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A Contribution to the Theory of Large Deviations for Sums of Independent Random Variables

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1. Introduction

Let the random variables X_i, $i = 1, 2, 3, \ldots$ be independent and identically distributed. Write $S_n = \sum_{i=1}^{n} X_i$ and $Z_n = S_n B_n^{-1} - A_n$ for normed and centered sums. The classical central limit formulation is for $\Pr(Z_n \leqq x_n)$ where $x_n = O(1)$ as $n \to \infty$ and thus gives only trivial information in the case where $x_n \to \infty$ as $n \to \infty$. However, we often require information on $\Pr(Z_n > x_n)$ under these circumstances. This type of problem is called a problem on the probability of large deviations and little in the way of comprehensive general results on such problems has so far been obtained. For references to the known results and also to various fields in which large deviation problems arise see, for example, Linnik [3] and Sethuraman [7].

Up till the work of Linnik [3], all the general large deviation results obtained had made use of the condition $E\{\exp(a|X_i|)\} < \infty$ for some $a > 0$ (that is, the X_i possess an analytic characteristic function) or something even stronger. The analytic characteristic function assumption was quite basic in much of the work as the results were obtained, in effect, from a use of the saddlepoint method in the theory of functions of a complex variable. Linnik broke away from the saddlepoint technique and the associated assumption of an analytic characteristic function. He has, however, only studied the problem of convergence to the normal distribution and, with the exception of one theorem, only cases where $x_n = o(\sqrt{n})$ as $n \to \infty$. In this paper we study, for the first time, large deviation problems associated with convergence to non-normal stable laws. A general expression is found for the order of magnitude of the large deviation probability in such cases.

2. Large Deviations in the Case of Attraction to a Non-normal Stable Law

Let $\{X_i, i = 1, 2, 3, \ldots\}$ be a sequence of independent and identically distributed random variables with law $\mathscr{L}(X)$ which belong, with normalizing constants B_n, to the domain of attraction of the stable law with characteristic function

$$\exp\left\{-a\,|t|^{\alpha}\left(1 + i\frac{t}{|t|}\,\frac{c_1 - c_2}{c_1 + c_2}\tan\frac{\pi\alpha}{2}\right)\right\},$$

$0 < \alpha < 2$, $\alpha \neq 1$, $a > 0$, $c_1 > 0$, $c_2 > 0$, and distribution function F. c_1, c_2 and a

are related according to

$$a = (c_1 + c_2)\,\alpha \int_0^\infty \frac{1 - \cos x}{x^{1+\alpha}}\, dx$$

(GNEDENKO and KOLMOGOROV [1], 169—171. We note in passing that an α is missing from the second and third terms on the right hand side of equation 12, 168 of [1]. This omission is perpetuated in the sequel). As a matter of convenience we shall exclude the case where $c_1(c_2)$ is zero although the problem is readily amenable to the same sort of treatment. Under these circumstances the stable law would be bounded on the left (right) (LUKACS [5], 106). If $\alpha > 1$ we suppose the X_i to be relocated so that $E X_i = 0$. Write $Z_n = B_n^{-1} \sum_{i=1}^{n} X_i = B_n^{-1} S_n$ and let $F_n(u) = \Pr(Z_n \leqq u)$. If $\alpha = 1$, the probability of interest cannot generally be put into the form $\Pr(Z_n > x_n)$ (GNEDENKO and KOLMOGOROV [1], 175). The problem in this case can be treated but it needs some special discussion and will be omitted here.

Under the above conditions we may write for $x > 0$, using GNEDENKO and KOLMOGOROV [1], Theorem 2, 175,

$$\Pr(X \leqq -x) = \frac{L_1(x)}{x^\alpha},$$

$$\Pr(X > x) = \frac{L(x)}{x^\alpha},$$

where $L_1(x)$ and $L(x)$ are slowly varying functions with

$$\frac{L_1(x)}{L(x)} \to \frac{c_1}{c_2} \quad \text{as} \quad x \to \infty .$$

$L(x)$ and the sequence B_n are related in such a way that

$$\frac{n L(B_n)}{B_n^\alpha} \to c_2 \quad \text{as} \quad n \to \infty .$$

We shall write $L_1(x) + L(x) = M(x)$ so that $M(x)$ is a slowly varying function, $M(x) \sim \left(1 + \frac{c_1}{c_2}\right) L(x)$ as $x \to \infty$ and

$$q(x) = \Pr(X \leqq -x) + \Pr(X > x) = \frac{M(x)}{x^\alpha}.$$

In the following discussion we shall restrict consideration to large deviations in the right hand tail. It is clear that corresponding results will hold for the left hand tail. We shall obtain the following theorem.

