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A Certain Class of Diffusion Processes Associated with Nonlinear Parabolic Equations

Tadahisa Funaki*

Department of Mathematics, Faculty of Science, Nagoya University, Nagoya, 464, Japan

Summary. We introduce a martingale problem to associate diffusion processes with a kind of nonlinear parabolic equation. Then we show the existence and uniqueness theorems for solutions to the martingale problem.

1. Introduction

H.P. McKean [6] discussed the following nonlinear parabolic equation:

$$\frac{\partial u}{\partial t}(t,x)=\tfrac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\,\partial x_j}\{a_{ij}(x,u(t,\cdot))\,u(t,x)\} - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\{b_i(x,u(t,\cdot))u(t,x)\},\qquad t>0,\ x=(x_i)_{i=1}^d\in\mathbb{R}^d, \tag{1.1}$$

with coefficients a and b depending on x and on probability densities u. He constructed a diffusion process X_t whose probability density $u(t,x)=P(X_t\in dx)/dx$ satisfies the Eq. (1.1) in the case where coefficients a and b are *tame* (see Sect. 3). The purpose of this paper is to construct diffusion processes associated with the Eq. (1.1) with more general coefficients than those treated by McKean. This enables us to treat, for example, the so-called Landau equation which appears concerning the problem of a diffusion approximation of the Boltzmann equation (see Sect. 5 and also Funaki [1, 2]). The method employed here is based on a martingale formulation. We shall introduce a martingale problem which corresponds to the Eq. (1.1) and then prove the existence and uniqueness theorems for solutions to the martingale problem.

Let $\mathscr{P}=\mathscr{P}(\mathbb{R}^d)$ be a family of Borel probability measures on $\mathbb{R}^d$ and let $\mathscr{P}_p(1\leqq p<\infty)$ be a class of all $u\in\mathscr{P}$ satisfying

$$\|u\|_p=\{\int_{\mathbb{R}^d}|x|^p u(dx)\}^{1/p}<\infty.$$

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The space $\mathscr{P}_p$ is equipped with a topology determined by an L^p-analogue ρ_p of the Vasershtein metric (see Sect. 2). Throughout this paper, $a=\{a_{ij}(x,u)\}_{1\leq i,j\leq d}$ and $b=\{b_i(x,u)\}_{1\leq i\leq d}$ are assumed to be functions of $\mathbb{R}^d\times\mathscr{P}_p$ into the space of symmetric non-negative definite $d\times d$ matrices respectively $\mathbb{R}^d$ with some fixed p. Given two such functions, we consider the weak version of (1.1):

$$\frac{d}{dt}\langle u(t),\psi\rangle=\langle u(t),\mathscr{L}_{u(t)}\psi\rangle,\qquad \psi\in C_0^\infty(\mathbb{R}^d), \tag{1.2}$$

where $C_0^\infty(\mathbb{R}^d)$ is the space of C^∞-functions on $\mathbb{R}^d$ with compact supports, $\langle u(t),\psi\rangle$ denotes the integral of ψ with respect to a probability measure solution $u(t)=u(t,dx)$ and

$$\mathscr{L}_u=\tfrac{1}{2}\sum_{i,j=1}^d a_{ij}(x,u)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(x,u)\frac{\partial}{\partial x_i},\qquad u\in\mathscr{P}_p.$$

To formulate the problem precisely, we introduce some further notations. Let C be the space of $\mathbb{R}^d$-valued continuous functions on $[0,\infty)$ and denote by $X_t=X_t(\omega)$ the value $\omega(t)$ of $\omega\in C$ at $t\in[0,\infty)$. We set $\mathscr{F}$ and $\mathscr{F}_t$, $0\leq t<\infty$, the smallest σ-fields generated by $\{X_s; 0\leq s<\infty\}$ and $\{X_s; 0\leq s\leq t\}$, respectively. Let $\mathscr{B}_p=B([0,\infty),\mathscr{P}_p)$ be the space of $\mathscr{P}_p$-valued Borel measurable functions $u(\cdot)$ on $[0,\infty)$ satisfying that $\sup_{0\leq t\leq T}\|u(t)\|_p<\infty$ for every $T<\infty$.

The problem is to find, for given $f\in\mathscr{P}_p$, a probability measure P on $(C,\mathscr{F})$ which satisfies the following condition:

(i) The distribution $u(t)=P\circ X_t^{-1}$ of X_t under P (i.e., $u(t,dx)=P(X_t\in dx)$) belongs to the space $\mathscr{B}_p$.

(ii) $u(0)=f$. (1.3)

(iii) For every $\psi\in C_0^\infty(\mathbb{R}^d)$, $\psi(X_t)-\int_0^t\mathscr{L}_{u(s)}\psi(X_s)ds$ is a martingale relative to $(P,\{\mathscr{F}_t\})$.

In Sect. 2, the existence theorem for the martingale problem (1.3) is shown under rather general assumptions, namely, the continuity of the coefficients a and b in $(x,u)\in\mathbb{R}^d\times\mathscr{P}_p$ and the existence of Lyapunov functions. While, to show the uniqueness, we need more restrictive assumption like as follows: the function $b(x,u)$ and the symmetric square root $a^{\frac{1}{2}}(x,u)$ of $a(x,u)$ satisfy

$$\begin{gathered}\|a^{\frac{1}{2}}(x,u)-a^{\frac{1}{2}}(y,v)\|+|b(x,u)-b(y,v)|\leq C|x-y|+\kappa(\rho_p(u,v)),\\ x,y\in\mathbb{R}^d,\qquad u,v\in\mathscr{P}_p,\end{gathered} \tag{1.4}$$

with a positive constant C and a strictly increasing continuous function κ on $[0,\infty)$ satisfying that $\kappa(0)=0$ and $\int_{0+}\kappa^{-2}(\sqrt{r})dr=\infty$, where $\|\cdot\|$ is the norm of matrices. In Sect. 3, the uniqueness theorem is established under a slightly milder assumption than (1.4). The uniqueness theorem is again discussed in Sect. 5 assuming that the coefficient a has a special form:

$$a(x,u)=\int_{\mathbb{R}^d}a(x,y)u(dy),\qquad x\in\mathbb{R}^d,\ u\in\mathscr{P}_2,$$

with a $d \times d$ matrix-valued function $a(x, y)$ which has a uniformly Lipschitz continuous square root $\sigma(x, y)$, $(x, y) \in \mathbb{R}^{2d}$. The notion of generalized Wiener processes is introduced in Sect. 4 to apply it in Sect. 5. Finally in Sect. 6, as consequences of the existence and uniqueness theorems, we discuss the Markov property, in the sense of McKean, of solutions to the martingale problem (1.3) and also the uniqueness of solutions to the Eq. (1.2).

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2. Existence Theorem

In the following we fix a number $p (1 \leqq p < \infty)$ and denote $\|\cdot\|_p$ simply by $\|\cdot\|$. A metric $\rho = \rho_p$ on the space $\mathscr{P}_p$ is introduced as follows:

$$\rho(u_1, u_2) = \inf\{ \int_{\mathbb{R}^{2d}} |x - y|^p F(dx\,dy)\}^{1/p}, \qquad u_1, u_2 \in \mathscr{P}_p,$$

where the infimum is taken over the space $\mathscr{P}(u_1, u_2)$ of all Borel probability measures F on $\mathbb{R}^{2d}$ which satisfy $F(B \times \mathbb{R}^d) = u_1(B)$ and $F(\mathbb{R}^d \times B) = u_2(B)$ for every Borel subset B of $\mathbb{R}^d$. To prove the existence of a solution to the martingale problem (1.3), we make the following assumption throughout this section.

Assumption I. (i) The functions a and b are continuous in $(x, u) \in \mathbb{R}^d \times \mathscr{P}_p$.

