty))$ a.s., but our analysis holds without making this assumption. The functions $\tilde{N}(.)$ and $\tilde{N}_0(.)$ defined by

$$\tilde{N}(x)=\sum_{t_j\in(0,x]} n_j((-t_j, x-t_j]), \quad (7)$$

$$\tilde{N}_0(x)=\sum_{t_j'\in(0,x]} n_j((-t_j', x-t_j']) \quad (8)$$

suffice for our discussion of transient processes: a more general function could be defined by

$$\sum_{t_j\in A} n_j(A-t_j).$$

The asymptotic behaviour we discuss concerns the behaviour as $x\to\infty$ of the random variables $\tilde{N}(x)$ and $\tilde{N}_0(x)$ at (7) and (8), and the random variables $N(x)$

and $N_0(x)$ obtained by taking $A=(0, x]$ in (2) and (6) and writing[1]

$$N(x)=N((0, x]), \qquad N_0(x)=N_0((0, x]). \tag{9}$$

For technical convenience we assume that the primary process is *orderly*, that is, $\mathrm{pr}\{.\}$ and $\mathrm{pr}_0\{.\}$ satisfy

$$\mathrm{pr}\{N^*(0, x]>1\}=o(x) \qquad (x\downarrow 0), \tag{10}$$

$$\mathrm{pr}_0\{N_0^*(-x, x]>1\}=o(1) \qquad (x\downarrow 0). \tag{11}$$

It is then a consequence of Korolyook's theorem that the intensity parameter μ at (4) satisfies

$$\lambda=\lim_{x\downarrow 0} \mathrm{pr}\{N^*(0, x]>0\}/x=\mu, \tag{12}$$

and as Ryll-Nardzewski (1961) showed,

$$\alpha=E_0(t'_{j+1}-t'_j)=\lambda^{-1}. \tag{13}$$

We assume that the primary processes $N^*(.)$ and $N_0^*(.)$ are *ergodic* in the sense that

$$\mathrm{pr}\{N^*(0, x)/x \to \lambda (x\to\infty)\}=1, \tag{14}$$

$$\mathrm{pr}_0\{t'_j/j \to \alpha (j\to\infty)\}=1. \tag{15}$$

The *expectation function*

$$U(x)=E_0(N_0^*(0, x])=E_0(N_0^*(x)) \tag{16}$$

will appear in the discussion of both $N(x)$ and $N_0(x)$, for when (as in this paper) $\mathrm{pr}_0\{.\}$ is determined by $\mathrm{pr}\{.\}$ as above, and $N^*(.)$ is orderly,

$$V^*(x)=\mathrm{var}(N^*(x))=\lambda \int_0^x [1+2U(y)-2\lambda y]\, dy. \tag{17}$$

Always, $U(x)\sim\lambda' x\ (x\to\infty)$ for some constant $\lambda'\geqq\lambda$ (see Lemma 9 of Daley, 1971), and when also the stationary process $N^*(.)$ is ergodic as at (14), $\lambda'=\lambda$ (Lemma 10, *op. cit.*), i.e.,

$$U(x)/x\to\lambda\ (x\to\infty). \tag{18}$$

Kaplan (1955) showed that $U(.)$ also has something of a Blackwell-type renewal theorem property, namely

$$\sup_{y>0} (U(y+x)-U(y))\leqq 2U(x)+1. \tag{19}$$

In the proofs of the theorems we use decompositions of the random variables $\tilde{N}(x)$ etc. as in

$$\tilde{N}(x)=N^+(x)-N^-(x), \qquad \tilde{N}_0(x)=N_0^+(x)-N_0^-(x), \tag{20}$$

where

$$N^+(x)=\sum_{t_j\in(0,x]} n_j(R), \qquad N_0^+(x)=\sum_{t'_j\in(0,x]} n_j(R), \tag{21}$$

$$N(x)=\tilde{N}(x)+N^=(x)=N^+(x)+N^=(x)-N^-(x) \tag{22}$$

[1] $N(\cdot)$ and $N_0(\cdot)$ can now denote either a set function (like $N(A)$) or a point function (like $N(x)$). It should be clear from the context which meaning is intended. Similarly, we shall use $N^*(x)$, $N_0^*(x)$, later.

