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Markov Solutions of Stochastic Differential Equations

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1. Introduction

Let (W_t) be the Wiener process. Then it is well known that for Lipschitz continuous f, g, the stochastic differential equation of K. Itô

$$(1.1)\quad X_t = X_0 + \int_0^t f(s, X_s)\, dW_s + \int_0^t g(s, X_s)\, ds$$

has a unique solution which is a Markov process with continuous paths. Moreover if f and g in (1.1) satisfy $f(t,x)=f(x)$, $g(t,x)=g(x)$ (i.e., they are *autonomous*) then X is a time homogeneous strong Markov process (cf., e.g. [10]). Kunita and Watanabe [14], Doléans-Dade [7], and Meyer [7, 16] have developed a martingale integral which includes Itô's integral for the Wiener process. C. Doléans-Dade [9] and the author [17, 18] have shown that unique solutions exist for equations of the form

$$(1.2)\quad X_t = X_0 + \int_0^t f(s, X_{s-})\, dZ_s + \int_0^t g(s, X_{s-})\, dA_s$$

where f and g are (say) jointly continuous and Lipschitz in the space variable, and Z and A are semimartingales. In this paper we will be interested in the cases where Z is a Markov process and a semimartingale, and A_t is an additive functional of Z. This allows one to consider models in which the (random) driving term is not white noise but, for example, only has stationary, independent increments (and so may have jumps), or is simply a Markov (e.g., Hunt) process. In this paper we determine the nature of the Markov properties which solutions of (1.2) have.

If $Z=(\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, P^z, Z_t)$ is a Hunt process [2, p. 45], in order for Equations (1.2) to be meaningful, Z must be a P^z semimartingale. If Z is a P^z semimartingale for every z, then a priori it may have a different decomposition for every z. In Section 3 we give a sufficient condition for Z to have a decomposition which holds for all P^z simultaneously. We call such a process a *universal semi-*

martingale. We show that if a universal semimartingale satisfies an additional condition then it remains a universal semimartingale after a change of time.

In Section 4 we prove several technical lemmas involving properties of solutions of equations of the form (1.2) as well as lemmas that arise from the Markov framework: the uncountable family of measures (P^z), $z \in \mathbb{R}$. We show that the solution X of (1.2) is independent of z.

In Section 5 we show that if $A_t = t$ and Z has independent increments and is a semimartingale then the solution X of (1.2) is strong Markov. Theorem (5.3) treats the cases in which Z either has independent increments alone or is a Lévy process. This result extends a classical result of Itô. Theorem (5.8) shows that if Z is merely assumed to be a Markov process then the vector process (X, Z) will be Markov, though X in (1.2) need not be Markov. Perhaps the most interesting situation is that treated in Theorem (5.9) which considers the case where Z is a Hunt process and f and g are autonomous. Then the vector process (X, Z) is a time-homogeneous strong Markov process with transition semigroup $P_t h(x, z) = E^{x,z}[h(X_t, Z_t)]$.

One can define a shift operator for the vector process (X, Z) of Theorem (5.9) and if one assumes the additive functional (A_s) in (1.2) is quasi-left-continuous then the process (X, Z) is a Hunt process. In Section 6 we calculate explicitly a Lévy system of (X, Z) in terms of the coefficients f, g, the jumps of (A_t), and a Lévy system of Z.

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2. Preliminaries

Although we assume that the reader is familiar with the stochastic integral for local martingales given in Kunita and Watanabe [14], Doléans-Dade [7], and Meyer [7, 16], we restate here some of the important definitions in this development. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a probability space where the filtration $(\mathscr{F}_t)$ is complete and right continuous. A right continuous adapted process M_t with left limits and $M_0 = 0$ is a *local martingale* if there exist stopping times T^n increasing to ∞ such that M^{T^n} is a uniformly integrable martingale for each n. The stopping times (T^n) are said to *reduce* M. We let $\mathscr{V}$ be the class of processes V which are adapted, $V_0 = 0$, and which have right continuous paths of bounded variation on compact intervals. We denote the total variation of a path between 0 and t by $|V(\omega)|_t = \int_0^t |dV_s(\omega)|$. A process Y is said to be a *semimartingale* (for a measure P) if it can be decomposed $Y_t = Y_0 + M_t + V_t$ where M is a local martingale and $V \in \mathscr{V}$. The above decomposition is not unique. However, there exists at most one decomposition in which the process $V \in \mathscr{V}$ is previsible [5] (also denoted "predictable"). Such a decomposition is called *canonical*, and semimartingales which have a canonical decomposition are called *special* (see [16, p. 310] for their properties). If a local martingale M is locally square integrable, there exists a unique increasing previsible process $\langle M, M \rangle_t \in \mathscr{V}$ such that $M_t^2 - \langle M, M \rangle_t$ is a local martingale and has value 0 at $t=0$. The process $\langle M, N\rangle$ is defined by polarization.

Equality of processes will mean indistinguishability relative to a measure. The letter Z will be reserved for Markov processes, and generally our notation is that of Blumenthal and Getoor [2]. If $Z=(\Omega, \mathscr{F}, \mathscr{F}_t, P^z, \theta_t, Z_t)$ is a Hunt process, equality of processes will mean indistinguishability for every measure P^μ. All of our Markov processes have state space $\mathbb{R}$, and $\mathscr{B}$, $\mathscr{B}^*$ denote, respectively, the Borel sets and the universally measurable sets on $\mathbb{R}$. We use the notation of Dellacherie for stochastic intervals: $[\![T,S)\!) = \{(t,\omega): T(\omega)\leqq t<S(\omega)\}$. For a process Y we let $Y_{t-}(\omega)=\lim_{\substack{s\to t\\ s<t}} Y_s(\omega)$, and we denote $\Delta Y_t(\omega)=Y_t(\omega)-Y_{t-}(\omega)$.

3. Universal Semimartingales

Throughout this section we will assume $Z=(\Omega, \mathscr{F}, \mathscr{F}_t, Z_t, \theta_t, P^z)$ is a Hunt process with state space $\mathbb{R}$ [2, p. 45]. We let $\mathscr{F}_t{}^0=\sigma(Z_s; s\leqq t)$ and $\mathscr{F}_t{}^\mu$ is the completion of the $\mathscr{F}_t{}^0$ under P^μ, where μ is a finite measure on $\mathscr{B}$, the Borel sets of $\mathbb{R}$. We let $\mathscr{F}_t=\bigcap_{\mu \text{ finite}} \mathscr{F}_t{}^\mu$.

(3.1) *Definition.* Let X be a process on Ω. Then

(i) X is a *complete semimartingale* if X is a P^z semimartingale for every z.

(ii) X is a *universal semimartingale* if it is a complete semimartingale, and if there is a decomposition

$$X_t=X_0+M_t+V_t$$

where M is a local martingale for every P^z, and $V_t\in\mathscr{V}$.

(iii) X is a *universally reducible semimartingale* if it is a universal semimartingale and if there exist stopping times (T^n) tending to ∞ which in a universal decomposition reduce the local martingale term for every P^z.

Suppose that X is a complete semimartingale. Define

(3.2) $J_t=\sum_{s\leqq t} \Delta X_s 1_{\{|\Delta X_s|\geqq 1\}}$,

(3.3) $Y_t=X_t-J_t$.

Then Y_t has bounded jumps, so the process $(\sum_{0<s\leqq t} \Delta Y_s^2)^{\frac{1}{2}}$ is locally integrable, and Y is hence a special semimartingale for every P^z [16, p. 310]. For a given P^z let

(3.4) $Y_t=Y_0+M_t^z+B_t^z$

be its canonical decomposition, where M_t^z is a P^z local martingale, and $B_t^z\in\mathscr{V}$ and is previsible.

(3.5) *Definition.* Let X be a process on Ω which is a complete semimartingale. Let J_t be as in (3.2), and Y_t as in (3.3). X is said to satisfy the *stopping condition* if there exists a sequence of stopping times (T^n) tending to ∞ which does not depend on z, and is such that for every z, $E^z(|B^z|_{T^n})<\infty$, where B^z is given in (3.4).