Theorem. *Let $\{x_n\}$ be a sequence with $x_n \to \infty$ as $n \to \infty$. Then,*

$$0 < \varliminf_{n\to\infty} \frac{x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(S_n > x_n B_n) \leqq \varlimsup_{n\to\infty} \frac{x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(S_n > x_n B_n) < \infty .$$

Proof. Denote by A_i and B_i the events $\{X_i > x_n B_n\}$ and $\left\{ \sum_{\substack{j=1\\ j \neq i}}^{n} X_j > 0 \right\}$

respectively, $i = 1, 2, \ldots, n$. If $\bar{E}$ is the complement of E, we then have

$$\begin{aligned}
\Pr(S_n > x_n B_n) &\geqq \Pr\left[\bigcup_{i=1}^{n}(A_i \cap B_i)\right] \\
&= \Pr\left[\bigcup_{i=1}^{n}\left\{\bigcap_{j=1}^{i-1}(\overline{A_j \cap B_j}) \cap (A_i \cap B_i)\right\}\right] \\
&= \sum_{i=1}^{n}\Pr\left[\bigcap_{j=1}^{i-1}(\overline{A_j \cap B_j}) \cap (A_i \cap B_i)\right] \\
&\geqq \sum_{i=1}^{n}\Pr\left[\bigcap_{j=1}^{i-1}\bar{A}_j \cap (A_i \cap B_i)\right] \geqq \\
&\geqq \sum_{i=1}^{n}\left\{\Pr(A_i \cap B_i) - \Pr\left(\bigcup_{j=1}^{i-1}A_j \cap A_i\right)\right\} \geqq \\
&\geqq \sum_{i=1}^{n}\Pr(A_i)\,[\Pr(B_i) - (i-1)\Pr(A_i)] \geqq \\
&\geqq n\Pr(A_i)\,[\Pr(B_i) - n\Pr(A_i)]\,.
\end{aligned} \tag{1}$$

Now $\Pr(B_i) \to A > 0$ as $n \to \infty$ where A is the probability that the corresponding stable law takes a positive value. Thus, given $\delta > 0$ with $A - 2\delta > 0$ we can choose N_1 so large that $\Pr(B_i) > A - \delta$ for $n \geqq N_1$. Also

$$n\Pr(A_i) = n\Pr(X > x_n B_n) = \frac{n L(x_n B_n)}{x_n^\alpha B_n^\alpha} = \frac{n L(B_n)}{B_n^\alpha}\,\frac{L(x_n B_n)}{x_n^\alpha L(B_n)}$$

and since $L(x)$ is a non-negative function of slow variation, it possesses a representation of the form

$$L(x) = \frac{a(x)}{x}\exp\left\{\int_1^x \frac{a(u)}{u}\,du\right\},$$

where $a(x) \to 1$ as $x \to \infty$ (Karamata [2]). Therefore,

$$\frac{L(x_n B_n)}{L(B_n)} = \frac{a(x_n B_n)}{a(B_n)}\,\frac{1}{x_n}\exp\left\{\int_{B_n}^{x_n B_n}\frac{a(u)}{u}\,du\right\},$$

and given arbitrarily small $\eta > 0$ we can choose N so large that for $n > N$,

$$\frac{L(x_n B_n)}{L(B_n)} < (1+\eta)\,\frac{1}{x_n}\exp\left\{(1+\eta)\int_{B_n}^{x_n B_n}\frac{du}{u}\right\} = (1+\eta)\,x_n^\eta\,.$$

Take $\eta < \alpha$. Then, for $n > N$,

$$n\Pr(A_i) < \frac{n L(B_n)}{B_n^\alpha}(1+\eta)\,x_n^{\eta-\alpha} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Thus, given $\delta > 0$ we can choose N_2 so large that $n\Pr(A_i) < \delta$ for $n \geqq N_2$ and hence, for $n \geqq \max(N_1, N_2)$, we have from (1)

$$\begin{aligned}
\Pr(S_n > x_n B_n) &\geqq n(A - 2\delta)\Pr(X > x_n B_n) = n(A - 2\delta)\,\frac{L(x_n B_n)}{x_n^\alpha B_n^\alpha} \\
&= (A - 2\delta)\,\frac{n L(B_n)}{B_n^\alpha}\,\frac{L(x_n B_n)}{x_n^\alpha L(B_n)}
\end{aligned}$$

so that

$$\frac{x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(S_n > x_n B_n) > (A - 2\delta) \frac{n L(B_n)}{B_n^\alpha}$$

and

$$\varliminf_{n\to\infty} \frac{x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(S_n > x_n B_n) > 0 .$$

In order to complete the proof we shall work in terms of the symmetrized random variables X_i^s, $i = 1, 2, 3, \ldots$, making continued use of the weak symmetrization inequalities (LOÈVE [4], 245) to transfer the results.