(ii) For $q = p, p'$ (p' is some number larger than p), there exist positive nondecreasing functions h_q defined on $[0, \infty)$ such that

$$\mathscr{L}_u \psi_q(x) \leqq h_q(\|u\|) \psi_q(x), \qquad x \in \mathbb{R}^d,\ u \in \mathscr{P}_p,$$

and

$$\int^{\infty} \{h_p(\alpha)\alpha\}^{-1} d\alpha = \infty, \tag{2.1}$$

where $\psi_q \in C^\infty(\mathbb{R}^d)$, $q = p, p'$, are functions satisfying

$$\psi_q(x) = |x|^q + 1, \qquad |x| \geqq 1,$$
$$|x|^q + 1 \leqq \psi_q(x) \leqq |x|^q + 2, \qquad |x| < 1.$$

In this section the following theorem will be shown.

Theorem 2.1. *Under Assumption I, the martingale problem* (1.3) *has a solution for every* $f \in \mathscr{P}_p$.

Remark. (i) Even if Assumption I-(ii) is satisfied only for $q = p$, Theorem 2.1 still holds for every initial distribution $f \in \mathscr{P}_q$, $q > p$.

(ii) The condition for $q = p$ in Assumption I-(ii) can be replaced by the following one:

$$\mathscr{L}_u \psi_p(x) \leqq C\{\psi_p(x) + \|u\|^p\}, \qquad x \in \mathbb{R}^d,\ u \in \mathscr{P}_p,$$

with $C > 0$. Indeed a similar condition is used in Funaki [2] for a martingale problem with jumps.

(iii) Without the condition (2.1), we can prove the existence of a local solution to the martingale problem (1.3).

For given $f \in \mathscr{P}$ and $u(\cdot) \in \mathscr{B}_p$ ($\mathscr{B}_p$ is the space introduced in Sect. 1), a probability measure P on $(C, \mathscr{F})$ will be called a solution to a martingale problem $[u(\cdot), f]$ if P satisfies $P \circ X_0^{-1} = f$ and the condition (iii) in (1.3). We introduce an incomplete metric $\bar{\rho}$ which gives a vague topology on the space $\mathscr{P}$ as follows:

$$\bar{\rho}(u, v) = \sum_{j=1}^{\infty} 2^{-j} \{1 \wedge |\int_{\mathbb{R}^d} \varphi_j(x) u(dx) - \int_{\mathbb{R}^d} \varphi_j(x) v(dx)|\}, \qquad u, v \in \mathscr{P},$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is a sequence satisfying

(i) $\varphi_j \in C_0^2(\mathbb{R}^d)$ for $j = 1, 2, \ldots,$

(ii) $\{\varphi_j\}_{j=1}^{\infty}$ is dense in the space $C_0(\mathbb{R}^d)$ with respect to the uniform topology, (2.2)

and $\alpha \wedge \beta = \min(\alpha, \beta)$ for $\alpha, \beta \in \mathbb{R}$. Put

$$\tau_N(\omega) = \inf\{t > 0; |\omega(t)| > N\}, \qquad \omega \in C, \; N = 1, 2, \ldots,$$

and

$$\|u(\cdot)\|_{[0, T]} = \sup_{0 \leqq t \leqq T} \|u(t)\|, \qquad u(\cdot) \in \mathscr{B}_p, \; T < \infty.$$

Assuming that there exists a solution P to the martingale problem $[u(\cdot), f]$ with $u(\cdot) \in \mathscr{B}_p$ and $f \in \mathscr{P}_q$ ($q = p$ or p'), several estimates on P are given a priori by the following lemma in which we set $\tilde{u}(t) = P \circ X_t^{-1}$.

Lemma 2.1. (i) *For $q = p$ or p' and for every $K, T > 0$, there exists $C_1 = C_1(q, K, T) > 0$ such that*

$$\|\tilde{u}(t)\|_q^q \leqq C_1 \{1 + \|f\|_q^q\}, \qquad 0 \leqq t \leqq T,$$

holds if $u(\cdot) \in \mathscr{B}_p$ satisfies $\|u(\cdot)\|_{[0, T]} \leqq K$.

(ii) $P(\tau_N < T) \leqq N^{-p} \{2 + \|f\|_p^p\} \exp\{T h_p(K)\}$ *if* $\|u(\cdot)\|_{[0, T]} \leqq K$.

(iii) *For every $\varepsilon, T > 0$ and every compact subset $\mathbf{K}$ in $\mathscr{P}_p$, there exists $\delta = \delta(\varepsilon, T, \mathbf{K}) > 0$ such that*

$$\bar{\rho}(\tilde{u}(t), \tilde{u}(s)) < \varepsilon, \qquad 0 \leqq s \leqq t \leqq T,$$

if $t - s < \delta$ and $u(t) \in \mathbf{K}$ for every $t \in [0, T]$.

(iv) *For every $N = 1, 2, \ldots,$ and every compact subset $\mathbf{K}$ in $\mathscr{P}_p$, there exists $C_2 = C_2(N, \mathbf{K}) > 0$ such that*

$$E[|X_{t \wedge \tau_N} - X_{s \wedge \tau_N}|^2 | \mathscr{F}_s] \leqq C_2(t - s), \qquad P\text{-a.s.}, \; 0 \leqq s \leqq t \leqq T,$$

if $u(t) \in \mathbf{K}$ for every $t \in [0, T]$.

Proof. (i) Since we see

$$E[\psi_q(X_{t \wedge \tau_N})] = E[\psi_q(X_0)] + E\left[\int_0^{t \wedge \tau_N} \mathscr{L}_{u(s)} \psi_q(X_s) ds\right]$$

$$\leqq E[\psi_q(X_0)] + \int_0^t h_q(\|u(s)\|) E[\psi_q(X_{s \wedge \tau_N})] ds,$$

Gronwall's lemma (see Hille [3]) implies that

$$E[\psi_q(X_{t\wedge\tau_N})]\leqq E[\psi_q(X_0)]\exp\left\{\int_0^t h_q(\|u(s)\|)ds\right\}.$$

Making N tend to infinity, by using Fatou's lemma, we have the desired estimate.

(ii) Assume that $\|u(\cdot)\|_{[0,T]}\leqq K$ and set $\lambda=h_p(K)$. Then

$$\exp\{-\lambda(t\wedge\tau_N)\}\psi_p(X_{t\wedge\tau_N}),\qquad 0\leqq t\leqq T,$$

is a supermartingale. Hence, we get

$$e^{-\lambda T}N^p P(\tau_N<T)\leqq E[\exp\{-\lambda(T\wedge\tau_N)\}\psi_p(X_{T\wedge\tau_N})]\leqq E[\psi_p(X_0)]\leqq 2+\|f\|_p^p,$$

which proves the conclusion.

(iii) Let $\mathbf{K}$ be a compact subset in $\mathscr{P}_p$. Then we see

$$K_j=\sup_{u\in\mathbf{K},\,x\in\mathbb{R}^d}|\mathscr{L}_u\varphi_j(x)|<\infty.$$

For $u(\cdot)\in\mathscr{B}_p$ satisfying $u(t)\in\mathbf{K}$, $t\in[0,T]$, we have

$$\begin{aligned}\bar\rho(\tilde u(t),\tilde u(s))&=\sum_{j=1}^{\infty}2^{-j}\{1\wedge|E[\varphi_j(X_t)-\varphi_j(X_s)]|\}\\&\leqq\sum_{j=1}^{\infty}2^{-j}\{1\wedge K_j(t-s)\},\qquad 0\leqq s\leqq t\leqq T,\end{aligned}$$

which proves the conclusion immediately.