where

$$N^{=}(x)=\sum_{t_j\in R\setminus(0,x]} n_j((-t_j, x-t_j]), \tag{23}$$

$$N_0(x)=\tilde{N}_0(x)+N_0^{=}(x)=N_0^{+}(x)+N_0^{=}(x)-N_0^{-}(x). \tag{24}$$

3. Finiteness and First Moments

The random variables $\tilde{N}(x)$ and $\tilde{N}_0(x)$ are a.s. finite because by assumption each is the sum of an a.s. finite number of finite random variables. It is not so trivial a matter to establish finiteness of $N(x)$ and $N_0(x)$ (see Theorem 3 of Westcott, 1971). Certainly they will be finite when the first moments are finite because the sums at (2) and (6) are sums of non-negative random variables, and we content ourselves with these weaker statements.

Theorem 1. *When the primary point process $N^*(.)$ is orderly and has finite mean, and $m_1<\infty$,*

$$\mathrm{pr}\{N(A)<\infty\}=1=\mathrm{pr}_0\{N_0(A)<\infty\} \tag{25}$$

for every bounded Borel set A.

Proof. Since each $n_j(.)$ is non-negative, $N(A)\leqq N(0,x]=N(x)$ when $A\subseteq(0,x]$, and similarly for $N_0(.)$. The modifications needed when $A\cap(-\infty,0)\neq\emptyset$ are easily made. Recalling that

$$0\leqq m_1(x)\uparrow m_1<\infty \quad (-\infty<x\uparrow\infty),$$

we have, finite or infinite, using $E(N^*(u,u+du])=\lambda\,du$,

$$\begin{aligned} E(N(x)) &= \int_{-\infty}^{\infty} E(n(-u,\,-u+x])\,\lambda\,du=\int_{-\infty}^{\infty}\lambda\,du\int_{-u}^{-u+x} dm_1(y)\\ &= \int_{-\infty}^{\infty} dm_1(y)\int_{-y}^{-y+x}\lambda\,du=\lambda\,m_1\,x \end{aligned} \tag{26}$$

and hence the first part of (25). Similarly, since $E_0(N_0^*(u,u+du])=|dU(u)|$ for $u\neq 0$,

$$\begin{aligned} E_0(N_0(x)) &= E(n(0,x])+\int_0^{\infty}[E(n(-u,\,-u+x])+E(n(u,u+x])]\,dU(u)\\ &= m_1(x)-m_1(0)+\int_0^{\infty} dU(u)\left(\int_{-u}^{-u+x} dm_1(y)+\int_u^{u+x} dm_1(y)\right)\\ &= \int_{-\infty}^{0}[U(x-y)-U(-y)]\,dm_1(y)+\int_0^x[U(x-y)+1+U(y)]\,dm_1(y)\\ &\quad+\int_x^{\infty}[U(y)-U(y-x)]\,dm_1(y) \end{aligned} \tag{27}$$

which by Kaplan's result (19) and the finiteness of m_1 is finite.

Theorem 2. *Under the conditions of Theorem* 1,

$$E(N(x))=\lambda m_1\,x\sim E_0(N_0(x))\sim E(\tilde{N}(x))\sim E_0(\tilde{N}_0(x)) \quad (x\to\infty). \tag{28}$$

Proof. We have shown the equality in (28) at (26). Use (27) to get an expression for $E_0(N_0(x))/x$ and note from (18) that (for example) $[U(x-y)-U(-y)]/x \to \lambda$ $(x \to \infty)$. Then by the dominated convergence theorem,

$$E_0(N_0(x))/x \to \int_{-\infty}^{0} \lambda\, dm_1(y) + \int_{0}^{\infty} \lambda\, dm_1(y) + 0 = \lambda m_1,$$

proving the next part of (28). The rest of (28) follows from similar dominated convergence theorem arguments based on

$$\begin{aligned} E(\tilde{N}(x)) &= \int_0^x E(n(-u, -u+x])\, \lambda\, du = \lambda \int_{-x}^{x} dm_1(y) \int_{\max(0,-y)}^{\min(x,x-y)} du \\ &= \lambda \int_{-x}^{x} (x-|y|)\, dm_1(y) = \lambda \int_{-\infty}^{\infty} (x-|y|)^+\, dm_1(y), \end{aligned} \tag{29}$$

$$\begin{aligned} E_0(\tilde{N}_0(x)) &= \int_0^x E(n(-u, -u+x])\, dU(u) \\ &= \int_{-x}^{0} [U(x)-U(-y)]\, dm_1(y) + \int_0^x U(x-y)\, dm_1(y). \end{aligned} \tag{30}$$