We now establish a useful lemma, which is an adaptation to our situation of one due to Doléans-Dade [9]. We will need a notion of previsibility on the space $(\Omega, \mathscr{F}, \mathscr{F}_t, P^z)$, and results due to Sharpe [19]. We let $\mathscr{N}^\mu = P^\mu$—evanescent processes (i.e., $Y \in \mathscr{N}^\mu$ if $P^\mu\{\exists\, t: Y_t \neq 0\}=0$). Let $\mathscr{N} = \bigcap \mathscr{N}^\mu$ be the *evanescent processes.* Let $\mathscr{P}$, the *previsible σ-algebra*, be the σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by $\mathscr{N}$ and the family of processes Y adapted to $\mathscr{F}_t$ and such that $t \to Y_t(\omega)$ is left continuous with right limits. If $\mathscr{P}^\mu$ denotes the previsible σ-algebra on $\mathbb{R}_+ \times \Omega$ relative to $(\Omega, \mathscr{F}^\mu, \mathscr{F}_t^\mu, P^\mu)$ for some finite μ, then clearly $\mathscr{P} \subseteqq \bigcap \mathscr{P}^\mu$.

(3.6) **Lemma.** *Let M be a universally reducible local martingale for all P^z, and $\beta>0$. Then M can be decomposed into $M=N+B$, where N is a local martingale for all P^z and $B\in\mathscr{V}$. There exists a null set Λ* (i.e., $P^z(\Lambda)=0$ *for all z*) *such that $|\Delta N_s| \leqq \beta$ for all $s\in\mathbb{R}_+$, off Λ.*

Proof. Following the technique of Doléans-Dade, we define stopping times (S_n) by:

$$S_1 = \inf\{t: |\Delta M_t| \geqq \beta\}$$
$$\vdots$$
$$S_n = \inf\{t: t > S_{n-1}, |\Delta M_t| \geqq \beta\}.$$

Let $C_t = \sum_n \Delta M_{S_n} 1_{\{t \geqq S_n\}}$, and let T_k be stopping times increasing to ∞ which universally reduce M. Let $R_k = T_k \wedge S_k$. Then $C_{t\wedge R_k}$ is of integrable variation for all P^z. By the results of Sharpe [19], we know there exists a process $\tilde{C}^k \in \mathscr{P}$ such that $\tilde{C}^k$ is the dual previsible projection (also known as the "compensator") for each system $(\Omega, \mathscr{F}^\mu, \mathscr{F}_t^\mu, P^\mu)$. Thus $C_{t\wedge R_k} - \tilde{C}_t^k$ is a local martingale for each P^z. By the uniqueness of the dual previsible projection, one can define $\tilde{C}$ on $[\![0,\infty))$ by

$$\tilde{C} = \tilde{C}^k \quad \text{on } [\![0, R_k]\!].$$

Moreover since the filtration $(\mathscr{F}_t)$ comes from a Hunt process, $\tilde{C}$ has continuous paths a.s. P^μ, each μ. Then the local martingale $N+M-C+\tilde{C}$ satisfies

$$\begin{aligned}|\Delta N_s| &\leqq |\Delta M_s - \Delta C_s| + |\Delta \tilde{C}_s| \\ &= |\Delta M_s - \Delta C_s| \\ &\leqq \beta.\end{aligned}$$

Taking $B = C - \tilde{C}$ completes the proof.

(3.7) **Theorem.** *A complete semimartingale X satisfies the stopping condition if and only if it is universally reducible.*

Proof. Necessity. Suppose X satisfies the stopping condition. Let J_t be as in (3.2) and Y_t be as in (3.3). Since X is a complete semimartingale it has "cadlag" paths (continue à droite, limites à gauche), and so $J_t \in \mathscr{V}$. We wish to show Y_t is universal. Choose K' large, let $R=\inf\{t: |Y_t| > K'\}$, and let $K=K'+1$. Then $|Y_t^R| \leqq K$. Let (T^n) be the stopping times tending to ∞ such that $E^z(B^z_{T^n})<\infty$, where

$B_t^z=\int_0^t |dA_s^z|$, and $Y_t=Y_0+M_t^z+A_t^z$ is the canonical decomposition. Let $S=S^n=R\wedge T^n$, so that S^n tends to ∞, and $M_{t\wedge S}^z$ is a local martingale such that $E^z(|M_{t\wedge S}^z|)<\infty$. Let Y be implicitly stopped at S. Then for $G\in b\mathscr{F}_s$, we have

$$E^z[G(Y_{t\wedge U_k}-Y_{s\wedge U_k})]=E^z[G(A_{t\wedge U^k}^z-A_{s\wedge U^k}^z)],$$

where the stopping times U^k tending to ∞ reduce $M_{t\wedge S}^z$. Taking limits as $k\to\infty$ yields

$$E^z[G(Y_t-Y_s)]=E^z[G(A_t^z-A_s^z)].$$

We conclude that $z\to E^z[G(A_t^z-A_s^z)]$ is $\mathscr{B}^*$ measurable, for $G\in b\mathscr{F}_s$. A monotone class argument shows that $z\to E^z\int_0^\infty H_s\,dA_s^z$ is $\mathscr{B}^*$ measurable for any bounded previsible integrand H.

We now use again the results of Sharpe [19]. We will show the existence of a process $A\in\mathscr{P}\cap\mathscr{V}$ which is the dual previsible projection of A_t^z for each system $(\Omega,\mathscr{F}^z,\mathscr{F}_t^z,P^z)$. Since the proofs of these results are omitted in [19], we will provide those details which are needed here. Let $G\in b\mathscr{F}^o$, and $H'=G\otimes 1_{[0,t]}=G(\omega)\,1_{[0,t]}^{(s)}$. Then by Sharpe [19] there exists a process $H_t={}^3H_t'$ such that H_t is previsible and P^μ-indistinguishable from H', for all μ. Let $\mathscr{F}^{oo}=\sigma\{f(Z_s);\ s\geqq 0,\ f\in\mathscr{B}\}$ and $\mathscr{F}^{o*}=\sigma\{f(Z_s);\ s\geqq 0,\ f\in\mathscr{B}^*\}$. We define a kernel Q_t^z on $\mathscr{F}^{oo}$ by $Q_t^z(G)=\mu^z(G\otimes 1_{[0,t]})=\mu^z(H_t)$, where $\mu^z(C)=E^z\int_0^\infty C_s\,dA_s^z$. Then $Q_t^z\ll P^z$, and Q_t^z is a finite kernel, since Y (and hence A^z) is stopped at S. Since $\mathscr{F}^{oo}$ is separable, we may use Doob's lemma [15, p. 154] and conclude that there exists $A_t'(z,\omega)\in\mathscr{B}^*\otimes\mathscr{F}^{oo}$ such that $Q_t^z(d\omega)=A_t'(z,\omega)\,P^z(d\omega)$ as (signed) measures on $\mathscr{F}^{oo}$. Since $Z_0\in\mathscr{F}_0^{o*}/\mathscr{B}^*$, we have $A_t'(\omega)=A_t'(Z_0(\omega),\omega)\in\mathscr{F}^{o*}$ and $A_t'(\omega)\doteq A_t'(z,\omega)$ a.s. P^z, for all z.

If $G\in b\mathscr{F}$, then given z we can find $G_1\leqq G\leqq G_2$, such that $G_i\in b\mathscr{F}^{oo}$ and $E^z(G_2-G_1)=0$. This allows us to show that $E^z(A_t'G)=\mu^z(G\otimes 1_{[0,t]})$ for $G\in b\mathscr{F}$. Define

$$A_t(\omega)=\lim A_s'(\omega)$$

as s decreases to t through $\mathbf{Q}$. By the right continuity of each A_t^z, we have $A_t(\omega)=A_t'(\omega)$ a.s. P^z, each z. For each z, A_t is P^z indistinguishable from A_t^z; hence

$$Y_t=M_t^z+A_t^z=M_t^z+A_t$$

and $Y_t-A_t=M_t^z$ implies M_t^z does not depend on z. Because the canonical decomposition is unique, we can define A_t on $[\![0,\infty))$ by defining it on each stochastic interval $[\![0,S]\!]$. We let A_t be equal on $[\![0,S]\!]$ to the process we obtain for $[\![0,S]\!]$.