(iv) Set $\psi(x,y)=|x-y|^2$ $(x,y\in\mathbb{R}^d)$. Then we see

$$C_2(N,\mathbf{K})=\sup\{\mathscr{L}_{u,x}\psi(x,y);|x|,|y|\leqq N,u\in\mathbf{K}\}<\infty,$$

where $\mathscr{L}_{u,x}\psi(x,y)$ is defined by $(\mathscr{L}_u\psi(\cdot,y))(x)$ for each y. Since

$$\psi(X_{t\wedge\tau_N},X_{s\wedge\tau_N})-\int_{s\wedge\tau_N}^{t\wedge\tau_N}\mathscr{L}_{u(r),x}\psi(X_r,X_{s\wedge\tau_N})dr,\qquad t\geqq s$$

is a martingale relative to $P(\cdot|\mathscr{F}_s)$ (P-a.s.) for every $s\geqq 0$, we have

$$\begin{aligned}E[|X_{t\wedge\tau_N}-X_{s\wedge\tau_N}|^2|\mathscr{F}_s]&=E\left[\int_{s\wedge\tau_N}^{t\wedge\tau_N}\mathscr{L}_{u(r),x}\psi(X_r,X_{s\wedge\tau_N})dr\,|\mathscr{F}_s\right]\\&\leqq C_2(N,\mathbf{K})(t-s),\qquad P\text{-a.s.},\ 0\leqq s\leqq t\leqq T,\end{aligned}$$

for $u(\cdot)\in\mathscr{B}_p$ satisfying $u(t)\in\mathbf{K}$, $t\in[0,T]$. □

We also need the next lemma to prove Theorem 2.1.

Lemma 2.2. *There exists a family $\{P_{x,g};\ x\in\mathbb{R}^d, g\in\mathscr{P}_p\}$ of probability measures on $(C,\mathscr{F})$ such that $P_{x,g}$ solves the martingale problem $[g,\delta_x]$, where δ_x is the δ-distribution at x, and the mapping $x\mapsto P_{x,g}(B)$ is Borel measurable on $\mathbb{R}^d$ for every $g\in\mathscr{P}_p$ and $B\in\mathscr{F}$.*

Proof. Noting the existence of a Lyapunov function ψ_p for the operator $\mathscr{L}_g$, techniques developed by Stroock and Varadhan [7] prove that the family $\mathscr{P}(x, g)$ of solutions to the martingale problem $[g, \delta_x]$ is non void for every x and g. Therefore, we can choose $P_{x,g}$ from $\mathscr{P}(x, g)$ so that $P_{x,g}(B)$ is Borel measurable in $x \in \mathbb{R}^d$ for every $B \in \mathscr{F}$ by applying Corollary 1 of Kuratowski and Ryll-Nardzewski [5]. □

We are now ready to prove Theorem 2.1. Let $\mathscr{C}_p = C([0, \infty), \mathscr{P}_p)$ be the space of $\mathscr{P}_p$-valued continuous functions on $[0, \infty)$.

Proof of Theorem 2.1. We divide the proof into six steps.

Step 1. Take a family $\{P_{x,g}; x \in \mathbb{R}^d, g \in \mathscr{P}_p\}$ given by Lemma 2.2 and fix it. For $s \geqq 0$, let $P^s_{x,g}$ be a unique probability measure on $(C, \mathscr{F})$ satisfying

$$\begin{aligned} &\text{(i)}\ P^s_{x,g}(X_t = x, 0 \leqq t \leqq s) = 1, \\ &\text{(ii)}\ P^s_{x,g} \circ \Xi_s^{-1} = P_{x,g}, \end{aligned} \tag{2.3}$$

where Ξ_s is a mapping of C defined by $\Xi_s(\omega) = \omega_s^+ (\omega \in C)$ and ω_s^+ is a shifted path:

$$\omega_s^+(t) = \omega(t+s), \ t \geqq 0.$$

For each $n = 1, 2, \ldots$, we define a sequence $\{P_i^{(n)}\}_{i=1}^\infty$ of probability measures on $(C, \mathscr{F})$ inductively in the following manner. For a given initial distribution $f \in \mathscr{P}_p$, we put

$$P_1^{(n)}(\cdot) = \int_{\mathbb{R}^d} P_{x,f}(\cdot) f(dx).$$

After determining $P_i^{(n)}$, we define $P_{i+1}^{(n)}$ by the unique probability measure which satisfies the following condition.

$$\begin{aligned} &\text{(i)}\ P_{i+1}^{(n)} = P_i^{(n)} \text{ on } \mathscr{F}_{i/n}. \\ &\text{(ii) A regular conditional probability distribution of } P_{i+1}^{(n)} \text{ given } \mathscr{F}_{i/n} \\ &\quad \text{is } P^{i/n}_{\omega(i/n), f^*} \text{ with } f^* = P_i^{(n)} \circ X_{i/n}^{-1}. \end{aligned} \tag{2.4}$$

By the condition (i) in (2.4), there exists a unique probability measure $P^{(n)}$ on $(C, \mathscr{F})$ such that

$$P^{(n)} = P_i^{(n)} \quad \text{on } \mathscr{F}_{i/n} \quad \text{for every } i = 1, 2, \ldots.$$

We set $\tilde{u}_n(t) = P^{(n)} \circ X_t^{-1}$ and $u_n(t) = \tilde{u}_n([nt]/n)$ for $t \geqq 0$, where $[nt]$ is the largest integer not exceeding nt. Note that $P^{(n)}$ is a solution to the martingale problem $[u_n(\cdot), f]$.

Step 2 is devoted to proving that

$$K(T) = \sup_n \|u_n(\cdot)\|_{[0,T]} < \infty, \qquad T < \infty. \tag{2.5}$$

First we note that a similar method used in the proof of Lemma 2.1-(i) shows the following:

$$E^{P^{(n)}}[\psi_p(X_{(i+1)/n})] \leqq E^{P^{(n)}}[\psi_p(X_{i/n})] \exp\{h_p(\|u_n(i/n)\|)/n\}. \tag{2.6}$$

Making use of this estimate repeatedly, we get

$$|||u_n(t)|||^p \leqq \alpha_0^p \exp\left\{\int_0^t h_p(|||u_n(s)|||)\,ds\right\}, \qquad t \geqq 0,$$

where

$$|||u|||^p = \int_{\mathbb{R}^d} \psi_p(x)u(dx) \quad \text{and} \quad \alpha_0 = |||f|||.$$

Let $\alpha(t)$, $0 \leqq t < \infty$, be an inverse function of a function A defined by

$$A(\alpha) = p \int_{\alpha_0}^{\alpha} \{h_p(\alpha')\alpha'\}^{-1} d\alpha', \qquad \alpha \geqq \alpha_0.$$

Note that $\alpha(t)$ is defined for every $t \geqq 0$ by the condition (2.1). Since the function $\alpha(t)$ satisfies

$$\alpha_0^p \exp\left\{\int_0^t h_p(\alpha(s))\,ds\right\} = \alpha^p(t), \qquad t \geqq 0,$$

we obtain by the induction in $i = 0, 1, \ldots,$

$$|||u_n(i/n)||| \leqq \alpha(i/n), \qquad n = 1, 2, \ldots,$$

and therefore

$$|||u_n(t)||| \leqq \alpha(t), \qquad t \geqq 0, \qquad n = 1, 2, \ldots.$$

This proves the estimate (2.5).

