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On the Growth of Stochastic Integrals*

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Introduction

Let $\beta(t)$ be a d-dimensional Brownian motion and suppose that $\sigma(t,\omega)$ is a bounded, measurable, real $d\times d$-matrix valued function which is independent of future increments of β. Then, following Itô [3], we know how to define the stochastic integral

$$\xi(t)=\int_0^t \sigma(s)\,d\beta(s);$$

and, following McKean [5], we can derive the bound

$$P\Big(\sup_{0\leqq t\leqq T}|\xi(t)|\geqq\varepsilon\Big)\leqq 2d\exp\big[-\varepsilon^2/2d^{\frac{1}{2}}AT\big]\text{[1]},\tag{1}$$

where

$$A=\sup_{\theta\in R^d}\sup_{\substack{0\leqq t\leqq T\\ \omega}}\frac{|\sigma^*(t,\theta)|^2}{|\theta|^2}\text{[2]}.$$

Furthermore, if $d=1$, then, once again following McKean, we can construct a one-dimensional Brownian motion $B(t)$ such that

$$\xi(t)=B\left(\int_0^t \sigma^2(s)\,ds\right).\tag{2}$$

From equation (2), we see that if $T'=\sup_\omega \int_0^T \sigma^2(s,\omega)\,ds$, then

$$P\Big(\sup_{0\leqq t\leqq T}|\xi(t)|\geqq\varepsilon\Big)\leqq P\Big(\sup_{0\leqq t\leqq T'}|B(t)|\geqq\varepsilon\Big).\tag{3}$$

The inequality in (3) leads to a much sharper estimate than that provided by (1) (see [2], p. 288, for a derivation of the exact analytic expression for the right side of (3)). In particular, (3) tells us that when $d=1$, $P\big(\sup_{0\leqq t\leqq T}|\xi(t)|\geqq\varepsilon\big)<1$ for all positive T and ε. Moreover, both (1) and (3) suggest that the rate at which $|\xi(t)|$ grows is governed by the largest eigenvalue of $\sigma\sigma^*$. The purpose of this note is to point out that it is not so much the upper eigenvalue of $\sigma\sigma^*$ as the ratio

$$\frac{(Tr\,\sigma\sigma^*)\,|\xi|}{|\sigma^*\xi|}$$

which determines whether $P\big(\sup_{0\leqq t\leqq T}|\xi(t)|\geqq\varepsilon\big)$ is always less than unity.

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[1] A derivation of this estimate is given on p. 22 of [5].

[2] We use σ^* to denote the transpose of σ.

Notation and Elementary Facts

The notation is the same as that used in [6], namely: $\Omega = C([0,\infty), R^d)$, $x(t,\omega) = x_t(\omega)$ is the position of the trajectory $\omega \in \Omega$ at time t, $\mathfrak{M}_t = \mathfrak{B}[x_u: 0 \leqq u \leqq t]$, and $\mathfrak{M} = \mathfrak{B}[x_u: u \geqq 0]$. A mapping η of $[0,\infty) \times \Omega$ into a measurable space is said to be *non-anticipating* if it is $\mathfrak{B}_{[0,\infty)} \times \mathfrak{M}$-measurable and if $\eta(t)$ is $\mathfrak{M}_t$-measurable for all $t \geqq 0$. If P is a probability measure on $\langle \Omega, \mathfrak{M} \rangle$, then η is called a *P-martingale* if it is a complex-valued non-anticipating function satisfying

$$E^P[\eta(t_2) | \mathfrak{M}_{t_1}] = \eta(t_1) \qquad \text{(a.s., } P)$$

for each $0 \leqq t_1 \leqq t_2$. A *P-Brownian motion* is a continuous R^d-valued non-anticipating function β such that

$$\exp\left[\langle \theta, \beta(t) - \beta(0) \rangle - \frac{|\theta|^2}{2} t\right]$$

is a P-martingale for all $\theta \in R^d$. If θ is a bounded R^d-valued non-anticipating function and β is a P-Brownian motion, then