Sufficiency. Suppose X is universally reducible. Let

$$X_t=X_0+M_t+V_t$$

be a universally reducible decomposition. Then by Lemma (3.6) we can write

$X_t = X_0 + N_t + C_t + V_t$ where N and C are complete local martingales such that $|\Delta N_s| \leqq 1$ and $C \in \mathscr{V}$. Let $J_t = \sum_{s \leqq t} \Delta X_s 1_{\{|\Delta X_s| \geqq 1\}}$, and

$$Y_t = X_0 + N_t + (C_t - J_t).$$

Then Y_t has bounded jumps and so is special for every P^z, and $|\Delta(C-J)_t| \leqq |\Delta Y_t| + |\Delta N_t| \leqq 2$. Thus $C-J$ is universally locally of integrable variation, which completes the proof.

We call a *change of time* a family $(\tau_t)_{t \geqq 0}$ of stopping times of the filtration $(\mathscr{F}_t)$, such that for each $\omega \in \Omega$ the function $\tau_.(\omega)$ is right continuous, non-decreasing, and *finite*. Kazamaki [13] has shown that semimartingales are preserved under changes of time. If X_t is a universal semimartingale and (τ_t) is a change of time, then X_{τ_t} is a complete semimartingale, but it is not clear that it is still universal. We do have however the following:

(3.8) **Theorem.** *If X is a universally reducible semimartingale, and τ_t is a change of time, then (X_{τ_t}) is a universally reducible semimartingale for the system $(\Omega, \mathscr{F}, \mathscr{F}_{\tau_t})$.*

Proof. Let $X_t = X_0 + M_t + B_t$ be a universal decomposition and let (T^n) be stopping times increasing to ∞ which universally reduce the local martingale M. Let $\hat{X}_t = X_{\tau_t}$, $\hat{\mathscr{F}}_t = \mathscr{F}_{\tau_t}$, etc. Then

$$\hat{X}_t = \hat{X}_0 + \hat{M}_t + \hat{B}_t.$$

It is easy to see that since $B \in \mathscr{V}$, so also $\hat{B} \in \mathscr{V}$. Thus it suffices to show that $\hat{M}_t$, which is a complete semimartingale, is also a universally reducible semimartingale. Let

$$\bar{J}_t = \sum_{s \leqq t} \Delta \hat{M}_s 1_{\{|\Delta \hat{M}_s| \geqq 1\}}.$$

Then $\bar{J} \in \mathscr{V}$, since $\hat{M}$ is a complete semimartingale. Denote

$$M_t^n = M_{t \wedge T^n},$$
$$\bar{J}_t^n = \sum_{s \leqq t} \Delta \hat{M}_s^n 1_{\{|\Delta \hat{M}_s^n| \geqq 1\}}.$$

It is easy to see that $\hat{M}_t^n$ is a uniformly integrable $\hat{\mathscr{F}}_t$ martingale. Let

$$S_1 = \inf\{s: |\Delta \hat{M}_s^n| \geqq 1\}$$
$$\vdots$$
$$S_{k+1} = \inf\{s: s > S_k; |\Delta \hat{M}_s^n| \geqq 1\}.$$

Then S_k are $\hat{\mathscr{F}}_t$ stopping times which increase to ∞ and satisfy $E^z(|\bar{J}^n|_{S_k}) < \infty$ for every z. Thus $\bar{J}^n$ is universally of locally integrable variation. Let

$$\hat{T}^n = \inf\{u: \tau_u \geqq T^n\}.$$

It is easy to check that $\hat{T}^n$ is an $\hat{\mathscr{F}}_t$ stopping time. Let

$$\hat{N}_t = \hat{M}_t - \bar{J}_t.$$

Then $\hat{N}$ has bounded jumps and so is a special semimartingale for every P^z.

Note that $\hat{M}_t = \hat{M}_t^n$ on $\{t < \hat{T}^n\}$, so $\bar{J}_t^n = \bar{J}_t$ on $\{t < \hat{T}^n\}$. Moreover since

$$\hat{M}_{t \wedge \hat{T}^n} = \hat{M}_t^n + (\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n}^n) 1_{\{t \geqq \hat{T}^n\}},$$

we have

$$(3.9) \quad \hat{N}_{t \wedge \hat{T}^n} = \hat{M}_t^n - \bar{J}_t^n 1_{\{t < \hat{T}^n\}} + [(\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n}^n) - (\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n -}^n) 1_A] 1_{\{t \geqq \hat{T}^n\}}$$

where $A = \{|\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n -}^n| \geqq 1\}$. Then

$$\begin{aligned}(3.10) \quad &|(\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n}^n) - (\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n -}^n) 1_A| \\ &= |(\hat{M}_{\hat{T}^n -}^n - \hat{M}_{\hat{T}^n}^n) + (\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n -}^n) - (\hat{M}_{\hat{T}^n} - \hat{M}_{\hat{T}^n -}^n) 1_A| \\ &\leqq |\Delta \hat{M}_{\hat{T}^n}^n| + 1.\end{aligned}$$

Rewrite (3.9) as

$$\hat{N}_{t \wedge \hat{T}^n} = \hat{M}_t^n + \bar{F}_t^n.$$

Then $\bar{J}_t^n$ is universally locally integrable, so (3.10) implies that $\bar{F}_t^n$ is universally locally integrable. Hence using Sharpe's results [19] we know $\tilde{\bar{F}}_t^n \in \mathscr{P} \cap \mathscr{V}$ exists and $\bar{F}_t^n - \tilde{\bar{F}}_t^n$ is a local martingale for every P^μ. Hence

$$(3.11) \quad \hat{N}_{t \wedge \hat{T}^n} = (\hat{M}_t^n + \bar{F}_t^n - \tilde{\bar{F}}_t^n) + \tilde{\bar{F}}_t^n$$

is a semimartingale and the decomposition (3.11) is the canonical one. By the uniqueness of the canonical decomposition, the decomposition (3.11) on $[\![0, \hat{T}^n]\!]$ agrees with the analogous one on $[\![0, \hat{T}^{n+m}]\!]$ for any $m > 0$ when the latter is restricted to $[\![0, \hat{T}^n]\!]$. We thus achieve a decomposition on $[\![0, \infty)\!)$. Hence $\hat{N}$ is a universal semimartingale, so $\hat{M} = \hat{N} + \bar{J}$ is one also, and so $\hat{X}$ is a universal semimartingale, and the result is established.

(3.12) *Example.* Suppose the Hunt process Z of this section is actually a Lévy process; that is, a process with stationary, independent increments. Let J_t and Y_t be as given in (3.2) and (3.3) respectively. Then Y_t is again a Lévy process and Y_t has finite mean. We have

$$Z_t = Z_0 + [Y_t - E^{Z_0}(Y_t)] + [J_t + E^{Z_0}(Y_t)]$$

where for each P^z the first term in brackets is a martingale and the second term is in $\mathscr{V}$, since the function $t \to E^{Z_0}(Y_t)$ is a.s. (P^z) an affine function. Thus a Lévy process is a universally reducible semimartingale, a fact pointed out, for example, in Doléans-Dade and Meyer [7].

Let C_t be a continuous additive functional [2, p. 148] of the Lévy process Z such that if

$$\tau_t = \inf\{s \colon C_s > t\}$$

then $\tau_t < \infty$ for each t. The process (τ_t) is a change of time. We denote

$$X_t = Z_{\tau_t}.$$

Then X_t is a strong Markov process [2, p. 212], where all of the obvious objects are time-changed. Since Z is universally reducible, by Theorem (3.8) X is a universally reducible semimartingale.

4. The Markov Framework

Let $Z=(\Omega_1, \mathscr{F}, \mathscr{F}_t, Z_t, \theta_t, P^z)$ be a Hunt process with state space $\mathbb{R}$. We let $\mathscr{F}_t^o=\sigma(Z_s;\, s\leqq t)$ and $\mathscr{F}_t^\mu$ is the completion of $\mathscr{F}_t^o$ under P^μ. Let $\mathscr{F}_t=\bigcap \mathscr{F}_t^\mu$, where the intersection is over all finite μ on $\mathscr{B}$, the Borel sets of $\mathbb{R}$. In order to allow more general initial conditions, we will need to consider a larger space. Define

$$
\begin{aligned}
(4.1) \qquad \Omega &= \mathbb{R}\times\Omega_1\\
\mathscr{G}_t^o &= \mathscr{B}\otimes\mathscr{F}_t^o\\
P^{x,z} &= \varepsilon_x \times P^z
\end{aligned}
$$

where ε_x is point mass at x. Let $\omega=(x,\omega_1)$ denote a point in Ω and define

$$(4.2) \quad X_0(\omega)=x, \qquad \text{when } \omega=(x,\omega_1).$$

We let $\mathscr{G}_t=\bigcap \mathscr{G}_t^\nu$, where the intersection is over all finite ν on $\mathscr{B}\otimes\mathscr{B}$. We assume $\mathscr{G}=\bigvee_t \mathscr{G}_t$. The process Z is defined on Ω_1, and we extend it to Ω by $Z(\omega)=Z(\omega_1)\,1_{\mathbb{R}}(x)$. By an *additive functional* of Z we will mean an adapted process A on Ω_1 which has right continuous paths of bounded variation such that $A_0=0$, and such that A satisfies for all s, t

$$A_{t+s}=A_t+A_s\circ\theta_t$$

with equality meaning P^z indistinguishability. We extend A to Ω by $A_t(\omega)=A_t(\omega_1)$ when $\omega=(x,\omega_1)$.