Step 3. Here we show

$$\lim_{N\to\infty} \sup_n \sup_{0 \leqq t \leqq T} \int_{|x|>N} |x|^p \tilde{u}_n(t, dx) = 0, \; T < \infty, \tag{2.7}$$

which implies that $\{\tilde{u}_n(t); n = 1, 2, \ldots, t \in [0, T]\}$ is a relatively compact subset of $\mathscr{P}_p$. In fact, we observe

$$\begin{aligned}
\int_{|x|>N} |x|^p \tilde{u}_n(t, dx) &= E^{P^{(n)}}[|X_t|^p; |X_t| > N] \\
&\leqq \int_{|x| \geqq \log N} E_x^{(n)}[|X_t|^p] f(dx) + \int_{|x| < \log N} E_x^{(n)}[|X_t|^p; |X_t| > N] f(dx) \\
&= I_1^{(n)}(N, t) + I_2^{(n)}(N, t),
\end{aligned}$$

where $E_x^{(n)}[\cdot]$ stands for the expectation relative to a regular conditional probability distribution $P_x^{(n)}$ of $P^{(n)}$ given $\mathscr{F}_0$ with $x = \omega(0)$. Since $P_x^{(n)}$ is a solution to the martingale problem $[u_n(\cdot), \delta_x]$ for f-a.e. x, Lemma 2.1-(i) and (2.5) prove that

$$I_1^{(n)}(N, t) \leqq C_1(p, K(T), T) \int_{|x| \geqq \log N} (1 + |x|^p) f(dx), \qquad t \leqq T,$$

which implies

$$\lim_{N\to\infty} \sup_n \sup_{0 \leqq t \leqq T} I_1^{(n)}(N, t) = 0.$$

While, since the integrand of $I_2^{(n)}(N, t)$ is bounded by

$$\{E_x^{(n)}[|X_t|^{p'}]\}^{p/p'} \{P_x^{(n)}(|X_t| > N)\}^{1 - p/p'},$$

by using Lemma 2.1-(i) again, we obtain

$$I_2^{(n)}(N,t) \leqq \int_{|x|<\log N} \{C_1(p', K(T), T)(1+|x|^{p'})\}^{p/p'} \times \{N^{-p} C_1(p, K(T), T)(1+|x|^p)\}^{1-p/p'} f(dx), \quad t \leqq T,$$

which proves

$$\lim_{N\to\infty} \sup_n \sup_{0\leqq t\leqq T} I_2^{(n)}(N,t)=0.$$

Therefore we get (2.7).

Step 4. For every $N=1, 2, \ldots$, Lemma 2.1-(iv) shows

$$E^{P^{(n)}}[|X_{t\wedge\tau_N}-X_{s\wedge\tau_N}|^2|\mathscr{F}_s] \leqq C_2(N, \mathbf{K})(t-s), \quad P^{(n)}\text{-a.s.}, \ 0\leqq s\leqq t\leqq T,$$

where $\mathbf{K}$ is a closure of the set $\{u_n(t); n=1,2,\ldots, t\in[0,T]\}$ in the space $\mathscr{P}_p$. While, noting (2.5), Lemma 2.1-(ii) shows that

$$\lim_{N\to\infty} \sup_n P^{(n)}(\tau_N<T)=0, \quad T<\infty.$$

Hence, the family $\{P^{(n)}\}_{n=1}^{\infty}$ is relatively weakly compact on the space C (see Lemma 4.2 of Funaki [1]).

Step 5. Lemma 2.1-(iii) combined with the result in Step 3 proves that $\{\tilde{u}_n(\cdot)\}_{n=1}^{\infty}$ is a family of $\mathscr{P}$-valued $\bar{\rho}$-equicontinuous functions. While Step 3 also shows that the set $\{\tilde{u}_n(t); n=1,2,\ldots, t\in[0,T]\}$, $T<\infty$, is relatively compact in the space $\mathscr{P}$ with the vague topology. Therefore Ascoli's theorem proves the relative compactness of the set $\{\tilde{u}_n(\cdot)\}_{n=1}^{\infty}$ in the space $C([0,\infty),\mathscr{P})$ with a topology determined by uniform convergence on each bounded interval of $[0,\infty)$. Since $\{P^{(n)}\}_{n=1}^{\infty}$ is relatively compact, there is a subsequence $\{P^{(n')}\}$ which converges weakly to a probability measure P on $(C,\mathscr{F})$. We also see that $\tilde{u}_{n'}(\cdot)$ converges to $u(\cdot)=P\circ X_\cdot^{-1}$ in the space $C([0,\infty),\mathscr{P})$. For each $t\geqq 0$, since

$$\bar{\rho}(u_{n'}(t), u(t))=\bar{\rho}(\tilde{u}_{n'}([n't]/n'), u(t)) \leqq \sup_{0\leqq s\leqq t} \bar{\rho}(\tilde{u}_{n'}(s), u(s))+\bar{\rho}(u([n't]/n'), u(t)) \to 0 \quad \text{as } n'\to\infty,$$

we see that $u_{n'}(t)$ converges to $u(t)$ vaguely, which proves the convergence of $u_{n'}(t)$ to $u(t)$ in the space $\mathscr{P}_p$ because $\{u_{n'}(t)\}$ is relatively compact in $\mathscr{P}_p$.

Step 6. Finally in this step we prove that the probability measure P given in Step 5 is a solution to the martingale problem (1.3). We may only show the condition (iii) in (1.3). Since

$$\psi(X_t)-\int_0^t \mathscr{L}_{u_{n'}(s)}\psi(X_s)\,ds, \quad t\geqq 0,$$

is a martingale relative to $P^{(n')}$ for every $\psi\in C_0^\infty(\mathbb{R}^d)$, it is sufficient to show that

$$\lim_{n'\to\infty} E^{P^{(n')}}[\Psi_{n'}\cdot\Phi]=E^P[\Psi\cdot\Phi] \tag{2.8}$$

holds for every bounded continuous function Φ on C, where

$$\Psi_{n'}(\omega)=\int_0^t \mathscr{L}_{u_{n'}(s)}\psi(X_s(\omega))\,ds$$

and

$$\Psi(\omega)=\int_0^t \mathscr{L}_{u(s)}\psi(X_s(\omega))\,ds.$$

The compactness of the family $\{u_{n'}(s); s\in[0,t],\ n=1,2,\dots\}$ and Assumption I-(i) show that $\{\Psi_{n'}\}$ is a family of uniformly bounded functions which are equicontinuous at every $\omega\in C$. While, since we have shown in Step 5 that $u_{n'}(s)$ converges to $u(s)$ in $\mathscr{P}_p$ for every $s\in[0,t]$, Lebesgue's dominated convergence theorem proves that $\Psi_{n'}(\omega)$ tends to $\Psi(\omega)$ as $n'\to\infty$ for every $\omega\in C$. Therefore $\Psi_{n'}$ converges to Ψ uniformly on each compact subset of C as $n'\to\infty$. Hence we can show (2.8) and this concludes the proof of Theorem 2.1. □

Noting that the condition (2.1) was used only to prove (2.5) in Step 2 in the proof, we get the following.

Corollary. *Suppose Assumption I except the condition* (2.1). *Assume also that there exists a solution P to the martingale problem* (1.3). *Then the function* $u(\cdot)=P\circ X_\cdot^{-1}$ *belongs to the space* $\mathscr{C}_p$.

Proof. By applying similar methods developed in Step 3 in the proof of Theorem 2.1, we can prove

$$\lim_{N\to\infty}\sup_{0\leqq t\leqq T}\int_{|x|>N}|x|^p u(t,dx)=0, \qquad T<\infty,$$

which shows that the set $\{u(t);\ t\in[0,T]\}$, $T<\infty$, is relatively compact in the space $\mathscr{P}_p$. Therefore the proof is completed since we see $u(\cdot)\in C([0,\infty),\mathscr{P})$ easily. □

3. Uniqueness Theorem – General Case

In this section the uniqueness theorem for the martingale problem (1.3) is shown. As in Sect. 2, a number p $(1\leqq p<\infty)$ is fixed. We assume the following.