Notice that $N^+(x)$ and $N_0^+(x)$ defined at (21) have

$$E(N^+(x)) = E(m_1 N^*(x)) = \lambda m_1 x, \tag{31}$$

$$E_0(N_0^+(x)) = E_0(m_1 N_0^*(x)) = m_1 U(x) \sim \lambda m_1 x \qquad (x \to \infty), \tag{32}$$

so from (20) we have the

Corollary. $E(N^-(x)) = o(x) = E_0(N_0^-(x)) \qquad (x \to \infty).$ (33)

4. Asymptotic Distributions

Lewis (1969, 1971) made extensive use of the type of decomposition at (20) and (22). When coupled with the results of the corollary it suggests that the asymptotic behaviour as $x \to \infty$ of $\tilde{N}(x)$ (or $\tilde{N}_0(x)$) should be similar to that of $N^+(x)$ (or $N_0^+(x)$). For simplicity below we assume that $m_2 = E(n^2(R)) < \infty$; it should be possible when $m_2 = \infty$ to prove analogous results with another limit law in place of the normal.

When $m_2 < \infty$, either $m_2 = m_1^2$ and $n_j(R) = m_1$ a.s., or we can set

$$\sigma = (m_2 - m_1^2)^{\frac{1}{2}} > 0, \qquad X_r = \left(\sum_{j=1}^{r} n_j(R) - m_1 r \right) \Big/ \sigma r^{\frac{1}{2}} \tag{34}$$

$(r = 1, 2, \ldots)$ with X_r asymptotically $(r \to \infty)$ normally distributed. Write

$$\begin{aligned} \tilde{N}(x) - \lambda m_1 x &= \sigma (N^*(x))^{\frac{1}{2}} X_{N^*(x)} + m_1(N^*(x) - \lambda x) - N^-(x) \\ &= \xi_x + \eta_x + \zeta_x \qquad \text{(say)}, \end{aligned} \tag{35}$$

and recall from the ergodicity assumption at (14) that $[N^*(x)/\lambda x]^{\frac{1}{2}} \to 1$ a.s. $(x \to \infty)$.

Theorem 3. *Let* $\sigma^2 = \mathrm{var}(n(R))$, $0 < \sigma < \infty$.

(i) *If*

$$\limsup_{x\to\infty} \mathrm{var}(N^*(x))/x^{1-\delta} < \infty \quad \textit{for some } \delta > 0, \tag{36}$$

$$E(N^-(x))/x^{\frac{1}{2}} \to 0 \quad (x \to \infty), \tag{37}$$

then

$$\lim_{x\to\infty} \mathrm{pr}\{\tilde{N}(x) - \lambda m_1 x \leqq \sigma(\lambda x)^{\frac{1}{2}} u\} = \Phi(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} e^{-t^2/2}\, dt. \tag{38}$$

(ii) *If*

$$\limsup_{x\to\infty} \mathrm{var}(N^*(x))/x < \infty, \tag{39}$$

$$\lim_{x\to\infty} \mathrm{pr}\{N^*(x) - \lambda x \leqq x^{\frac{1}{2}} u\} = F(u) \tag{40}$$

for all points of continuity of $F(.)$ *where* $F(.)$ *is a distribution function on* R, *and* (37) *is satisfied, then*

$$\lim_{x\to\infty} \mathrm{pr}\{\tilde{N}(x) - \lambda m_1 x \leqq \sigma(\lambda x)^{\frac{1}{2}} u\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(\sigma \lambda^{\frac{1}{2}}(u-v)/m_1)\, e^{-v^2/2}\, dv. \tag{41}$$