The process Z extended to Ω is still a Hunt process, and the σ-fields $\mathscr{G}_t$ are right continuous. Z is a semimartingale on Ω_1 for P^z if and only if it is a semimartingale on Ω for $P^{x,z}$ for all x. Throughout this section Z will be a Hunt process and X_0 will be the random variable described in (4.2).

Suppose the Hunt process Z is a complete semimartingale (i.e., Z is a semimartingale for every $P^{x,z}$). Let f and g map $\mathbb{R}_+\times\mathbb{R}$ to $\mathbb{R}$ and satisfy for all $t\in\mathbb{R}_+$ and $x, y\in\mathbb{R}$:

(4.3) (i) $|f(t,x)-f(t,y)|+|g(t,x)-g(t,y)|\leqq K|x-y|$,

(ii) f and g are left continuous in t with finite right limits.

Then it is known [9, 17, 18] that a unique solution exists for each $P^{x,z}$ of

$$(4.4) \quad X_t^{x,z}=X_0+\int_0^t f(s, X_{s-}^{x,z})\,dZ_s+\int_0^t g(s, X_{s-}^{x,z})\,dA_s$$

which, a priori, depends on $P^{x,z}$. The process (A_t) in (4.4) can be taken to be any complete semimartingale, but we will only be interested here in the case where A_t is an additive functional of Z.

Doléans-Dade [8] has shown that if Y is a complete semimartingale (for the family $P^{x,z}$) and H is a locally bounded previsible integrand, then there exists a version of $\int_0^t H_s\,dY_s$ which does not depend on $P^{x,z}$. We will use a similar technique to show that $X_t^{x,z}$ in (4.4) does not depend on $P^{x,z}$. We first establish a useful result.

(4.5) **Lemma.** *Let* $\eta_t^0 = W_0$, *and*

$$(4.6)\quad \eta_t^{n+1} = W_0 + \sum_{i=1}^{m} \int_0^t f_i(s, \eta_{s-}^n)\,dY_s^i$$

where Y^i *are P-semimartingales, and each* f_i *satisfies* (4.3). *Let* W_t *be a solution of*

$$W_t = W_0 + \sum_{i=1}^{m} \int_0^t f_i(s, W_{s-})\,dY_s^i.$$

Then $\eta_t^n \to W_t$ *in probability.*

Proof. We only treat the case $m=1$. Suppose first that Y is a VS semimartingale; that is, $Y_t = Y_0 + M_t + B_t$, with M locally square integrable, and B previsible and locally of square integrable total variation (see [17, 18]). Then it follows from the proof of Theorem (3.1) in [18] that $\eta_t^n \to W_t$ in probability.

If Y has bounded jumps it is special (cf. [16]). Let $Y_t = Z_0 + M_t + B_t$ be its canonical decomposition. Select a t and $\varepsilon > 0$, and choose μ so large that $P\{S \leqq t\} < \varepsilon$, where

$$S = \inf\{s : |B|_s \geqq \mu\}.$$

Then S is previsible. Let (S_n) announce S, and choose v so large that $P\{T \leqq t\} < \varepsilon$, where

$$T = S_n \wedge \inf\{t : |M_s| \geqq v\}.$$

Then Y^T is very special. We have

$$\begin{aligned} P\{|\eta_t^n - W_t| > \delta\} &\leqq P(\{|\eta_{t\wedge T}^n - W_{t\wedge T}| > \delta\} \cap \{T \geqq t\}) + P(T < t) \\ &\leqq P\{|\eta_{t\wedge T}^n - W_{t\wedge T}| > \delta\} + \varepsilon. \end{aligned}$$

Thus $\eta_t^n \to W_t$ in probability when Y has bounded jumps.

For arbitrary Y, let $R = \inf\{s : |\Delta Y_s| \geqq \lambda\}$, where λ is chosen so large that $P\{R < t\} < \varepsilon$. Let $V_s = \bar{Y}_s^R = Y_s 1_{\{s<R\}} + Y_{R-} 1_{\{s \geqq R\}}$ so that V is a semimartingale with bounded jumps. Let $\gamma_t^o = W_0$, and

$$\begin{aligned} \gamma_t^{n+1} &= W_0 + \int_0^t f(s, \gamma_{s-}^n)\,dV_s \\ U_t &= W_0 + \int_0^t f(s, U_{s-})\,dV_s. \end{aligned}$$

It is easy to check that $\gamma_t^n = \eta_t^n$ and $U_t = W_t$ on $[\![0, R))$. We have

$$P\{|\eta_t^n - W_t| > \delta\} \leqq P(\{|\gamma_t^n - U_t| > \delta\} \cap \{R \geqq t\}) + P(R < t)$$
$$\leqq P\{|\gamma_t^n - U_t| > \delta\} + \varepsilon,$$

and $\lim_{n\to\infty} P\{|\gamma_t^n - U_t| > \delta\} = 0$, since $V = \bar{Y}^R$ has bounded jumps. This completes the proof of the lemma.

(4.7) **Theorem.** *Let f, g satisfy condition* (4.3). *Then there is a version of the solution $X_t^{x,z}$ in* (4.4) *which does not depend on $P^{x,z}$.*

Proof. Doléans-Dade [8] has shown that if previsible H is locally bounded and Y is a complete semimartingale, then there exists a version of the stochastic integral $\int_0^t H_s\, dY_s$ which does not depend on $P^{x,z}$. Induction shows that $(\eta_t^n)_{n\geqq 0}$ do not depend on (x, z). Moreover $\eta_t^n \to X_t^\mu$ in P^μ-probability for every P^μ. We now use the technique of Mokobodzki ("rapid" filters [5, p. 45]). Assuming the continuum hypothesis, Mokobodzki has shown the existence of a rapid filter $\mathscr{R}$ on $\mathbb{N}$ such that if one denotes

$$X_t = \liminf_{\mathscr{R}} \eta_t^n$$

then for each P^μ, $X_t \in \mathscr{G}_t^\mu$ and $X_t = X_t^\mu$ a.s. (P^μ). Thus $X_t \in \mathscr{G}_t$. We define X_s as above for rationals and let

$$X_t = \lim_{s\to t} X_s$$

for $s \in \mathbf{Q}$, $s > t$. Since X_t^μ has right continuous paths, X_t and X_t^μ are P^μ indistinguishable for each μ. Since for each n η_t^n does not depend on (x, z), we deduce X_t does not depend on (x, z). This completes the proof.

The next theorem makes use of Meyer's result on the local character of stochastic integrals [16, p. 308] to reveal an intuitively pleasing dependence of the solution on the (random) initial value.

(4.8) **Theorem.** *Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $(\mathscr{F}_t)$ satisfying the "usual conditions"* [5, p. 183]. *Let Y^i, $1 \leqq i \leqq k$ be semimartingales, $Y_0^i = 0$, and H, K be finite $\mathscr{F}_0$-measurable random variables. Let V, W respectively solve*

$$V_t = H + \sum_{i=1}^k \int_0^t f_i(s, V_{s-})\, dY_s^i,$$
$$W_t = K + \sum_{i=1}^k \int_0^t f_i(s, W_{s-})\, dY_s^i$$

where f_i satisfy conditions (4.3), $1 \leqq i \leqq k$. *Let $\Lambda = \{H = K\}$. Then* a.s. *on Λ, $t \to V_t(\omega)$ and $t \to W_t(\omega)$ agree.*

Proof. Let $\eta_t^0 = H$, $\mu_t^0 = K$, and

$$\eta_t^{n+1} = H + \sum_{i=1}^k \int_0^t f_i(s, \eta_{s-}^n)\, dY_s^i,$$
$$\mu_t^{n+1} = K + \sum_{i=1}^k \int_0^t f_i(s, \mu_{s-}^n)\, dY_s^i.$$

By the local character of the stochastic integral [16, p. 308] and a standard induction argument, $t\to\eta_t^n$ agrees with $t\to\mu_t^n$ a.s. on Λ. Since η_t^n and μ_t^n tend respectively in probability to V_t and W_t by Lemma (4.5), we have $V_t=W_t$ a.s. on Λ. Since V_t and W_t have right continuous paths, the result follows.