$$\exp\left[\int_0^t \langle \theta(s), d\beta(s) \rangle - \tfrac{1}{2} \int_0^t |\theta(s)|^2 \, ds\right]$$

is a P-martingale, where the stochastic integral

$$\int_0^t \langle \theta(s), d\beta(s) \rangle$$

is defined in the sense of Itô (cf. [3] or [5]). Finally, if ξ is a continuous, real-valued, non-anticipating function such that

$$\exp\left[\lambda(\xi(t \vee t_0) - \xi(t_0)) - \frac{\lambda^2}{2} \int_{t_0}^{t \vee t_0} \alpha(s) \, ds\right]$$

is a P-martingale for all $\lambda \in R$, where α is a bounded, non-negative, non-anticipating function, then (after enlarging the sample space) we can construct a one-dimensional Brownian motion B such that

$$\xi(t) - \xi(t_0) = B\left(\int_{t_0}^t \alpha(s) \, ds\right) \qquad \text{(a.s., } P).$$

For the detail of this construction see p. 29 of [5].

The Main Result

Let $\beta(t)$ be a P-Brownian motion and let σ be a bounded $d \times d$-matrix valued non-anticipating function. Set

$$\xi(t) = \int_0^t \sigma(s) \, d\beta(s)$$

and

$$p(T, \varepsilon) = P\left(\sup_{0 \leqq t \leqq T} |\xi(t)| \geqq \varepsilon\right).$$

Theorem. *If*

$$\sup_{\substack{s\geqq 0\\ \omega}} \frac{(Tr\,\sigma\,\sigma^*(s,\omega))\,|\xi(s,\omega)|}{|\sigma^*(s,\omega)\,\xi(s,\omega)|}<\infty,$$

then $p(T,\varepsilon)<1$ *for all positive* T *and* ε.

Proof. Suppose there exists $T_0>0$ and $\varepsilon_0>0$ such that $p(T_0,\varepsilon_0)$ equals one. Let

$$\tau=\inf\{t\geqq 0: |\xi(t)|\geqq \varepsilon_0/2\}$$

and

$$\tau'=\inf\{t\geqq \tau: \big||\xi(t)|-|\xi(\tau)|\big|\geqq \varepsilon_0/4\}.$$

Then $0\leqq\tau\leqq\tau'\leqq T_0$ (a.s., P). Clearly we may assume, without loss of generality, that $\sigma(s)=0$ for $|\xi(s)|\geqq 2\varepsilon_0$.

Now let $\eta(t)=|\xi(t)|^2$. Then, by Itô's formula (cf. p. 32 of [5]),

$$\eta(t)=2\int_0^t \langle\sigma^*(s)\,\xi(s), d\beta(s)\rangle+\int_0^t Tr\,\sigma\,\sigma^*(s)\,ds.$$

Take

$$b(t)=\mathfrak{X}_{|\xi(t)|\geqq\varepsilon_0/4}\,\frac{Tr\,\sigma\,\sigma^*(t)}{2\,|\sigma^*(t)\,\xi(t)|}\,\sigma^*(t)\,\xi(t)$$

and

$$\theta(t)=2\lambda\,\sigma^*(t)\,\xi(t)+b(t).$$

Then we see that

$$\exp\left[\lambda\Big(\eta(t)-\int_0^t \mathfrak{X}_{|\xi(s)|\leqq\varepsilon_0/4}\,Tr\,\sigma\,\sigma^*(s)\,ds\Big)-2\lambda^2\int_0^t|\sigma^*(s)\,\xi(s)|^2\,ds\right]$$

is a Q-martingale for all $\lambda\in R$, where

$$\frac{dQ|_{\mathfrak{M}_t}}{dP|_{\mathfrak{M}_t}}=R(t),\qquad t\geqq 0,$$

and

$$R(t)=\exp\left[\int_0^t\langle b(s), d\beta(s)\rangle-\tfrac{1}{2}\int_0^t|b(s)|^2\,ds\right].$$