(iii) *If for some* $\delta > 0$

$$\lim_{x\to\infty} \mathrm{pr}\{N^*(x) - \lambda x \leqq x^{\frac{1}{2}(1+\delta)} u\} = F(u) \tag{42}$$

for all points of continuity of $F(.)$ *where* $F(.)$ *is a distribution function on* R, *and*

$$\lim_{x\to\infty} E(N^-(x))/x^{\frac{1}{2}(1+\delta)} = 0, \tag{43}$$

then

$$\lim_{x\to\infty} \mathrm{pr}\{\tilde{N}(x) - \lambda m_1 x \leqq x^{\frac{1}{2}(1+\delta)} u\} = F(u/m_1) \tag{44}$$

for all points of continuity of $F(.)$.

Proof. We make use of Slutsky's theorem (e.g. p. 254 of Cramer, 1946) applied to appropriate normalizations of $\xi_x + \eta_x + \zeta_x$.

(i) The random variable $N^-(x)$ is non-negative so (37) implies that $\zeta_x/x^{\frac{1}{2}} = N^-(x)/x^{\frac{1}{2}} \xrightarrow{p} 0$ $(x \to \infty)$. (Here and below, $\xrightarrow{p}$ denotes convergence in probability.) Eq. (36) implies that $\eta_x/x^{\frac{1}{2}} \xrightarrow{p} 0$ $(x \to \infty)$. Now $(N^*(x)/\lambda x)^{\frac{1}{2}} \xrightarrow{\text{a.s.}} 1$ $(x \to \infty)$, so by Slutsky's theorem $(\tilde{N}(x) - \lambda m_1 x)/\sigma(\lambda x)^{\frac{1}{2}}$ has the same limit law as $X_{N^*(x)}$. But $N^*(x)/[\lambda x] \xrightarrow{\text{a.s.}} 1$ and X_r is the standardized sum of r independent identically distributed random variables with finite positive variance, so by a result of Anscombe (1952) $X_{N^*(x)}$ has limit law $\Phi(.)$ as at (38).

(ii) As in (i), $\zeta_x/x^{\frac{1}{2}} \xrightarrow{p} 0$ $(x \to \infty)$. Then the limit law (if any) of $(\tilde{N}(x) - \lambda m_1 x)/x^{\frac{1}{2}}$ coincides with that of $(\xi_x + \eta_x)/x^{\frac{1}{2}}$. Let $\phi_x(.)$ denote the characteristic function of $(N^*(x) - \lambda x)/x^{\frac{1}{2}}$ and set

$$\phi(t) = \lim_{x\to\infty} \phi_x(t) = \int_{-\infty}^{\infty} e^{itu}\, dF(u).$$

Note that with $\psi(t) = E(e^{it(n(R)-m_1)})$,

$$(\psi(t/x^{\frac{1}{2}}))^x \to e^{-t^2\sigma^2/2} \quad (x \to \infty). \tag{45}$$

Then

$$
\begin{aligned}
&E(\exp\{i t(\xi_x+\eta_x)/x^{\frac{1}{2}}\})\\
&=E\left(E\left[\exp\left\{i t \sum_{j=1}^{N^*(x)} (n_j(R)-m_1)/x^{\frac{1}{2}}\right\} \exp\{i t m_1(N^*(x)-\lambda x)/x^{\frac{1}{2}}\} \mid N^*(x)\right]\right)\\
&=E(\exp\{N^*(x)\log\psi(t/x^{\frac{1}{2}})+i t m_1(N^*(x)-\lambda x)/x^{\frac{1}{2}}\})\\
&=(\psi(t/x^{\frac{1}{2}}))^{\lambda x} E(\exp\{i[t m_1 - i x^{\frac{1}{2}}\log\psi(t/x^{\frac{1}{2}})]\,(N^*(x)-\lambda x)/x^{\frac{1}{2}}\})\\
&=(\psi(t/x^{\frac{1}{2}}))^{\lambda x}\,\phi_x(t m_1 - i x^{\frac{1}{2}}\log\psi(t/x^{\frac{1}{2}})).
\end{aligned}
$$