We are now in a position to record some trivial but useful relationships among the measures P^z and $P^{x,z}$. Let Z be the Hunt process and X_0 the random variable described at the beginning of this section. Let X_t^x and X_t respectively solve

$$(4.9)\quad X_t^x=x+\int_0^t f(s,X_{s-}^x)\,dZ_s+\int_0^t g(s,X_{s-}^x)\,dA_s,$$

$$(4.10)\quad X_t=X_0+\int_0^t f(s,X_{s-})\,dZ_s+\int_0^t g(s,X_{s-})\,dA_s$$

where f and g satisfy (4.3), Z is a Hunt process and a complete semimartingale and $A\in\mathscr{V}$.

(4.11) **Proposition.** *Let* (X_t^x) *and* (X_t) *be as in* (4.9) *and* (4.10). *Let* $H\in b\mathscr{F}$, *and* $\hat{H}(\omega)=H(\omega_1)\,1_{\mathbb{R}}(x)$, *where* $\omega=(x,\omega_1)$. *Then*

(i) X_t *and* X_t^x *are* $P^{x,z}$ *indistinguishable for all* z.

(ii) *For any* $f\in b\mathscr{B}\otimes\mathscr{B}$, $E^{x,z}[f(X_t,\hat{H})]=E^z[f(X_t^x,H)]$.

(iii) $E^{x,z}[\hat{H}\,|\,\mathscr{G}_t]=E^z[H\,|\,\mathscr{F}_t]\,1_{\mathbb{R}}$.

Proof. Part (i) is an application of Theorem (4.8). (No problems are caused by the lack of completeness of each $\mathscr{G}_t$; results are to be interpreted as "up to evanescence"). Part (ii) follows from part (i) and a monotone class argument, and part (iii) is clear, given part (ii).

5. Markov Solutions

A diffusion D_t can be defined as a strong Markov process with continuous paths. If one requires conditions on the conditioned increments so that they are approximately Gaussian, then one can express D_t as the solution of an Itô type stochastic differential equation

$$(5.1)\quad D_t=D_0+\int_0^t f(s,D_s)\,dW_s+\int_0^t g(s,D_s)\,ds$$

where W_t is the Wiener process. (See Gihman and Skorhod [10, p. 70].)

If one considers a model in which the continuity of the paths is not essential, one can consider Markov processes other than Brownian motion, and random measures, as differentials. Let f,g satisfy conditions (4.3), Z be (say) a Hunt process which is a complete semimartingale relative to $P^{x,z}$ (see Sections 3 and 4 for definitions) and A_t an additive functional of Z. Let X_0 be as in (4.2), and let X_t solve

$$(5.2)\quad X_t=X_0+\int_0^t f(s,X_{s-})\,dZ_s+\int_0^t g(s,X_{s-})\,dA_s.$$

Then one might hope that X_t would be a Markov process. This is not true in general, as simple examples show. (Use a Markov chain, so that X_t becomes the solution of a difference equation, and extend to continuous time.)

Processes with independent increments need not be semimartingales. Indeed, as is pointed out in [16, p. 298], if Z_t has independent increments,

$$Y_t = Z_t - \sum_{s \leqq t} \Delta Z_s 1_{\{|\Delta Z_s| \geqq 1\}},$$

then Z is a semimartingale if and only if the function $t \to E[Y_t]$ is of bounded variation. Using the notation established at the beginning of Section 4, we obtain the following extension of Itô's classical result:

(5.3) **Theorem.** *Let Z have independent increments, $Z_0 = 0$, and be a semimartingale. Let f and g satisfy conditions* (4.3). *Let X_0 be as in* (4.2) *and let X_t be a solution of*

$$(5.4) \quad X_t = X_0 + \int_0^t f(s, X_{s-})\, dZ_s + \int_0^t g(s, X_s)\, ds.$$

Then X_t is a strong Markov process.

If Z is a Lévy process and f and g are autonomous (i.e., $f(t,x) = f(x)$, $g(t,x) = g(x)$), *then X_t is a (time-homogeneous) strong Markov process, with its transition semigroup given by*

$$P_t h(x) = E^{x,0}[h(X_t)].$$

Proof. Let T be a stopping time, and $\mathscr{H}^T = \sigma\{Z_{T+u} - Z_T;\, u \geqq 0\}$. Then $\mathscr{H}^T$ is a σ-algebra on Ω_1 and $\mathscr{H}^T$ is independent of $\mathscr{F}_T$ under P^0 (cf. [3]). Let $\eta^0(x,s) = x$; define $X(x,t,s)$ and for $s > t$ inductively define $\eta^n(x,s)$ by

$$(5.5) \quad X(x,t,s) = x + \int_t^s f(u, X(x,t,u-))\, dZ_u + \int_t^s g(u, X(x,t,u-))\, du$$

$$\eta^{n+1}(x,s) = x + \int_t^s f(u, \eta^n(x,u-))\, dZ_u + \int_t^s g(u, \eta^n(x,u-))\, du.$$

Since η^n is a semimartingale it has cadlag paths; hence (as is easy to check)

$$\eta^{n+1}(x,s) = \lim \sum_{u_i \in \mathscr{P}^m} f(u_i, \eta^n(x,u_i))(Z_{u_{i+1}} - Z_{u_i}) + \sum_{u_i \in \mathscr{P}^m} g(u_i, \eta^n(x,u_i))(u_{i+1} - u_i)$$

where the convergence is in P^0-probability and the limit is taken as mesh $(\mathscr{P}^m) \to 0$, where $\mathscr{P}^m$ are partitions of $(t,s]$. An inductive argument shows $\eta^n \in \mathscr{H}^t$, and Lemma (4.5) shows $X(x,t,s) \in \mathscr{H}^t$. By the uniqueness of the solutions (see [9, 18]), one can show $X_S^x = X(X(x,0,T),T,S)$ for stopping times S, T with $S \geqq T$. If X_t is the solution of (5.4), we write $X_t = X(X_0, 0, t)$ and also $X_t^x = X(x,0,t)$. By the independence of $\mathscr{F}_T$ and $\mathscr{H}^T$ and using Proposition (4.11) we have for any $h \in b\mathscr{B}$ and stopping times $S \geqq T$,

$$\begin{aligned}(5.6) \quad E^{x,0}[h(X_S) | \mathscr{G}_T] &= E^0[h(X_S^x) | \mathscr{F}_T]\, 1_{\mathbb{R}} \\ &= E^0[h(X(X_T^x, T, S)) | \mathscr{F}_T]\, 1_{\mathbb{R}} \\ &= j(X_T^x)\, 1_{\mathbb{R}}\end{aligned}$$

where $j(y)=E^0[h(X(y,T,S))]=E^{y,0}[h(X(X_0,T,S))]$, and the last equality above is a consequence of an elementary lemma in Gihman and Skorohod [10, p. 67]. We finally observe that under $P^{x,0}$ we have

(5.7) $j(X_T^x)\,1_{\mathbb{R}}=j(X_T)$.

Suppose now that f and g are autonomous, Z is a Lévy process, and X_t is a solution of (5.4). It is well known that for a Lévy process Z, the process $Z_{T+s}-Z_T$ is identical in law to Z_s (cf. [3]). It is then easy to check that $X(x,T,T+u)$, $u\geqq 0$ is independent of $\mathscr{F}_T$ and is identical in law (under P^0) to X_u^x, $u\geqq 0$. By (5.6) and (5.7) we have

$$E^{x,0}[h(X_S)\,|\,\mathscr{G}_T]=j(X_T),$$

but in this case we have

$$\begin{aligned} j(y)&=E^0[h(X(y,T,S))]\\ &=E^0[h(X_{S-T}^y)]\\ &=E^{y,0}[h(X_{S-T})]\end{aligned}$$

where the second equality above is due to the identification in law of X_u^x and $X(x,T,T+u)$. This completes the proof of Theorem (5.3).