Define

$$X_{kn}^s = \begin{cases} X_k^s & \text{if } \quad |X_k^s| \leqq x_n B_n \\ 0 & \text{otherwise,} \end{cases}$$

and write $S_n^s = \sum_{k=1}^{n} X_k^s$, $S_{nn}^s = \sum_{k=1}^{n} X_{kn}^s$. We have

(2) $$\Pr(|S_n^s| > x_n B_n) \leqq n \Pr(|X^s| > x_n B_n) + \Pr(|S_{nn}^s| > x_n B_n)$$

and, using the weak symmetrization inequalities,

$$\begin{aligned} \frac{n x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(|X^s| > x_n B_n) &\leqq 2 \frac{n x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr\left(|X| > \frac{x_n B_n}{2}\right) \leqq \\ &\leqq 2^{1+\alpha} \frac{n L(B_n)}{L(x_n B_n)} \frac{M(\frac{1}{2} x_n B_n)}{B_n^\alpha} \\ &= 2^{1+\alpha} \frac{n L(B_n)}{B_n^\alpha} \frac{M(\frac{1}{2} x_n B_n)}{M(x_n B_n)} \frac{M(x_n B_n)}{L(x_n B_n)}, \end{aligned}$$

so that

(3) $$\varlimsup_{n\to\infty} \frac{n x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(|X^s| > x_n B_n) < \infty .$$

We now turn our attention to $\Pr(|S_{nn}^s| > x_n B_n)$. From Markov's inequality we obtain

$$\begin{aligned} \Pr(|S_{nn}^s| > x_n B_n) &\leqq x_n^{-2} B_n^{-2} E(S_{nn}^s)^2 = n x_n^{-2} B_n^{-2} E(X_{kn}^s)^2 \\ &= n x_n^{-2} B_n^{-2} \int_{|x| \leqq x_n B_n} x^2 \, d \Pr(X^s \leqq x) \\ &= - n x_n^{-2} B_n^{-2} \int_0^{x_n B_n} x^2 \, d \Pr(|X^s| > x) \\ &\leqq 2 n x_n^{-2} B_n^{-2} \int_0^{x_n B_n} x \Pr(|X^s| > x) \, dx \leqq 4 n x_n^{-2} B_n^{-2} \int_0^{x_n B_n} x \, q(\tfrac{1}{2} x) \, dx \leqq \\ &\leqq 16 n x_n^{-2} B_n^{-2} \int_0^{\frac{1}{2} x_n B_n} M(x) \, x^{1-\alpha} \, dx \end{aligned}$$

using integration by parts to obtain the third last inequality and the weak symmetrization inequalities to obtain the second last. Thus,

$$\frac{x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(|S_{nn}^s| > x_n B_n) \leqq 16 \frac{n L(B_n)}{B_n^\alpha} \frac{1}{L(x_n B_n)(x_n B_n)^{2-\alpha}} \int_0^{\frac{1}{2} x_n B_n} M(x) \, x^{1-\alpha} \, dx$$

$$(4) \qquad = 16\,\frac{nL(B_n)}{B_n^\alpha}\,\frac{1}{L(x_nB_n)(x_nB_n)^{2-\alpha}}\int_0^A M(x)\,x^{1-\alpha}\,dx +$$

$$+\,16\,\frac{nL(B_n)}{B_n^\alpha}\,\frac{M(\frac{1}{2}x_nB_n)}{M(x_nB_n)}\,\frac{M(x_nB_n)}{L(x_nB_n)}\,\frac{1}{M(\frac{1}{2}x_nB_n)(x_nB_n)^{2-\alpha}}\int_A^{\frac{1}{2}x_nB_n} M(x)\,x^{1-\alpha}\,dx$$

for fixed A. The first term on the right hand side becomes arbitrarily small for sufficiently large n since L is a slowly varying function and $\alpha < 2$. Further, making use of the representation result for non-negative slowly varying functions, we have for $A \leqq x \leqq \frac{1}{2}x_nB_n$,

$$\frac{M(x)}{M(\frac{1}{2}x_nB_n)} = \frac{b(x)}{b(\frac{1}{2}x_nB_n)}\,\frac{x_nB_n}{2x}\exp\left\{-\int_x^{\frac{1}{2}x_nB_n}\frac{b(u)}{u}\,du\right\}$$

where $b(u) \to 1$ as $u \to \infty$, and given $0 < \delta < 2 - \alpha$ we can choose A so large that

$$\frac{M(x)}{M(\frac{1}{2}x_nB_n)} < (1+\delta)\,\frac{x_nB_n}{2x}\exp\left\{-(1-\delta)\int_x^{\frac{1}{2}x_nB_n}\frac{du}{u}\right\} = (1+\delta)\left(\frac{x_nB_n}{2x}\right)^{\delta}.$$