Now by (45) for every fixed t, $h(x)=-i x^{\frac{1}{2}}\log\psi(t/x^{\frac{1}{2}})\to 0$ $(x\to\infty)$ and

$$|\phi_x(t m_1+h(x))-\phi_x(t m_1)|=|h(x)|\cdot|\phi_x'(t m_1+\vartheta h(x))|$$

for some ϑ with $|\vartheta|\leqq 1$. Moreover,

$$|\phi_x'(t m_1+\vartheta h(x))|\leqq[\operatorname{var}(N^*(x)/x^{\frac{1}{2}})]^{\frac{1}{2}}$$

which by (39) is uniformly bounded as $x\to\infty$. So

$$\lim_{x\to\infty}\phi_x(t m_1+h(x))=\lim_{x\to\infty}\phi_x(t m_1)=\phi(t m_1),$$

implying that

$$\lim_{x\to\infty} E(\exp\{i t(\xi_x+\eta_x)/x^{\frac{1}{2}}\})=e^{-t^2\lambda\sigma^2/2}\,\phi(t m_1).$$

Written in terms of distribution functions we recover (41).

(iii) As in the proof of (38), $\zeta_x/x^{\frac{1}{2}(1+\delta)}\xrightarrow{p}0$, and now $\xi_x/x^{\frac{1}{2}(1+\delta)}\xrightarrow{p}0$. The result (43) follows from Slutsky's theorem.

Remarks. Since

$$E(N^-(x))=\int_0^x E(n(R\setminus(-u,\,-u+x]))\,\lambda\,du=\lambda\int_0^x[m_1-m_1(x-u)+m_1(-u)]\,du,$$

the conditions at (37) and (43) relate to the rate of convergence to zero of $m_1(-u)$ and $m_1-m_1(u)$ as $u\to\infty$. These conditions are obviously satisfied when

$$\lim_{x\to\infty}E(N^-(x))=\int_0^\infty[m_1-m_1(u)+m_1(-u)]\,du<\infty. \tag{46}$$

We have stated the conditions on $N^-(x)$ as these are usually easier to verify (e.g. by (46)) than the property of convergence in probability actually needed in the proof.

The second moment conditions at (36) and (39) are also stronger than need be. The properties actually used were that $(N^*(x)-\lambda x)/x^{\frac{1}{2}}\xrightarrow{p}0$ and the uniform boundedness as $x\to\infty$ of $E(|N^*(x)-\lambda x|/x^{\frac{1}{2}})$.

Observe that case (i) of the theorem is a special case of (ii) (viz., by taking $F(u)=0$ or 1 as $u<$ or >0).

The theorem is equally true for $\tilde{N}_0(x)$ on replacing $N^*(.)$ by $N_0^*(.)$, etc. The only difference to be noted is that in the remark above concerning $E(N^-(x))$, we should have instead

$$E(N_0^-(x)) = \int_0^x [m_1 - m_1(x-u) + m_1(-u)]\, dU(u)$$
$$= [m_1 - m_1(x) + m_1(-x)]\, U(x) + \int_{-x}^0 U(-y)\, dm_1(y)$$
$$+ \int_0^x [U(x) - U(x-y)]\, dm_1(y).$$

Then by (19) and (18) it follows that convergence of $E(N^-(x))/x^\gamma$ for any given $\gamma \geqq 0$ implies that $\limsup_{x\to\infty} E(N_0^-(x))/x^\gamma < \infty$, and in particular convergence to zero implies that the lim sup = 0.

Asymptotic distributions for $N(x)$ and $N_0(x)$ follow from the theorem by using the decompositions at (22) and (24). We find easily that $E(N^=(x)) = E(N^-(x))$, so the assumptions sufficient for the conclusions about $\tilde{N}(.)$ suffice also for $N(.)$. A little more algebra shows that

$$E(N_0^=(x)) = \int_{-\infty}^{-x} [U(x-y) - U(-y)]\, dm_1(y) + \int_x^\infty [U(y) - U(y-x)]\, dm_1(y)$$
$$+ \int_{-x}^0 [U(x-y) - U(x)]\, dm_1(y) + \int_0^x [1 + U(y)]\, dm_1(y),$$

so by (19) and (18) it follows as above that convergence to zero of $E(N_0^-(x))/x^\gamma$ for any given $\gamma \geqq 0$ implies that $E(N_0^=(x))/x^\gamma$ also converges to zero. Thus,