In Theorem (5.3) we assumed the differential Z had independent increments and were able to conclude the solution X of (5.4) was a strong Markov process. If we weaken the conditions on Z so that it is merely a strong Markov process, the solution need not be Markov. However, the vector process (X,Z) is a strong Markov process.

(5.8) **Theorem.** *Let $Z=(\Omega,\mathscr{M},\mathscr{M}_t,Z_t,P)$ be a (strong) Markov process and a semimartingale. Let f and g satisfy conditions* (4.1) *and let X_t be a solution of*

$$X_t=X_0+\int_0^t f(s,X_{s-})\,dZ_s+\int_0^t g(s,X_{s-})\,ds$$

where $X_0\in\mathscr{M}_0$. Then the vector process (X,Z) is (strong) Markov for $(\Omega,\mathscr{M},\mathscr{M}_t,P)$.

Proof. Let $X(x,t,s)$ and $\eta^{n+1}(x,s)$ be as given in (5.5). Then the results of Doléans [6] and an inductive argument show that $(x,t,\omega)\to\eta^n(x,t,\omega)$ is jointly measurable for each n. Since $\eta^n(x,t,\omega)\to X(x,t,\omega)$ in P-probability for each x by Lemma (4.5), $-K\vee(\eta^n\wedge K)$ converges in $\sigma(L^1,L^\infty)$ to $-K\vee(X\wedge K)$ for each K. An application of Doob's lemma [15, p. 154] yields that $(x,t,\omega)\to X(x,t,\omega)$ is jointly measurable. Indeed, this yields $X(x,t,\omega)\in\mathscr{B}\otimes\mathscr{H}^t$, where $\mathscr{B}$ is the Borel sets on $\mathbb{R}$ and $\mathscr{H}^t=\sigma\{Z_{t+u}-Z_u;u\geqq 0\}$. By the uniqueness of the solutions, one easily checks that for stopping times $S\geqq T$, $X_S=X(X_T,T,S)$. Let $h\in b\mathscr{B}$ and $K\in b\mathscr{H}^t$. Then

$$\begin{aligned} E\{h(X_t)\,K\,|\,\mathscr{M}_t\}&=h(X_t)\,E\{K\,|\,\mathscr{M}_t\}\\ &=h(X_t)\,E\{K\,|\,Z_t\}\\ &=j(X_t,Z_t).\end{aligned}$$

Therefore $E\{h(X_t)K|\mathcal{M}_t\}=E\{h(X_t)K|X_t,Z_t\}$. If Z is assumed to be strong Markov, the preceding holds for stopping times S,T. The theorem now follows by an application of the monotone class theorem.

We now state our main result. Observe that time changed Lévy processes such as those described in example (3.11) satisfy the conditions imposed on the process Z in the following theorem.

(5.9) **Theorem.** *Let Z be a Hunt process and a universally reducible semimartingale. Let A be an additive functional of Z. Let autonomous f and g satisfy conditions* (4.3), *X_0 be as given in* (4.2), *and let X_t be the solution of*

$$(5.10)\quad X_t=X_0+\int_0^t f(X_{s-})\,dZ_s+\int_0^t g(X_{s-})\,dA_s.$$

Then the vector process (X,Z) is strong Markov, with transition semigroup $P_t h(x,z)=E^{x,z}[h(X_t,Z_t)]$.

Before proving this result, we establish some notation and a lemma. For fixed u, let $\tilde{M}_t=M_t\circ\theta_u$ for a process M. Let $\tilde{\mathcal{F}}_t=\theta_u^{-1}(\mathcal{F}_t)$. Following Meyer, we let $C\cdot Y$ denote the stochastic integral $\int_0^t C_s\,dY_s$ for a semimartingale Y. The following lemma is used in the proof of Theorem (5.9).

(5.11) **Lemma.** *Let Y be a universally reducible semimartingale. Let C be a previsible integrand which is universally locally bounded. Then $\tilde{C}\cdot\tilde{Y}=\widetilde{C\cdot Y}$, for any fixed u.*

Proof. Let $Y_t=Y_0+M_t+B_t$ be a universal decomposition and (T^n) stopping times tending to ∞ such that M^{T^n} is a P^z martingale for each n. Implicitly stopping Y at T^n for some fixed n, by Lemma (3.6) we can write

$$(5.12)\quad M=N+B$$

where N is a (universally) locally bounded martingale, $B\in\mathcal{V}$, and $N_0=B_0=0$. Let $G\in\tilde{\mathcal{F}}_s$, where $G=H\circ\theta_u$, $H\in\mathcal{F}_s$. By stopping N if necessary we assume without loss of generality that N is bounded. Then

$$E^z[(\tilde{N}_t-\tilde{N}_s)G]=E^z[E^{Z_u}(N_t-N_s)H]=0,$$

consequently $\tilde{N}$ is an $\tilde{\mathcal{F}}_s$ martingale. If M is a square integrable $\mathcal{F}_t$ martingale we have

$$\begin{aligned}&E^z[(\tilde{M}_t\tilde{N}_t-\widetilde{\langle M,N\rangle}_t)G]\\&\quad=E^z[E^{Z_u}[(M_tN_t-\langle M,N\rangle_t)H]]\\&\quad=E^z[E^{Z_u}[(M_sN_s-\langle M,N\rangle_s)H]]\\&\quad=E^z[(\tilde{M}_s\tilde{N}_s-\widetilde{\langle M,N\rangle}_s)G]\end{aligned}$$

and if $\widetilde{\langle M,N\rangle}_t$ is $\tilde{\mathcal{F}}_t$-previsible, by the uniqueness of $\langle\cdot,\cdot\rangle$ we can conclude

$$(5.13)\quad \langle\tilde{M},\tilde{N}\rangle_t=\widetilde{\langle M,N\rangle}_t.$$

Let $\mathscr{P}(\mathscr{F}_t)$ denote the previsible σ-algebra for a filtration $(\mathscr{F}_t)$. Let $\mathscr{H}=\{Y\in b\mathscr{P}(\mathscr{F}_t): \tilde{Y}\in b\mathscr{P}(\tilde{\mathscr{F}}_t)\}$. Then $\mathscr{H}$ clearly contains the left-continuous and $\mathscr{F}_t$-adapted processes, and therefore a monotone class argument shows that shifting preserves previsibility. For a process $B\in\mathscr{V}$, the statement $\tilde{C}\cdot\tilde{B}=\widetilde{C\cdot B}$ is merely notation. For N locally bounded, using (5.13) we have

$$
\begin{aligned}
(5.14)\quad &\langle \tilde{C}\cdot\tilde{N}-\widetilde{C\cdot N}, \tilde{C}\cdot\tilde{N}-\widetilde{C\cdot N}\rangle\\
&=(\tilde{C})^2\cdot\langle\tilde{N},\tilde{N}\rangle-2\tilde{C}\cdot\langle\tilde{N},\widetilde{C\cdot N}\rangle+\langle\widetilde{C\cdot N},\widetilde{C\cdot N}\rangle\\
&=(\tilde{C})^2\cdot\langle\widetilde{N,N}\rangle-2\tilde{C}\cdot(\widetilde{C\cdot\langle N,N\rangle})+(\widetilde{C^2\cdot\langle N,N\rangle})\\
&=(\tilde{C})^2\cdot\langle\widetilde{N,N}\rangle-2\tilde{C}\cdot\tilde{C}\cdot\langle\widetilde{N,N}\rangle+(\widetilde{C^2})\cdot\langle\widetilde{N,N}\rangle\\
&=0.
\end{aligned}
$$

Since $\tilde{C}\cdot\tilde{N}_0-\widetilde{C\cdot N_0}=0$, (5.14) implies that $\tilde{C}\cdot\tilde{N}=\widetilde{C\cdot N}$. Using the decomposition (5.12) we have

$$\widetilde{C\cdot M}=\widetilde{C\cdot N}+\widetilde{C\cdot B}=\tilde{C}\cdot\tilde{N}+\tilde{C}\cdot\tilde{B}=\tilde{C}\cdot\tilde{M}$$

and the lemma is proved.