Assumption II. There exist a positive integer d' and an $\mathbb{R}^d\otimes\mathbb{R}^{d'}$-valued function $\sigma(x,u)=\{\sigma_{ij}(x,u)\}_{1\leqq i\leqq d,\,1\leqq j\leqq d'}$ defined on $\mathbb{R}^d\times\mathscr{P}_p$ such that

(i) $$a_{ij}(x,u)=\sum_{k=1}^{d'}\sigma_{ik}(x,u)\sigma_{jk}(x,u), \qquad 1\leqq i,\ j\leqq d,$$

and

(ii) for every $K>0$,

$$\|\sigma(x,u)-\sigma(y,v)\|+|b(x,u)-b(y,v)|\leqq C_1|x-y|+\kappa(\rho(u,v)),$$
$$x,y\in\mathbb{R}^d, \qquad u,v\in\mathscr{P}_p^K,$$

with some positive constant $C_1 = C_1(K)$ and a strictly increasing continuous function $\kappa(\cdot) = \kappa(\cdot\,; K)$ on $[0, \infty)$ satisfying that $\kappa(0) = 0$ *and* $\int_{0+} \kappa^{-2}(\sqrt{r})\,dr = \infty$, where $\mathscr{P}_p^K = \{u \in \mathscr{P}_p;\ \|u\| \leqq K\}$.

Remark. Assume that, for every $K > 0$, there exist positive constants C and α such that

(i) $\quad \|a(x, u) - a(y, v)\| \leqq C\{|x - y| + \rho(u, v)\}, \qquad x, y \in \mathbb{R}^d,\ u, v \in \mathscr{P}_p^K,$

and

(ii) $\quad \inf\{\langle \theta, a(x, u)\theta\rangle;\ x \in \mathbb{R}^d, u \in \mathscr{P}_p^K, \theta \in \mathbb{R}^d\colon |\theta| = 1\} \geqq \alpha.$

Then Theorem 5.2.2 of Stroock and Varadhan [7] shows that the positive definite square root $a^{\frac{1}{2}}(x, u)$ of $a(x, u)$ satisfies that

$$\|a^{\frac{1}{2}}(x, u) - a^{\frac{1}{2}}(y, v)\| \leqq C\{|x - y| + \rho(u, v)\}/2\sqrt{\alpha}, \qquad x, y \in \mathbb{R}^d,\ u, v \in \mathscr{P}_p^K.$$

Theorem 3.1. *Under Assumption II, for each $f \in \mathscr{P}_p$, there is at most one solution to the martingale problem* (1.3).

The proof of the theorem will be completed by showing the uniqueness of solutions P which satisfy $P \circ X_\cdot^{-1} \in \mathscr{C}_p$, since Assumption II implies Assumption I except the condition (2.1). Since Assumption II also implies the uniform Lipschitz continuity in x of the coefficients, the martingale problem $[u(\cdot), f]$ has a unique solution $P_{f, u(\cdot)}$ for every $u(\cdot) \in \mathscr{C}_p$ and $f \in \mathscr{P}_p$. A property of the mapping $u(\cdot) \mapsto \tilde{u}(\cdot) = P_{f, u(\cdot)} \circ X_\cdot^{-1}$ of the space $\mathscr{C}_p$ is investigated by the following lemma.

Lemma 3.1. *For every $K > 0$, there exist positive constants $t_0 = t_0(K) \leqq 1$ and $C_2 = C_2(K)$ such that*

$$\rho^2(\tilde{u}_1(t), \tilde{u}_2(t)) \leqq C_2 \int_0^t \kappa^2(\rho(u_1(s), u_2(s)); K)\,ds, \qquad 0 \leqq t \leqq t_0,$$

holds if $u_l(\cdot) \in \mathscr{C}_p$, $l = 1, 2$, satisfy $\|u_l(\cdot)\|_{[0,1]} \leqq K$, where $\tilde{u}_l(t) = P_{f, u_l(\cdot)} \circ X_t^{-1}$.

Proof. For $u_l(\cdot) \in \mathscr{C}_p$; $\|u_l(\cdot)\|_{[0,1]} \leqq K$ $(l = 1, 2)$, let $(X^1(t), X^2(t)) \in \mathbb{R}^{2d}$, $t \geqq 0$, be a unique solution to the following stochastic differential equation:

$$\begin{aligned} &dX^l(t) = \sigma(X^l(t), u_l(t))\,dB_t + b(X^l(t), u_l(t))\,dt, \qquad l = 1, 2, \\ &X^1(0) = X^2(0) = X, \end{aligned} \tag{3.1}$$

where B_t is a d'-dimensional Brownian motion and X is an f-distributed $\mathbb{R}^d$-valued random variable. B_t and X are defined on a proper probability space $(\Omega, \mathscr{G}, P)$ and taken to be mutually independent. Note that distributions on C of $X^l(\cdot)$ are $P_{f, u_l(\cdot)}$ for $l = 1, 2$. We set

$$A_t = \int_0^t \{b(X^1(s), u_1(s)) - b(X^2(s), u_2(s))\}\,ds$$

and

$$M_t = X^1(t) - X^2(t) - A_t.$$

Then, since M_t is a d-dimensional martingale relative to P, Burkholder-Davis-Gundy's inequality (see, e.g., Ikeda and Watanabe [4] p. 110) yields that

$$E[\sup_{0\leqq s\leqq t}|M_s|^p]\leqq C_3\sum_{i=1}^{d}E\left[\left\{\int_0^t\sum_{k=1}^{d'}\{\sigma_{ik}(X^1(s),u_1(s))-\sigma_{ik}(X^2(s),u_2(s))\}^2ds\right\}^{p/2}\right],$$

with some positive constant C_3, which shows that there exists a positive constant $C_4=C_4(K)$ such that

$$E[\sup_{0\leqq s\leqq t}|M_s|^p]\leqq C_4t^{p/2}E[\sup_{0\leqq s\leqq t}|X^1(s)-X^2(s)|^p]$$
$$+C_4\left\{\int_0^t\kappa^2(\rho(u_1(s),u_2(s)))ds\right\}^{p/2},\qquad t\in[0,1].$$

As for the process A_t, we obtain a similar bound:

$$E[\sup_{0\leqq s\leqq t}|A_s|^p]\leqq C_5t^pE[\sup_{0\leqq s\leqq t}|X^1(s)-X^2(s)|^p]$$
$$+C_5\left\{\int_0^t\kappa(\rho(u_1(s),u_2(s)))ds\right\}^p,\qquad t\in[0,1].$$

Noting that

$$I(t)\equiv E[\sup_{0\leqq s\leqq t}|X^1(s)-X^2(s)|^p]\leqq 2^{p-1}\{E[\sup_{0\leqq s\leqq t}|M_s|^p]+E[\sup_{0\leqq s\leqq t}|A_s|^p]\},$$

we get

$$I(t)\leqq\tfrac{1}{2}C_2^{p/2}\left[t^{p/2}I(t)+\left\{\int_0^t\kappa^2(\rho(u_1(s),u_2(s)))ds\right\}^{p/2}\right],$$
$$t\in[0,1],\qquad C_2=4(C_4+C_5)^{2/p},$$

which proves that

$$I(t)^{2/p}\leqq C_2\int_0^t\kappa^2(\rho(u_1(s),u_2(s)))ds,\qquad t\in[0,t_0],\ t_0=1\wedge C_2^{-1}.$$

This completes the proof since we see that

$$\rho(\tilde{u}_1(t),\tilde{u}_2(t))\leqq I(t)^{1/p}.\quad\square$$

Proof of Theorem 3.1. For $f\in\mathscr{P}_p$, assume that there are two different solutions P_1 and P_2 to the martingale problem (1.3). We set $u_l(t)=P_l\circ X_t^{-1}$ $(l=1,2)$. The uniqueness of solutions to the martingale problem $[u(\cdot),f]$ for every $u(\cdot)\in\mathscr{C}_p$ and $f\in\mathscr{P}_p$ implies that $t^*=\inf\{t\geqq 0;\ u_1(t)\neq u_2(t)\}$ is finite. We put $f^*=u_1(t^*)$ $(=u_2(t^*))$. Since $P_l\circ\Xi_{t^*}^{-1}$ $(l=1,2)$ are two solutions to the martingale problem (1.3) with f replaced by f^*, we may assume $t^*=0$ without loss of generality. Noting that $\tilde{u}_l(\cdot)=u_l(\cdot)$ $(l=1,2)$, by the assumption on κ, Lemma 3.1 shows that $\rho(u_1(t),u_2(t))=0$ holds for every sufficiently small t (see Hille [3]) and this leads us to a contradiction. Hence we have the conclusion. $\square$

Remark. (i) Assuming $d=1$ for simplicity, McKean [6] discussed the case where a and b are *tame* functionals of degree m:

$$a(x,u)=\left\{\int_{\mathbb{R}^m} \sigma(x; y_1, \ldots, y_m) \prod_{k=1}^{m} u(dy_k)\right\}^2,$$

$$b(x,u)=\int_{\mathbb{R}^m} b(x; y_1, \ldots, y_m) \prod_{k=1}^{m} u(dy_k).$$

If the functions $\sigma(x; y_1, \ldots, y_m)$ and $b(x; y_1, \ldots, y_m)$ are uniformly Lipschitz continuous on $\mathbb{R}^{m+1}$, then $a(x,u)$ and $b(x,u)$ satisfy Assumption II for every $p \geqq 1$.