Hence, if $\omega\to Q_\omega$ is a regular conditional probability distribution of Q given $\mathfrak{M}_\tau$ (cf. [6]), then for Q-almost every ω

$$\exp\left[\lambda\Big(\eta(t\vee\tau(\omega))-\eta(\tau(\omega))-\int_{\tau(\omega)}^{t\vee\tau(\omega)}\mathfrak{X}_{|\xi(s)|\leqq\varepsilon_0/4}\,Tr\,\sigma\,\sigma^*(s)\,ds\Big)\right.$$
$$\left.-\frac{\lambda^2}{2}\int_{\tau(\omega)}^{t\vee\tau(\omega)}|\sigma^*(s)\,\xi(s)|^2\,ds\right]$$

is a Q_ω-matingale. In particular,

$$Q(\tau'\leqq T_0|\mathfrak{M}_\tau)\leqq Q_\omega\Big(\sup_{\tau(\omega)\leqq t\leqq T_0}|\eta(t)-\eta(\tau(\omega))|\geqq\frac{3\varepsilon_0^2}{16}\Big)$$
$$=Q_\omega\Big(\sup_{\tau(\omega)\leqq t\leqq T}\Big|\eta(t)-\eta(\tau(\omega))-\int_{\tau(\omega)}^t\mathfrak{X}_{|\xi(s)|\leqq\varepsilon_0/4}\,Tr\,\sigma\,\sigma^*(s)\,ds\Big|\geqq\frac{3\varepsilon_0^2}{16}\Big)$$
$$<1$$

since

$$\eta(t)-\eta(\tau(\omega))-\int_{\tau(\omega)}^{t}\mathfrak{X}_{|\xi(s)|\leqq\varepsilon_0/4}\, Tr\,\sigma\,\sigma^*(s)\,ds$$

is distributed under Q_ω like a one-dimensional Brownian motion run with the clock

$$4\int_0^t|\sigma^*(s)\,\xi(s)|^2\,ds.$$

This proves that $Q(\tau'\leqq T_0)<1$. But P and Q are equivalent on $\mathfrak{M}_{T_0}$, and therefore $P(\tau'\leqq T_0)<1$, which is a contradiction. Q.E.D.

Remark. The preceding theorem shows that if $\sigma^{-1}(s,\omega)$ exists and is uniformly bounded, then $p(T,\varepsilon)<1$ for all $T,\varepsilon>0$.

An Example. We will now give an example which shows that the conditions given in our theorem are in some sense necessary. Assume $d>1$ and let $\bar{\sigma}(x)$ be an infinitely differentiable, non-negative definite, symmetric $d\times d$-matrix valued extention of the function $\sigma(x)$ which is defined on $|x|\geqq\frac{1}{2}$ by

$$\sigma^{ij}(x)=\delta_{ij}-\frac{x_i\,x_j}{|x|^2}.$$

Take $x^0=(1,0,\ldots,0)$ and

$$\bar{\xi}(t)=x^0+\int_0^t\bar{\sigma}(\bar{\xi}(s))\,d\beta(s),\qquad t\geqq 0,$$

where β is a P-Brownian motion. Then, by Itô's formula,

$$\begin{aligned}|\bar{\xi}(t)|^2&=|x^0|^2+2\int_0^t\langle\bar{\sigma}(\bar{\xi}(s))\,\bar{\xi}(s),d\beta(s)\rangle+(d-1)\,t\\&=|x^0|^2+(d-1)\,t,\end{aligned}$$

since $\bar{\sigma}(x)\,x=0$ for $|x|\geqq\frac{1}{2}$. Thus if $\xi(t)=\bar{\xi}(t)-x^0$ and $\sigma(s)=\bar{\sigma}(\bar{\xi}(s))$, then

$$\xi(t)=\int_0^t\sigma(s)\,d\beta(s)$$

and for each $\varepsilon>0$, $p(T,\varepsilon)=1$ for large enough T.