Proof of Theorem (5.9). We define $X(x,t,s)$ and inductively define $\mu^n(x,t,s)$ by $\mu^0(x,t,s)\equiv x$ and for $s>t$,

$$
\begin{aligned}
(5.15)\quad \mu^{n+1}(x,t,s)&=x+\int_t^s f(\mu^n(x,t,u-)\,dZ_u+\int_t^s g(\mu^n(x,t,u-))\,dA_u\\
X(x,t,s)&=x+\int_t^s f(x,t,u-))\,dZ_u+\int_t^s g(X(x,t,u-))\,dA_u.
\end{aligned}
$$

We also write $X(x,t)$ for $X(x,0,t)$ and $\mu^n(x,t)$ for $\mu^n(x,0,t)$. Observe that

$$
\begin{aligned}
\mu^1(x,t,s)&=x+f(x)(Z_s-Z_t)+g(x)(A_s-A_t)\\
&=(x+f(x)(Z_{s-t}-Z_0)+g(x)(A_{s-t}-A_0))\circ\theta_t\\
&=\mu^1(x,s-t)\circ\theta_t.
\end{aligned}
$$

Assume $\mu^n(x,t,s)=\mu^n(x,s-t)\circ\theta_t$. Then

$$
\begin{aligned}
\mu^{n+1}(x,t,s)&=x+\int_t^s f(\tilde{\mu}^n(x,(u-t)-)\,d\tilde{Z}_{u-t}+\int_t^s g(\tilde{\mu}^n(x,(u-t)-)\,d\tilde{A}_{u-t}\\
&=x+\int_0^{s-t} f(\tilde{\mu}^n(x,u-))\,d\tilde{Z}_u+\int_0^{s-t} g(\tilde{\mu}^n(x,u-))\,d\tilde{A}_u\\
&=\left(x+\int_0^{s-t} f(\mu^n(x,u-))\,dZ_u+\int_0^{s-t} g(\mu^n(x,u-))\,dA_u\right)\circ\theta_t
\end{aligned}
$$

where the last equality uses Lemma (5.11). Induction shows then that for all n

$$(5.16)\quad \mu^n(x,t,s)=\mu^n(x,s-t)\circ\theta_t.$$

We next establish the equality

$$(5.17)\quad E^z\{h(X(x,t,s),Z_s)\,|\,\mathscr{F}_t\}=E^{Z_t}\{h(X(x,s-t),Z_{s-t})\}$$

for $h\in b\mathscr{B}\otimes\mathscr{B}$. First, consider h of the form $h(x,y)=h_1(x)\,h_2(y)$, with h_i continuous with compact support. By Lemma (4.5) and the uniform continuity of h_1, $h_1(\mu^n(x,t,s))\to h_1(X(x,t,s))$ in the mean. Using (5.16), we have

$$\begin{aligned}E^z\{h_1(X(x,t,s))\,h_2(Z_s)\,|\,\mathscr{F}_t\}&=\lim_{n\to\infty}E^z\{h_1(\mu^n(x,t,s)\,h_2(Z_s)\,|\,\mathscr{F}_t\}\\&=\lim_{n\to\infty}E^z\{h_1(\mu^n(x,s-t))\,h_2(Z_{s-t})\circ\theta_t\,|\,\mathscr{F}_t\}\\&=\lim_{n\to\infty}E^{Z_t}\{h_1(\mu^n(x,s-t))\,h_2(Z_{s-t})\}\\&=E^{Z_t}\{h_1(X(x,s,t))\,h_2(Z_{s-t})\}.\end{aligned}$$

A monotone class argument now yields (5.17). Note that (5.17) also holds for stopping times $S\geqq T$.

Let (X_t) be as given in (5.10), and fix a measure $P^{x,z}$. Let X_t^x denote the solution of

$$(5.18)\quad X_t^x=x+\int_0^t f(X_{u-}^x)\,dZ_u+\int_0^t g(X_{u-}^x)\,dA_u$$

for the law P^z on Ω_1. Let $h\in b\mathscr{B}\otimes\mathscr{B}$, $F\in b\mathscr{F}_t{}^o$, and $k\in b\mathscr{B}$. Using Proposition (4.11) we have

$$\begin{aligned}(5.19)\quad &E^{x,z}[h(X_s,Z_s)\,F\,k(X_0)]\\&=E^z[h(X_s^x,Z_s)\,F]\,k(x)\\&=E^z[h(X(X_t^x,t,s),Z_s)\,F]\,k(x)\end{aligned}$$

by the uniqueness of the solutions. As was shown in the proof of Theorem (5.8), X_t^x is jointly measurable in (x,t,ω). A monotone class argument then yields

$$\begin{aligned}(5.20)\quad &E^z[h(X_t^x,t,s),Z_s)\,F]\,k(x)\\&=E^z[E^z[h(X_t^x,t,s),Z_s)\,|\,\mathscr{F}_t{}^o]\,F]\,k(x)\\&=E^z[E^z[h(X(y,k,s),Z_s)\,|\,\mathscr{F}_t{}^o]|_{y=X_t^x}\,F]\,k(x)\\&=E^z[E^{X_t^x,z}[h(X(X_0,t,s),Z_s)\,|\,\mathscr{G}_t^o]\,F]\,k(x)\\&=E^{x,z}[E^{X_t,z}[h(X(X_0,t,s),Z_s)\,|\,\mathscr{G}_t^o]\,F\,k(X_0)].\end{aligned}$$

Together (5.19) and (5.20) establish that

$$(5.21)\quad E^{x,z}[h(X_s,Z_s)\,|\,\mathscr{G}_t]=E^{X_t,z}[h(X(X_0,t,s),Z_s)\,|\,\mathscr{G}_t^o].$$

Let

$$(5.22)\quad j(y)=E^{y,z}[h(X(X_0,t,s),Z_s)\,|\,\mathscr{G}_t^o].$$

Then $j(y)$ is also a version of $E^z[h(X(y,t,s),Z_s)\,|\,\mathscr{F}_t]$, and so

$$\begin{aligned}(5.23)\quad j(y)&=E^{Z_t}[h(X(y,s-t),Z_{s-t})]\\&=E^{y,Z_t}[h(X_{s-t},Z_{s-t})]\end{aligned}$$

where we have used (5.17) and (4.11). Combining (5.21), (5.22), and (5.23) yields

$$E^{x,z}[h(X_s, Z_s)|\mathscr{G}_t] \\ = E^{X_t, Z_t}[h(X_{s-t}, Z_{s-t})].$$

To show that (X, Z) is strong Markov it suffices to show $E^{x,z}[h(X_{T+s}, Z_{T+s})|\mathscr{G}_T] = E^{x,z}[h(X_{T+s}, Z_{T+s})|X_T, Z_T]$ for any stopping time T, and $s>0$. The proof of (5.17) is valid for stopping times. For $h \in b\mathscr{B} \otimes \mathscr{B}$ we have

$$(5.24) \quad E^{x,z}[h(X_{T+s}, Z_{T+s})|\mathscr{G}_T\} \\ = E^z\{h(X^x_{T+s}, Z_{T+s})|\mathscr{F}_T\} \\ = E^z\{h(X(X^x_T, T, T+s), Z_{T+s})|\mathscr{F}_T\}.$$

For a fixed P^z we know that $X(x, T, T+s, \omega)$, the solution relative to $(\Omega, \mathscr{F}^z, \mathscr{F}^z_t, P^z)$, is jointly measurable; it suffices to observe that for $h_1, h_2 \in b\mathscr{B}$ we have

$$E^z\{h_1(X^x_T)\, h_2(X(y, T, T+s), Z_{T+s})|\mathscr{F}_T\} \\ = h_1(X^x_T)\, E^{Z_T}\{h_2(X(y, s), Z_s)\} \\ = j(s, X_T, Z_T) \quad \text{a.s., } P^{x,z}.$$

This completes the proof of Theorem (5.9).

6. The Lévy System

In this section we assume $Z = (\Omega_1, \mathscr{F}, \mathscr{F}_t, Z_t, \theta_t, P^z)$ is a Hunt process and a universally reducible semimartingale (see § 3 for definitions). We will reserve A_t to denote a *quasileft-continuous additive functional* of Z. (That is, for any sequence (T_n) of stopping times increasing to T, $A_{T_n} \to A_T$ a.s.).