(ii) Let a and b have the following form:

$$\begin{aligned} a(x,u)&=\int_{\mathbb{R}^d} a(x,y)u(dy), \\ b(x,u)&=\int_{\mathbb{R}^d} b(x,y)u(dy), \end{aligned} \tag{3.2}$$

where $a(x,y)$ and $b(x,y)$ are functions of $\mathbb{R}^{2d}$ into $\mathbb{R}^d \otimes \mathbb{R}^d$ and $\mathbb{R}^d$, respectively. If $a(x,y)$ and $b(x,y)$ are uniformly Lipschitz continuous on $\mathbb{R}^{2d}$ and the matrix $a(x,y)$ is symmetric and uniformly positive definite on $\mathbb{R}^{2d}$, then $a(x,u)$ and $b(x,u)$ satisfy Assumption II for every $p \geqq 1$. This case will be discussed again in Sect. 5.

4. Generalized Wiener Processes and Stochastic Integrals

We introduce a notion of generalized Wiener processes and define stochastic integrals to apply them in the next section. Take $u(\cdot) \in B([0,\infty), \mathscr{P})$, the space of $\mathscr{P}$-valued Borel measurable functions, and fix it throughout this section. Let $B_b(\mathbb{R}^d)$ be the space of bounded Borel measurable functions on $\mathbb{R}^d$.

Definition. A family of stochastic processes $\{B_t(\varphi), t \in [0,\infty); \varphi \in B_b(\mathbb{R}^d)\}$ defined on a probability space $(\Omega, \mathscr{G}, P)$ with a reference family $\{\mathscr{G}_t; t \geqq 0\}$ is called a $\{\mathscr{G}_t\}$-generalized Wiener process (with intensity $u(\cdot)$) if it satisfies the following three conditions.

(i) For every $t \geqq 0$, $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in B_b(\mathbb{R}^d)$,

$$B_t(\alpha\varphi + \beta\psi) = \alpha B_t(\varphi) + \beta B_t(\psi), \qquad P\text{-a.s.}$$

(ii) For every $\varphi \in B_b(\mathbb{R}^d)$, $B_.(\varphi)$ is a $\{\mathscr{G}_t\}$-adapted continuous process such that $B_0(\varphi)=0$.

(iii) For every $\varphi \in B_b(\mathbb{R}^d)$ and $0 \leqq s \leqq t$,

$$E[\exp\{i(B_t(\varphi)-B_s(\varphi))\} \mid \mathscr{G}_s] = \exp\left\{-\int_s^t \langle \varphi, \varphi \rangle_r dr/2\right\}, \qquad P\text{-a.s.},$$

where

$$\langle \varphi, \psi \rangle_r = \int_{\mathbb{R}^d} \varphi(x)\psi(x)u(r,dx), \qquad \varphi, \psi \in B_b(\mathbb{R}^d).$$

A generalized Wiener process exists on a probability space $(\Omega, \mathscr{G}, P)$ taken properly. We shall sometimes denote $B_t(\varphi)$ by $\langle B_t, \varphi\rangle$. By the property (iii), for every bounded measurable function $\varphi(x;\omega)$ on $\mathbb{R}^d \times \Omega$ which is $\mathscr{G}_s$-measurable, we can define

$$B_t(\varphi(\cdot\,;\omega)) - B_s(\varphi(\cdot\,;\omega)), \qquad t \geqq s \geqq 0,$$

also denoted by

$$\langle B_t - B_s, \varphi(\cdot\,;\omega)\rangle.$$

Let $\mathscr{M}_2^c$ be a family of continuous square integrable martingales $X=\{X_t\}$ on $(\Omega, \mathscr{G}, P)$ with respect to $\{\mathscr{G}_t\}$. For $X \in \mathscr{M}_2^c$, we set

$$\| X \|_T = E[X_T^2]^{\frac{1}{2}}, \qquad T < \infty,$$

and

$$\| X \| = \sum_{n=1}^{\infty} 2^{-n}(\| X \|_n \wedge 1).$$

Now we define stochastic integrals with respect to the $\{\mathscr{G}_t\}$-generalized Wiener process. Let $\mathscr{L}_2 = \mathscr{L}_2(u(\cdot))$ be the space of measurable functions $f=\{f(t,x;\omega)\}$ defined on $[0,\infty)\times\mathbb{R}^d\times\Omega$ such that for each $x\in\mathbb{R}^d$, f is a $\{\mathscr{G}_t\}$-predictable process and

$$\|f\|_{T,\mathscr{L}_2}^2 = \int_0^T dt \int_{\mathbb{R}^d} E[f^2(t,x;\omega)]\,u(t,dx) < \infty$$

for every $T<\infty$. For $f\in\mathscr{L}_2$, we set

$$\|f\|_{\mathscr{L}_2} = \sum_{n=1}^{\infty} 2^{-n}(\|f\|_{n,\mathscr{L}_2} \wedge 1).$$

Let $\mathscr{L}_0$ be a family of functions $f\in\mathscr{L}_2$ having the property that there exist a sequence of increasing numbers $0=t_0<t_1<\ldots<t_n<\ldots\to\infty$ and a sequence of measurable functions $\{f_i(x;\omega)\}_{i=0}^{\infty}$ on $\mathbb{R}^d\times\Omega$ such that $f_i(x;\cdot)$ is $\mathscr{G}_{t_i}$-measurable for every $x\in\mathbb{R}^d$,

$$\sup_{i,x,\omega} |f_i(x;\omega)| < \infty$$

and

$$f(t,x;\omega) = f_0(x;\omega)\mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} f_i(x;\omega)\mathbb{1}_{(t_i,t_{i+1}]}(t).$$

For $f\in\mathscr{L}_0$, we define

$$I(f)(t) = \sum_{i=0}^{\infty} \langle B_{t_{i+1}\wedge t} - B_{t_i\wedge t}, f_i(\cdot\,;\omega)\rangle, \qquad t\geqq 0.$$

Then we see that $I(f)\in\mathscr{M}_2^c$ and

$$\| I(f) \| = \|f\|_{\mathscr{L}_2} \qquad \text{for every } f\in\mathscr{L}_0.$$

Using this isometry, since $\mathscr{L}_0$ is dense in $\mathscr{L}_2$, the mapping $f\in\mathscr{L}_0\mapsto I(f)\in\mathscr{M}_2^c$ is extended to $f\in\mathscr{L}_2\mapsto I(f)\in\mathscr{M}_2^c$ as usual. $I(f)$ is called a stochastic integral of

$f \in \mathscr{L}_2$ with respect to the generalized Wiener process B_t. We shall also denote $I(f)(t)$ by $\int_0^t \langle f(s, \cdot\,; \omega), dB_s \rangle$. We see easily that, for f and $g \in \mathscr{L}_2$, the quadratic variational process corresponding to $I(f)$ and $I(g)$ is given by

$$\langle I(f), I(g) \rangle (t) = \int_0^t \langle f(s, \cdot\,; \omega), g(s, \cdot\,; \omega) \rangle_s \, ds.$$

Finally we give a representation theorem for martingales in terms of generalized Wiener processes, which is an infinite dimensional analogue of the well-known martingale representation theorem (see, e.g., Ikeda and Watanabe [4] p. 90). We call $\{B^k_\cdot\}_{k=1}^n$ an n-dimensional generalized Wiener process with intensity $u(\cdot)$ if $\{B^k_\cdot\}_k$ is an independent system and $B^k_\cdot$ is a generalized Wiener process with intensity $u(\cdot)$ for each k.