Corollary. *Under any of the conditions of the theorem, all or none of* $N(x)$, $N_0(x)$, $\tilde{N}(x)$, $\tilde{N}_0(x)$ *have the same limiting distribution with a given pair of norming functions.*

Similar techniques can be used to deduce the asymptotic joint distribution of $N^*(.)$ and (say) $N(.)$. For example, under the conditions of part (ii) of the theorem, it is not difficult to show that as $x \to \infty$,

$$\mathrm{pr}\{N(x) - \lambda m_1 x \leqq \sigma(\lambda x)^{\frac{1}{2}} u,\ N^*(x) - \lambda x \leqq x^{\frac{1}{2}} v\} \to \int_{-\infty}^{v} \Phi\big(u - m_1 y/(\sigma \lambda^{\frac{1}{2}})\big)\, dF(y). \quad (47)$$

5. Second Moment Properties

In the statistical analysis of a stationary point process $N^*(.)$ much interest centres on there being a finite limit as $x \to \infty$ of $\mathrm{var}(N^*(x))/x$ (see Cox and Lewis, 1966, Section 4.5). In the case of the stationary cluster process $N(.)$ we have the following result.

Theorem 4. *If*

$$\mathrm{var}(N^*(x)) \sim \lambda_2 x \qquad (x \to \infty) \quad (48)$$

for some finite constant λ_2 *and if* $m_2 < \infty$, *then*

$$\mathrm{var}(N(x)) \sim [m_1^2 \lambda_2 + \lambda(m_2 - m_1^2)]\, x \qquad (x \to \infty). \quad (49)$$

Before proving this result observe that it can be written as

$$\operatorname{var}(N(x)) \sim [E(n(R))]^2 \operatorname{var}(N^*(x)) + \operatorname{var}(n(R))\, E(N^*(x)) = \operatorname{var}(N^+(x)), \tag{50}$$

in which form it is consistent with the normalization used at (41) when the distribution function $F(.)$ there has finite second moment.

Proof. From the definition at (2), the stationarity and orderliness of $N^*(.)$ and the consequent interpretation

$$\lambda\, du\, dU(v) = \operatorname{pr}\{N^*(u, u+du] = 1, N^*(u+v, u+v+dv] = 1\} + o(du\, dv),$$

$$\begin{aligned} E(N^2(x)) = \lambda \int_{-\infty}^{\infty} E(n^2(-u, -u+x])\, du \\ + 2\lambda \int_{-\infty}^{\infty} du \int_0^{\infty} E(n_1(-u, -u+x]\, n_2(-u-v, x-u-v])\, dU(v) \end{aligned} \tag{51}$$

where $n_1(.)$ and $n_2(.)$ denote independent subsidiary processes. Now using the non-negativity of $n(.)$ and Fubini's theorem

$$\begin{aligned} \int_{-\infty}^{\infty} E(n^2(-u, -u+x])\, du &= E\left(\int_{-\infty}^{\infty} du \int_{-u}^{x-u} n(dv) \int_{-u}^{x-u} n(dw) \right) \\ &= E\left(\int_{-\infty}^{\infty} n(dv) \int_{v-x}^{v+x} (x - |w-v|)\, n(dw) \right), \end{aligned}$$

so

$$\begin{aligned} \int_{-\infty}^{\infty} E(n^2(-u, x-u])\, du/x &= E\left(\int_{-\infty}^{\infty} n(dv) \int_{-\infty}^{\infty} (1 - |w-v|/x)^+\, n(dw) \right) \\ &\to E(n^2(R)) = m_2 \qquad (x \to \infty) \end{aligned}$$

by dominated convergence. Thus we have explained the term $\lambda m_2 x$ in (49). For the rest of (51), the independence of $n_1(.)$ and $n_2(.)$ yields