A Property of Stochastic Differentials

The preceeding example displays an interesting property of stochastic differentials. At first sight one might expect that the process $\bar{\xi}(t)$ should live on the sphere $S^{d-1}(|x^0|)$. However, as we have just seen, $\bar{\xi}(t)$ feels a "centrifugal drift" which pushes it off the sphere. The origin of this "drift" is, of course, the fact that $d\beta(s)$ is not a true differential in that "$(d\beta_i(s))^2=ds$". In [7] it is shown that if $\sigma(x)$ is a smooth $d\,x\,d$-matrix valued function and if

$$\xi(t)=x+\int_0^t\sigma(\xi(s))\,d\beta(s),\qquad t\geqq 0,$$

then the process $\xi(t)$ experiences a drift $1/2(\sigma'\sigma)(x)$, where

$$(\sigma'\sigma)_i(x)=\sum_{j,k=1}^{d}\frac{\partial\sigma^{ij}(x)}{\partial x_k}\,\sigma^{kj}(x).$$

(A similar drift term arises naturally in the work of Fichera [1] (also see [4]) on degenerate elliptic boundary value problems.)

For the above example

$$(\bar\sigma'\bar\sigma)(x) = -\frac{(d-1)\,x}{|x|^2}, \qquad |x| \geqq \tfrac{1}{2}.$$

If we now let

$$\bar{\bar\xi}(t) = x^0 + \int_0^t \bar\sigma(\bar{\bar\xi}(s))\,d\beta(s) + \tfrac{1}{2}\int_0^t (\bar\sigma'\bar\sigma)(\bar{\bar\xi}(s))\,ds, \qquad t \geqq 0,$$

we obtain a process which does live on $S^{d-1}(|x^0|)$. In fact, an application of Itô's formula shows that

$$\begin{aligned} |\bar{\bar\xi}(t)|^2 &= |x^0|^2 + 2\int_0^t \langle \bar\sigma(\bar{\bar\xi}(s))\,\bar{\bar\xi}(s), d\beta(s)\rangle + \tfrac{1}{2}\int_0^t (\bar\sigma'\bar\sigma)(\bar{\bar\xi}(s))\,ds \\ &\quad + \tfrac{1}{2}\int_0^t \mathit{Tr}\,\sigma^2(\bar{\bar\xi}(s))\,ds \\ &= |x^0|. \end{aligned}$$

Moreover, as K. Itô pointed out to me, the process $\bar{\bar\xi}(t)$ is Brownian motion on $S^{d-1}(|x^0|)$. That is, the generator of $\bar{\bar\xi}(t)$ is the Laplace-Beltrami operator on $S^{d-1}(|x^0|)$. Finally, it is easy to show that $\bar{\bar\xi}(t)$ doesn't escape from balls in a finite time. To see this, observe that

$$|\bar{\bar\xi}(t) - x^0|^2 = 2\int_0^t \langle \bar\sigma(\bar{\bar\xi}(s))(\bar{\bar\xi}(s) - x^0), d\beta(s)\rangle + \frac{(d-1)}{|x^0|^2}\int_0^t \langle x^0, \bar{\bar\xi}(s)\rangle\,ds.$$

Since $\bar{\bar\xi}(t)$ is always on $S^{d-1}(|x^0|)$, for each $0<\alpha<1$ there is a $\delta>0$ such that $|\bar\sigma(\bar{\bar\xi}(s))(\bar{\bar\xi}(s)-x^0)| \geqq \alpha\,|\bar{\bar\xi}(s)-x^0|$ whenever $|\bar{\bar\xi}(s)-x^0|<\delta$. Thus, by an argument like the one used to prove our theorem, we can show that

$$P\big(\sup_{0\leqq t\leqq T} |\bar{\bar\xi}(t) - x^0| \geqq \varepsilon\big) < 1$$

for all $T, \varepsilon > 0$.

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