Proposition 4.1. *Let $(\Omega, \mathscr{G}, P)$ be a probability space with a reference family $\{\mathscr{G}_t\}$ and $M_i \in \mathscr{M}_2^c$, $i = 1, 2, \ldots, n$ such that $M_i(0) = 0$. Assume that there exist functions $\sigma_{ik} \in \mathscr{L}_2(u(\cdot))$, $1 \leqq i \leqq n$, $1 \leqq k \leqq n'$ with some n' such that the quadratic variational processes corresponding to M_i and M_j are given by*

$$\langle M_i, M_j \rangle (t) = \sum_{k=1}^{n'} \int_0^t ds \int_{\mathbb{R}^d} \sigma_{ik}(s, x; \omega) \sigma_{jk}(s, x; \omega) u(s, dx).$$

Then on an extension $(\tilde{\Omega}, \tilde{\mathscr{G}}, \tilde{P})$ and $(\tilde{\mathscr{G}}_t)$ of $(\Omega, \mathscr{G}, P)$ and $(\mathscr{G}_t)$. there exists an n'-dimensional generalized Wiener process $\{B^k_\cdot\}_{k=1}^{n'}$ with intensity $u(\cdot)$ such that

$$M_i(t) = \sum_{k=1}^{n'} \int_0^t \langle \sigma_{ik}(s, \cdot\,; \omega), dB^k_s \rangle, \qquad i = 1, 2, \ldots, n.$$

As the generalized Wiener process, in a simple case:

$$n = n' = 1 \text{ and } \langle \sigma(s, \cdot\,; \omega), \sigma(s, \cdot\,; \omega) \rangle_s \neq 0 \qquad \text{a.s. for every } s \geqq 0,$$

we may take

$$B_t(\varphi) = \int_0^t f(s; \varphi) dM_s + \int_0^t \langle \varphi - f(s; \varphi) \sigma(s, \cdot\,; \omega), dB'_s \rangle, \qquad \varphi \in B_b(\mathbb{R}^d),$$

$$f(s; \varphi) \equiv f(s, \omega; \varphi) = \{ \int_{\mathbb{R}^d} \sigma^2(s, x; \omega) u(s, dx) \}^{-1} \int_{\mathbb{R}^d} \varphi(x) \sigma(s, x; \omega) u(s, dx),$$

with a generalized Wiener process B'_t with intensity $u(\cdot)$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathscr{G}}, \tilde{P})$ properly extended in such a way that B'_t and M_t are mutually independent. In general case, although we need arguments in infinite dimensional spaces, the proposition is shown by applying similar methods developed in the book of Ikeda and Watanabe so that we omit the proof.

5. Uniqueness Theorem – Special Case

In this section, taking $p = 2$, the uniqueness theorem for the martingale problem (1.3) is shown assuming that the coefficient a has a special form:

$$a(x, u) = \int_{\mathbb{R}^d} a(x, y) u(dy), \qquad x \in \mathbb{R}^d, \ u \in \mathscr{P}_2,$$

with an $\mathbb{R}^d \otimes \mathbb{R}^d$-valued function $a(x, y) = \{a_{ij}(x, y)\}_{1 \leqq i, j \leqq d}$ defined on $\mathbb{R}^{2d}$. We also assume the following.

Assumption III. (i) There exist a positive integer d' and an $\mathbb{R}^d \otimes \mathbb{R}^{d'}$-valued function $\sigma(x, y) = \{\sigma_{ij}(x, y)\}_{1 \leqq i \leqq d, 1 \leqq j \leqq d'}$ defined on $\mathbb{R}^{2d}$ such that

$$a_{ij}(x, y) = \sum_{k=1}^{d'} \sigma_{ik}(x, y) \sigma_{jk}(x, y), \qquad 1 \leqq i, j \leqq d.$$

(ii) There exists $C > 0$ such that

$$\|\sigma(x^1, y^1) - \sigma(x^2, y^2)\| \leqq C\{|x^1 - x^2| + |y^1 - y^2|\}, \qquad x^1, x^2, y^1, y^2 \in \mathbb{R}^d,$$

and

$$|b(x^1, u_1) - b(x^2, u_2)| \leqq C\{|x^1 - x^2| + \rho_2(u_1, u_2)\}, \qquad x^1, x^2 \in \mathbb{R}^d, \ u_1, u_2 \in \mathscr{P}_2.$$

Theorem 5.1. *Under Assumption III, the martingale problem* (1.3) *with* $p = 2$ *has a unique solution for every* $f \in \mathscr{P}_2$.

We need two lemmas to prove the theorem.

Lemma 5.1. *For every* $u(\cdot) \in \mathscr{C}_2$ *and* $f \in \mathscr{P}$, *the martingale problem* $[u(\cdot), f]$ *has a unique solution* $P_{f, u(\cdot)}$.

Proof. We may only prove the assertion of the uniqueness of solutions. Let P be a solution to the martingale problem $[u(\cdot), f]$. We set

$$M_t = \{M_i(t)\}_{i=1}^d = X_t - X_0 - \int_0^t b(X_s, u(s)) ds.$$

Then $M_i(t)$, $1 \leqq i \leqq d$, are continuous square integrable martingales on the probability space $(C, \mathscr{F}, P)$ with respect to $(\mathscr{F}_t)$ and quadratic variational processes are given by

$$\langle M_i, M_j \rangle(t) = \int_0^t a_{ij}(X_s, u(s)) ds = \sum_{k=1}^{d'} \int_0^t ds \int_{\mathbb{R}^d} \sigma_{ik}(X_s, y) \sigma_{jk}(X_s, y) u(s, dy).$$

Therefore, by Proposition 4.1, on an extension $(\tilde{C}, \tilde{\mathscr{F}}, \tilde{P})$ and $(\tilde{\mathscr{F}}_t)$ of $(C, \mathscr{F}, P)$ and $(\mathscr{F}_t)$, there exists a d'-dimensional generalized Wiener process $\{B^k_\cdot\}_{k=1}^{d'}$ with intensity $u(\cdot)$ such that

$$M_i(t) = \sum_{k=1}^{d'} \int_0^t \langle \sigma_{ik}(X_s, \cdot), dB^k_s \rangle, \qquad 1 \leqq i \leqq d,$$

which gives

$$X_i(t) = X_i(0) + \sum_{k=1}^{d'} \int_0^t \langle \sigma_{ik}(X_s, \cdot), dB^k_s \rangle + \int_0^t b_i(X_s, u(s)) ds. \tag{5.1}$$

Since the Lipschitz continuity in the variable x of coefficients σ and b implies the pathwise uniqueness of solutions to the stochastic integral equation (5.1), we get the conclusion by imitating the arguments due to Yamada and Watanabe (see Ikeda and Watanabe [4]). □

Lemma 5.2. *There exists a positive constant C such that, for every $u_l(\cdot)\in\mathscr{C}_2(l=1, 2)$,*

$$\rho_2^2(\tilde{u}_1(t), \tilde{u}_2(t)) \leqq C e^{Ct} \int_0^t \rho_2^2(u_1(s), u_2(s))ds, \quad t\geqq 0,$$

holds, where $\tilde{u}_l(t)=P_{f,u_l(\cdot)}\circ X_t^{-1}$.