$$\begin{aligned} &2\lambda \int_{-\infty}^{\infty} du \int_0^{\infty} E(n_1(-u, x-u]\, n_2(-u-v, x-u-v])\, dU(v) \\ &= 2\lambda \int_{-\infty}^{\infty} du \int_0^{\infty} dU(v) \int_{-u}^{x-u} dm_1(y) \int_{-u-v}^{x-u-v} dm_1(z) \\ &= 2\lambda \int_{-\infty}^{\infty} dm_1(y) \left(\int_{-\infty}^{y} dm_1(z) \int_{-y}^{x-y} du + \int_{y}^{x+y} dm_1(z) \int_{-y}^{x-z} du \right) \int_{(-u-z)^+}^{x-u-z} dU(v) \end{aligned} \tag{52}$$

after some interchanges (note that all integrands and measures are non-negative, so Fubini's theorem applies here, as above). We now use Eq. (17) and set $V^*(x) = \operatorname{var}(N^*(x))$ in writing

$$\begin{aligned} &2\lambda \int_{-y}^{x-y} [U(x-u-z) - U((-u-z)^+)]\, du \\ &= 2\lambda \int_{y-z}^{x+y-z} U(v)\, dv - 2\lambda \int_{y-z-x}^{y-z} U(v^+)\, dv \\ &= V^*(y-z+x) - 2V^*(y-z) + V^*((y-z-x)^+) \\ &\quad - \lambda \left(\int_{y-z}^{y-z+x} - \int_{(y-z-x)^+}^{y-z} \right) (1 - 2\lambda u)\, du. \end{aligned}$$

The other term in (52) can be treated similarly and enables us to write the second term in (51) in the form

$$\int_{-\infty}^{\infty} dm_1(y) \int_{-\infty}^{x+y} dm_1(z) \{V^*(y-z+x)-2V^*((y-z)^+)+V^*((y-z-x)^+)$$
$$-\lambda(x-|z-y|)^+ + \lambda^2[(x+y-z)^2-2((y-z)^+)^2+((y-z-x)^+)^2]\}$$
$$= \int_{-\infty}^{\infty} dm_1(z) \int_{z-x}^{\infty} dm_1(y) \ \{\text{integrand as above}\}$$
$$=\tfrac{1}{2} \int_{-\infty}^{\infty} dm_1(y) \int_{-\infty}^{\infty} dm_1(z) \{V^*(y-z+x)-2V^*(y-z)+V^*(y-z-x)$$
$$-2\lambda(x-|y-z|)^+ + 2\lambda^2 x^2\},$$

where as usual $V^*(-x)=V^*(x)\,(x>0)$. The term in x^2 equals $\lambda^2 m_1^2 x^2 = (E(N(x)))^2$, so finally we have

$$\operatorname{var}(N(x)) = \lambda \int_{-\infty}^{\infty} E(n^2(-u, x-u])\,du + \int_{-\infty}^{\infty} dm_1(y) \int_{-\infty}^{\infty} dm_1(z) \{-\lambda(x-|y-z|)^+ + [V^*(y-z+x)-2V^*(y-z)+V^*(y-z-x)]\}. \tag{53}$$

The term involving $V^*(.)$ here equals $\operatorname{cov}(N^*(0,x], N^*(y-z, y-z+x])$ which has modulus $\leq V^*(x)$ for all y and z. Thus the dominated convergence theorem can be applied along with (48) in asserting that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} dm_1(y) \int_{-\infty}^{\infty} dm_1(z) \{-\lambda(1-|y-z|/x)^+$$
$$+[V^*(y-z+x)-2V^*(y-z)+V^*(y-z-x)]/2x\}$$
$$= \int_{-\infty}^{\infty} dm_1(y) \int_{-\infty}^{\infty} dm_1(z) \{-\lambda + \tfrac{1}{2}[\lambda_2 - 0 + \lambda_2]\}$$
$$= m_1^2(-\lambda+\lambda_2).$$

The theorem is proved.

We note that Vere-Jones (1970) has given the covariance density form of Eq. (53) in his Eq. (17).

The same method as above can be used to find an expression for $\operatorname{var}(\tilde{N}(x))$, but the limits of integration in the analogue even of the integrals at (52) are more complicated. Work of Lewis (1971) on the case where $N^*(.)$ is renewal process indicates that the statement for $\tilde{N}(x)$ analogous to Theorem 4 may involve conditions on the rate of convergence to zero of $m_1 - m_1(x) + m_1(-x)$ (cf. Eq. (46)).