Let $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfy conditions (4.3), let X_0 be as given in (4.2), and let X_t be the solution of

$$(6.1) \quad X_t = X_0 + \int_0^t f(X_{s-})\, dZ_s + \int_0^t g(X_{s-})\, dA_s.$$

By Theorem (5.9), the process (X, Z) is a time-homogeneous strong Markov process with semigroup $P_t h(x, z) = E^{x,z}[h(X_t, Z_t)]$, where the measures $P^{x,z}$ are as given in (4.1). Since (A_t) is quasi-left-continuous, X_t and hence (X, Z) are also quasi-left-continuous.

As we have seen in § 4, we can define $\Omega = \mathbb{R} \times \Omega_1$, $\mathscr{G}^o_t = \mathscr{B} \otimes \mathscr{F}^o_t$, etc. Let $\boldsymbol{\omega} \in \Omega$, where $\boldsymbol{\omega} = (x, \omega)$, with $x \in \mathbb{R}$ and $\omega \in \Omega_1$. For each $t \in \mathbb{R}_+$ we define

$$(6.2) \quad \hat{\theta}_t(\boldsymbol{\omega}) = (X_t(\boldsymbol{\omega}), \theta_t(\omega)).$$

One easily checks that for $H \in b\mathscr{G}$

$$(6.3) \quad E^{x,z}[H \circ \hat{\theta}_u] = E^{X_u, Z_u}[H].$$

Equation (6.3) allows the generalization of Lemma (5.11), which in turn gives that for each fixed s

$$X_t = X_{t-s} \circ \hat{\theta}_s(\omega)$$

are indistinguishable as processes in t (the null sets depend on s). A "perfection argument" in the style of Walsh [20] shows that there exists a process $\bar{X}$ which is indistinguishable from X and for ω not in Λ (with $P^\mu(\Lambda)=0$ for every probability μ on $\mathscr{B}^2$),

$$\bar{X}_t = \bar{X}_{t-s} \circ \hat{\theta}_s(\omega).$$

We define $\mathscr{H}_t^o = \sigma((\bar{X}_s, Z_s): s \leqq t)$, $\mathscr{H}^o = \bigvee_s \mathscr{H}_s^o$, and $\mathscr{H}_t = \bigcap_{\mu \text{ finite}} \mathscr{H}_t^\mu$ where $\mathscr{H}_t^\mu$ denotes the completion of $\mathscr{H}_t^o$ under P^μ, with μ a probability on $\mathscr{B}^2$, the Borel sets of $\mathbb{R}^2$. *We conclude that* $(\bar{X}, Z) = (\Omega, \mathscr{H}, \mathscr{H}_t, (X_t, Z_t), \hat{\theta}_t, P^{x,z})$ *is a Hunt process.* For the rest of this section we will write (X_t) for $(\bar{X}_t)$.

Cinlar [4] and Jacod [11, 12] have considered processes which are similar in structure to the process (X, Z). Indeed, the process (X, Z) is a Markov Additive Process in the sense of Cinlar [4, p. 86]. To see this let $F, G \in \mathscr{B}$, the Borel sets of $\mathbb{R}$. Let $Q_t(z, F \times G) = E^{0,z}[1_F(X_t)\, 1_G(Z_t)]$. Let $X(x, t) = X(x, 0, t)$ be the solution of (6.1) starting at x (which is defined rigorously by (5.15)), and one easily checks that $X(0, t)$ is P^z indistinguishable from $X(x, t) - x$. Thus $Q_t(z; (F-x) \times G) = P_t(x, z; F \times G)$. One can also easily check that the process (X, Z) is a Semi-direct Markov Process Product in the sense of Jacod [11, 12].

A pair (K, H) is said to be a Lévy system of the Hunt process Z if $K(z; dz')$ is a kernel on $\mathbb{R} \times \mathbb{R}$ such that $K(z, \{z\}) = 0$ for every $z \in \mathbb{R}$, and H is a continuous additive functional of Z such that for any nonnegative Borel function F on $\mathbb{R} \times \mathbb{R}$ we have

$$E^z\{\sum_{s \leqq t} F(Z_{s-}, Z_s)\, 1_{\{Z_{s-} \neq Z_s\}}\} = E^z\left\{\int_0^t dH_s \int_0^t K(Z_s, dz')\, F(Z_s, z')\right\}.$$

Every Hunt process has a Lévy system [1]. Jacod [12] has related a Lévy system of a semi-direct Markov process product (Y, Z) to a Lévy system of the (say) Hunt process Z. In our situation we can obtain a more explicit relationship by expressing a Lévy system of (X, Z) in terms of the coefficients f, g; the jumps of (A_t); and a Lévy system of Z.

Let $B_t = \sum_{s \leqq t} \Delta A_s$, where A_t is the quasi-left-continuous additive functional of (6.1). Then Motoo's theorem (see, e.g., [1]) states that there exists a Borel function h on $\mathbb{R} \times \mathbb{R}$ such that B_t is equivalent to the AF

$$\text{(6.4)} \quad B_t = B'_t \equiv \sum_{\substack{s \leqq t \\ s \in J}} h(Z_{s-}, Z_s)$$

where equality means up to indistinguishability and $J = \{(s, \omega): Z_{s-}(\omega) \neq Z_s(\omega)\}$.

(6.5) **Theorem.** *Let Z, X, and A be as given at the beginning of this section, and let h be as given in* (6.4). *Let (K, H) be a Lévy system for Z. Then (N, H) is a Lévy system for (X, Z), where $N(x, z; dx' \times dz') = K(z; dz')\, \varepsilon_{k(x,z,z')}(dx')$ and $k(x, z, z') = x + f(x)(z'-z) + g(x)\, h(z, z')$ and ε_a denotes point mass at a.*

Proof. Recall that $\hat{\theta}_t$ as given in (6.2) is the shift for (X, Z). Extend H_t, the AF of Z, to Ω by $H_t(x,\omega)=H_t(\omega)$. Then H is also an AF of (X, Z).

Let Y_t be a nonnegative previsible process and let F be nonnegative Borel on $\mathbb{R}^2$. It is well known that $\int dH_s \int K(Z_s, dz')\, F(Z_s, z')$ is the dual previsible projection of $\sum_{s\in J} F(Z_{s-}, Z_s)$. Thus

$$E^z\{\sum_{s\leq t,\, s\in J} Y_s F(Z_{s-}, Z_s)\}$$
$$=E^z\left\{\int_0^t Y_s \int K(Z_s, dz')\, F(Z_s, z')\, dH_s\right\}.$$

Let $Y_t = W_t G(X_{s-})$ where W_t is nonnegative previsible, and $G\in\mathscr{B}_+$. A monotone class argument then yields

$$\text{(6.6)}\quad E^{x,z}\{\sum_{s\leq t,\, s\in J} W_s F(X_{s-}, Z_{s-}, Z_s)\}$$
$$=E^{x,z}\left\{\int_0^t W_s \int K(Z_s, dz') F(X_s, Z_s, z')\, dH_s\right\}$$

for nonnegative Borel F on $\mathbb{R}^3$. From properties of the stochastic integral [16, p. 300] we have (assuming h vanishes on the diagonal of $\mathbb{R}^2$)

$$\text{(6.7)}\quad X_t = X_{t-} + f(X_{t-})\,\Delta Z_t + g(X_{t-})\,\Delta A_t$$
$$= X_{t-} + f(X_{t-})\,\Delta Z_t + g(X_{t-})\, h(Z_{t-}, Z_t)$$
$$= k(X_{t-}, Z_{t-}, Z_t).$$

Let nonnegative Borel F be defined on $\mathbb{R}^4$. Equations (6.6) and (6.7) imply

$$E^{x,z}\{\sum_{s\leq t,\, s\in J} W_s F(X_{s-}, X_s, Z_s, Z_{s-})\}$$
$$=E^{x,z}\left\{\int_0^t W_s \int_0^t K(Z_s, dz')\, F(X_s, k(X_{s-}, Z_s, z'), Z_s, z')\, dH_s\right\}$$
$$=E^{x,z}\left\{\int_0^t W_s \int N(X_s, Z_s; dx'\, dz')\, F(X_s, x', Z_s, z')\, dH_s\right\}.$$

This completes the proof.

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