Proof. For every $u_l(\cdot)\in\mathscr{C}_2$ $(l=1,2)$, by applying the theory of measurable selections (see Chap. 12 of Stroock and Varadhan [6]), we can take a family $\{F(t), t\in[0,\infty)\}$ which satisfies

(i) The mapping $F: t\in[0,\infty)\mapsto F(t)\in\mathscr{P}(\mathbb{R}^{2d})$ is Borel measurable, where the space $\mathscr{P}(\mathbb{R}^{2d})$ is equipped with the weak topology.

(ii) For every $t\geqq 0$, $F(t)\in\mathscr{P}(u_1(t), u_2(t))$ and (5.2)

$$\int_{\mathbb{R}^{2d}} |x-y|^2 F(t, dx\,dy) = \rho_2^2(u_1(t), u_2(t)).$$

Let $\{B_t^k\}_{k=1}^{d'}$ be a d'-dimensional generalized Wiener process with intensity $F(\cdot)$ defined on a probability space $(\Omega, \mathscr{G}, P)$. Note that we can take $2d$ instead of d in Sect. 4. Let X be an f-distributed random variable independent of $\{B_t^k\}_{k=1}^{d'}$. We now consider the following stochastic differential equation:

$$\begin{aligned} dX_i^l(t) &= \sum_{k=1}^{d'} \langle \sigma_{ik}^l(X^l(t);\cdot), dB_t^k\rangle + b_i(X^l(t), u_l(t))dt, \quad 1\leqq i\leqq d, \quad l=1,2, \\ X^1(0) &= X^2(0) = X, \end{aligned} \tag{5.3}$$

where $X^l(t)=\{X_i^l(t)\}_{i=1}^d$ and $\sigma_{ik}^l(x; y^1, y^2)=\sigma_{ik}(x, y^l)$, $(x, y^1, y^2)\in\mathbb{R}^{3d}$. By using a usual iteration technique, we can show the existence and uniqueness of solutions to (5.3). By Assumption III-(ii) and (5.2)-(ii), we obtain

$$E[|X^1(t)-X^2(t)|^2] \leqq C\int_0^t \{E[|X^1(s)-X^2(s)|^2] + \rho_2^2(u_1(s), u_2(s))\}ds$$

with some positive constant C. Since Gronwall's lemma shows

$$E[|X^1(t)-X^2(t)|^2] \leqq C e^{Ct} \int_0^t \rho_2^2(u_1(s), u_2(s))ds,$$

we get the conclusion by noting

$$\rho_2^2(\tilde{u}_1(t), \tilde{u}_2(t)) \leqq E[|X^1(t)-X^2(t)|^2]. \quad \square$$

Since Assumption III implies Assumption I $(p=2)$ except the condition (2.1), by the corollary of Theorem 2.1, the proof of Theorem 5.1 is concluded by showing the existence and uniqueness of fixed points of the mapping $u(\cdot)\in\mathscr{C}_2\mapsto\tilde{u}(\cdot)=P_{f,u(\cdot)}\circ X_\cdot^{-1}\in\mathscr{C}_2$. However this is an easy consequence of Lemma 5.2. Therefore the proof of the theorem is now complete.

Remark. The coefficients $a(x,u)$ and $b(x,u)$ of the 3-dimensional spatially homogeneous Landau equation have the forms in (3.2) with kernels:

$$a(x, y)=\{a_{ij}(x, y)\}_{1 \leqq i, j \leqq 3},$$
$$a_{ij}(x, y)=\{\delta_{ij}|x-y|^2-(x_i-y_i)(x_j-y_j)\}k(x, y),$$
$$b(x, y)=-2(x-y)k(x, y).$$

Here $k(x, y)$ is a non-negative function on $\mathbb{R}^6$. If the functions

$$(x_i-y_i)\sqrt{k(x, y)} \quad \text{and} \quad (x_i-y_i)k(x, y), \qquad 1 \leqq i \leqq 3,$$

are uniformly Lipschitz continuous in $(x, y) \in \mathbb{R}^6$, then the coefficients $a(x, u)$ and $b(x, u)$ satisfy Assumption III. We remark that the non-negative definite matrix $a(x, u)$, $(x, u) \in \mathbb{R}^3 \times \mathscr{P}_2$, degenerates if and only if there exists a line L in $\mathbb{R}^3$ such that $x \in L$ and $u(L)=1$. See Funaki [2] for the derivation of the Landau equation.

6. Markov Property and Uniqueness Theorem for the Nonlinear Parabolic Equation

As consequences of the existence and uniqueness theorems for the martingale problems (1.3) and $[u(\cdot), f]$, we study the Markov property of solutions to (1.3) and also the uniqueness of solutions to the Eq. (1.2). We make the following three assumptions.

(A) The martingale problem (1.3) with an initial distribution $f \in \mathscr{P}_p$ $(1 \leqq p < \infty)$ has a unique solution P_f.

(B) For every $f \in \mathscr{P}_p$ and $u(\cdot) \in \mathscr{B}_p$, the martingale problem $[u(\cdot), f]$ has a unique solution $P_{f, u(\cdot)}$.

(C) For every $f \in \mathscr{P}_p$ and $u(\cdot) \in \mathscr{B}_p$, the following *linear* equation (6.1) has a unique solution $v(\cdot) \in \mathscr{B}_p$.

$$\begin{aligned} &\frac{d}{dt}\langle v(t), \psi\rangle = \langle v(t), \mathscr{L}_{u(t)}\psi\rangle, \qquad t \geqq 0,\ \psi \in C_0^\infty(\mathbb{R}^d), \\ &v(0)=f. \end{aligned} \tag{6.1}$$

First it is shown that P_f has a Markov property in the sense of McKean. We define $u_t^+(\cdot) \in \mathscr{B}_p$ by $u_t^+(s)=u(t+s)$, $s \geqq 0$, for $t \geqq 0$ and $u(\cdot) \in \mathscr{B}_p$.

Proposition 6.1. *Assume* (A), (B) *and put* $u(t)=P_f \circ X_t^{-1}$.

(i) *The probability measure P_f has a Markov property in the following sense:*

$$P_f(\omega_t^+ \in \cdot \,|\, \mathscr{F}_t)(\omega) = P_{\delta_{\omega(t)}, u_t^+(\cdot)}(\cdot), \qquad P_f\text{-a.s. } \omega,\ t \geqq 0,$$

where $\omega_t^+ \in C$ is the shifted path introduced in the proof of Theorem 2.1.

(ii) *The function $u(t)$ solves the Eq.* (1.2).

Proof. Since $P_f(\omega_t^+ \in \cdot \,|\, \mathscr{F}_t)(\omega)$ solves the martingale problem $[u_t^+(\cdot), \delta_{\omega(t)}]$ for $t \geqq 0$, we get the assertion (i). The assertion (ii) follows from the condition (iii) of (1.3) immediately. □

Next we show the uniqueness of solutions to the nonlinear equation (1.2) follows from that to the linear equation (6.1).

Proposition 6.2. *Assume* (A), (B) and (C). *Then the Eq.* (1.2) *with an initial condition* $u(0)=f$ *has a unique solution* $u(\cdot)\in\mathscr{B}_p$.

Proof. The existence of a solution was already shown. Let $u(\cdot)\in\mathscr{B}_p$ be a solution to the Eq. (1.2) with $u(0)=f$. Since $v(t)=P_{f,u(\cdot)}\circ X_t^{-1}$ solves the Eq. (6.1), the assumption (C) shows $v(\cdot)=u(\cdot)$ which proves $P_{f,u(\cdot)}=P_f$. Therefore we get $u(t)=P_f\circ X_t^{-1}$ and this implies the uniqueness of solutions to the Eq. (1.2). □

Remark. By using results of Echeverria [8], S.R.S. Varadhan pointed out that the existence and uniqueness of solutions to a usual (linear) martingale problem are equivalent to those to a corresponding linear weak forward equation. His remark shows that the assumption (C) follows from the assumption (B).

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