Expressions for $\operatorname{var}(N_0(x))$ and $\operatorname{var}(\tilde{N}_0(x))$ involve joint probabilities of a higher order than $U(.)$ as above (51) (cf. the discussion in Lewis, 1971 when $N_0^*(.)$ is a renewal process with a density function).

6. Almost Sure Convergence

The assumption of non-negativity of the subsidiary processes $n_j(.)$ affords a fairly simple proof of the following result.

Theorem 5. *When the primary process $N^*(.)$ is ergodic and $m_1 < \infty$, $N(x)/x$ and $\tilde{N}(x)/x$ converge almost surely to λm_1 as $x \to \infty$.*

Proof. It is simplest to prove first that

$$\lim_{x \to \infty} \tilde{N}(x)/x = \lambda m_1 \qquad \text{a.s.} \tag{54}$$

Observe first from (20) and the non-negativity of $n(.)$ that

$$\tilde{N}(x) = N^+(x) - N^-(x) \leqq N^+(x) = \sum_{j=1}^{N^*(x)} n_j(R),$$

so

$$\tilde{N}(x)/x \leqq \left(\sum_{j=1}^{N^*(x)} n_j(R)/N^*(x) \right) \left(N^*(x)/(\lambda x) \right) \lambda. \tag{55}$$

With probability one, as $x \to \infty$, the second bracket converges to 1 by the ergodicity assumption at (14), and since then $N^*(x) \to \infty$, the first bracket is the sample mean of $N^*(x)$ independent identically distributed random variables with finite mean m_1. Thus the strong law of large numbers applies to show that the first bracket converges to m_1, and hence

$$\limsup_{x \to \infty} \tilde{N}(x)/x \leqq \lambda m_1 \qquad \text{a.s.} \tag{56}$$

On the other hand, since $E(n(R)) = m_1 < \infty$ and $n(.)$ is nonnegative, for any $\varepsilon > 0$ there exists an interval $(-a, b]$ for which $E(n(-a, b]) > m_1 - \varepsilon$, where without loss of generality we can assume that $a \geqq 0$, $b > 0$. Then taking $x > a + b$,

$$\begin{aligned} \tilde{N}(x) &= \sum_{t_j \in (0, x]} n_j(-t_j, x - t_j] \\ &\geqq \sum_{t_j \in (a, x-b]} n_j(-t_j, x - t_j] \\ &\geqq \sum_{t_j \in (a, x-b]} n_j(-a, b], \end{aligned}$$

so by the earlier argument,

$$\begin{aligned} \liminf_{x \to \infty} \tilde{N}(x)/x &\geqq \liminf_{x \to \infty} \sum_{t_j \in (a, x-b]} n_j(-a, b]/x \\ &= \lambda E(n(-a, b]) \qquad \text{a.s.} \\ &> \lambda (m_1 - \varepsilon). \end{aligned} \tag{57}$$

The inequalities at (56) and (57) imply (54).

From (22), $N(x) = \tilde{N}(x) + N^=(x)$ and $N^=(x) \geqq 0$, so from (53) it follows that

$$\liminf_{x \to \infty} N(x)/x \geqq \lambda m_1 \qquad \text{a.s.} \tag{58}$$

But the non-negative stochastic process $\{N(t, t+1]: -\infty < t < \infty\}$ is stationary with $E(N(0, 1]) = \lambda m_1 < \infty$, so by the ergodic theorem for strictly stationary processes, the limit

$$\lim_{x \to \infty} N(x)/x = \xi$$

exists almost surely for each sample function, with the random variable ξ having $E(\xi)=\lambda m_1$. This result, in conjunction with (58), implies that $\xi=\lambda m_1$ a.s., and proves the theorem.

If instead of $N^*(.)$ we have $N_0^*(.)$, the same arguments can be adapted to show that $\tilde{N}_0(x)/x$ and $N_0(x)/x \to \lambda m_1$ a.s. $(x\to\infty)$. These arguments are simpler than those used by Brown and Ross (1969) to prove similar a.s. convergence properties for a particular cluster process, namely the output of a transient $M/G/\infty$ queue. Brown and Ross (1971) have also studied a cluster process with $\{n_j(.)\}$ not necessarily independent of $N^*(.)$.

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