Therefore, for suitably large A,

$$\frac{1}{M(\frac{1}{2}x_nB_n)(x_nB_n)^{2-\alpha}}\int_A^{\frac{1}{2}x_nB_n} M(x)\,x^{1-\alpha}\,dx < \frac{1+\delta}{2^\delta(x_nB_n)^{2-\alpha-\delta}}\int_A^{\frac{1}{2}x_nB_n} x^{1-\alpha-\delta}\,dx <$$

$$< \frac{1}{2^{2-\alpha}}\,\frac{1+\delta}{2-\alpha-\delta} < \infty,$$

and it follows from (4) that

$$(5) \qquad \overline{\lim_{n\to\infty}}\,\frac{x_n^\alpha L(B_n)}{L(x_nB_n)}\,\Pr(|S_{nn}^s| > x_nB_n) < \infty.$$

Using (3) and (5) in (2) we obtain

$$\overline{\lim_{n\to\infty}}\,\frac{x_n^\alpha L(B_n)}{L(x_nB_n)}\,\Pr(|S_n^s| > x_nB_n) < \infty$$

which, using the weak symmetrization inequalities, implies in particular that

$$(6) \qquad \overline{\lim_{n\to\infty}}\,\frac{x_n^\alpha L(B_n)}{L(x_nB_n)}\,\Pr(S_n - \operatorname{med} S_n > x_nB_n) < \infty.$$

Now $B_n^{-1}S_n$ converges in distribution so $x_n^{-1}B_n^{-1}S_n$ converges in probability to zero. Hence, $x_n^{-1}B_n^{-1}\operatorname{med} S_n \to 0$ as $n \to \infty$. It follows then from (6) by a simple transformation that

$$(7) \qquad \overline{\lim_{n\to\infty}}\,\frac{x_n^\alpha L(B_n)}{L(x_nB_n)}\,\Pr(S_n > x_nB_n) < \infty$$

and the proof of the theorem is complete.

In large deviation problems it is usual to compare the large deviation behaviour of the normed sums with that of the corresponding stable law. In this context, that amounts to looking at

$$\frac{1-F_n(x_n)}{1-F(x_n)} = \frac{x_n^\alpha}{c_2 + \alpha_2(x_n)}\,\Pr(S_n > x_nB_n)$$

where $\alpha_2(x) \to 0$ as $x \to \infty$ ([1], Theorem 5, 181—182), and we shall have

$$0 < \varliminf_{n\to\infty} \frac{1-F_n(x_n)}{1-F(x_n)} \leqq \varlimsup_{n\to\infty} \frac{1-F_n(x_n)}{1-F(x_n)} < \infty$$

if and only if

$$0 < \varliminf_{n\to\infty} \frac{L(B_n)}{L(x_n B_n)} \leqq \varlimsup_{n\to\infty} \frac{L(B_n)}{L(x_n B_n)} < \infty .$$

This is the case, for example, when the random variables $\{X_i, i = 1, 2, 3, \ldots\}$ actually belong to the domain of normal attraction of the particular stable law. In that case,

$$L(x) = c_2 + \beta_2(x)$$

where $\beta_2(x) \to 0$ as $x \to \infty$. In general, however, different distributions belonging to the same domain of attraction can possess quite different asymptotic behaviour.

It seems plausible that in the theorem the result

$$\varliminf_{n\to\infty} \frac{x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(S_n > x_n B_n) = \varlimsup_{n\to\infty} \frac{x_n^\alpha L(B_n)}{L(x_n B_n)} \Pr(S_n > x_n B_n)$$

should hold. In this connection it should be mentioned that McLAREN [6] has shown, using combinatorial arguments rather like those of LINNIK [3], 302, that in the case of normal attraction,

$$\lim_{n\to\infty} \frac{1-F_n(x_n)}{1-F(x_n)} = 1$$

provided $\frac{x_n}{n^{1+\delta}} \to \infty$ as $n \to \infty$ for some $\delta > 0$.

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