Z. Wahrscheinlichkeitstheorie verw. Gebiete
65, 99–114 (1983)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete
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Laws of the Iterated Logarithm for Brownian Motions on Compact Manifolds

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§ 1. Introduction and Summary

Let $(M, \mathscr{g})$ be a compact C^∞ Riemannian manifold. It is well-known that $(M, \mathscr{g})$ supports a "Brownian motion", i.e. a strong Markov process

$$\{\Omega, \mathscr{A}; Pr_x, x \in M; X_t: \Omega \to M, \mathscr{F}_t, t \geqq 0\}$$

with continuous sample paths such that $Pr_x\{X_t \in B\} = \int_B p(t, x, y)\, dm(y)$ for all $t \geqq 0$, $x \in M$, B Borel set $\subseteq M$. Here $p: (0, \infty) \times M \times M \to \mathbb{R}$ is the fundamental solution of

$$\frac{1}{2} \Delta_y p(t, x, y) = \frac{\partial}{\partial t} p(t, x, y), \tag{1.1}$$

and dm and Δ are volume element and Laplace operator on M induced by the metric. Since $\frac{m}{m(M)}$ is the invariant probability measure for Brownian motion on M, the well-known ergodic theorem implies for all $f \in L^1(M)$, all $x \in M$

$$Pr_x\left\{\omega;\ \lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)\, ds = m_0^{-1} \int_M f\, dm\right\} = 1, \tag{1.2}$$

where $m_0 = m(M)$. A trivial consequence is

$$Pr_x\left\{\omega;\ \lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)\, ds = m_0^{-1} \int_M f\, dm, \quad \text{all} \quad f \in C(M)\right\} = 1, \tag{1.3}$$

all $x \in M$.

In [1] Baxter and I proved that for bounded measurable $f: M \to \mathbb{R}$

$$Pr_x\left\{\omega;\ \overline{\lim_{t \to \infty}} \frac{\int_0^t f(X_s) ds - m_0^{-1} t \int_M f\, dm}{\sqrt{2t \log\log t}} = \sqrt{2 m_0^{-1} (Gf, f)}\right\} = 1, \tag{1.4}$$

all $x \in M$.

Here

(1.5) $$(Gf)(x)=\int_M g(x,y)\, f(y)\, dm(y),$$

where the kernel g is uniquely determined by the differential equation

(1.6a) $$\tfrac{1}{2}\Delta_y g(x,y)=-\delta_x(y)+m_0^{-1}, \qquad x,y\in M$$

and the normalisation

(1.6b) $$\int_M g(x,y)\, dm(y)=0, \qquad x\in M.$$

Equation (1.5) by the way, defines a bounded linear operator $G\colon L^2(M)\to L^2(M)$ which is nonnegative and symmetric.

We posed the question whether there is an intrinsic class of functions on M, for which the $\log_2$-law (1.4) holds simultaneously. The existence of such a class for classical Brownian motion on the circle follows from a result of Stackelberg [9]. For the special case of the *flat* d-dimensional torus T^d it has been shown recently by Bolthausen [2] that the Sobolev spaces $H^\alpha(T^d)$ which include $C^\infty(T^d)$ are such classes if $\alpha>\frac{d}{2}$. Bolthausen's proof uses our result (1.4) and a $\log_2$-law by Kuelbs [4] for Banach space-valued random variables.

It is the purpose of this paper, to prove a simultaneous $\log_2$-law for Brownian motion on any compact $(M,\mathscr{g})$. A simple version of such a theorem is

(1.7) **Theorem.** *For any compact C^∞ Riemannian manifold $(M,\mathscr{g})$ and associated Brownian motion X we have for all $x\in M$*

(1.8) $$Pr_x\left\{\overline{\lim_{t\to\infty}}\frac{\int_0^t f(X_s)\,ds-m_0^{-1}t\int_M f\,dm}{\sqrt{2t\log\log t}}=\sqrt{2m_0^{-1}(Gf,f)}\quad \text{all}\quad f\in C^\infty(M)\right\}=1,$$

where $m_0=m(M)$.

This theorem follows immediately from our Theorem (3.16) which generalizes the result of [2] for the flat torus as far as the manifold is concerned. We have not been able to improve on the index $\alpha>\max\left(d-\frac{3}{2},\frac{d}{2}\right)$ of the Sobolev spaces H_0^α in our $\log_2$-laws. The proof of our Theorem (3.16) follows the one for the flat torus in that it relies on our result (1.4) and the boundedness of a certain H_0^α-valued process (Theorem (3.8)). In order to get the estimates we need for general compact manifolds, we shall use a version of Weyl's theorem on the asymptotic distribution of the eigenvalues of Δ as well as a result of Hörmander [3] that provides bounds for the eigenfunctions of Δ. It also seems convenient to define Sobolev spaces $H_0^\alpha(M)$ in terms of the kernel (1.6). It can be shown that these $H_0^\alpha(M)$ are essentially the same as the ones in the sense of [7]. Once the key boundedness result (3.8) has been obtained our argument differs slightly from that of Bolthausen, in that we give a direct proof of Theorem (3.16), rather than use the $\log_2$-law of Kuelbs for Banach space valued random variables referred to earlier. In the last section however, we do use Kuelbs' method to obtain a function space version of the $\log_2$-law as was done in [2].

Theorem (1.7) has an intriguing implication, regarding the information about the geometry of M, that can be obtained from a typical Brownian path with arbitrary starting point.

Let $\phi: \Omega \times C^\infty(M) \to C(\mathbb{R}^+)$ be defined by $\phi(\omega,f)(t)=f(X_t(\omega))$, $t \geqq 0$. For $\omega \in \Omega$, let $C_\omega = \phi(\omega, C^\infty(M))$. Obviously C_ω is a subspace of the vector space $C(\mathbb{R}^+)$ (also: $f \in C_\omega \Rightarrow f^2 \in C_\omega$), and Theorem (1.9) below states, that for $x \in M$, Pr_x-a.a. paths the spectrum of G or equivalently of its "inverse" $\frac{1}{2}\Delta$ can be obtained from C_ω. Thus for such paths ω all the information on the geometry of M, that is furnished by the spectrum of Δ can be extracted from C_ω, by simply using the ergodic theorem and the universal $\log_2$-law (1.8).

(1.9) **Theorem.** *For all $x \in M$, Pr_x-a.a. ω the following hold:*

(1) *For all $f \in C_\omega$, $a_\omega(f) \underset{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,ds$ exists and*

$$b_\omega(f) \underset{\text{def}}{=} \overline{\lim_{t\to\infty}} \frac{\int_0^t f(s)\,ds - t\,a_\omega(f)}{\sqrt{2t\log\log t}}$$

is finite and nonnegative.

(2) *The function $\|f\|_\omega \underset{\text{def}}{=} \sqrt{a_\omega(f^2)}$ is a norm on the vector space C_ω with an inner product, say $(\cdot,\cdot)_\omega$. The function*

$$\langle f_1, f_2\rangle_\omega \underset{\text{def}}{=} \tfrac{1}{4}\{b_\omega^2(f_1+f_2) - b_\omega^2(f_1-f_2)\}$$

on $C_\omega \times C_\omega$ is bilinear, and symmetric. Moreover $\sup_{f\in C_\omega} \frac{\langle f,f\rangle_\omega}{\|f\|_\omega^2} < \infty$.

(3) *If $\{L_\omega, (\cdot,\cdot)_\omega\}$ denotes the completion of the inner product space $\{C_\omega, (\cdot,\cdot)_\omega\}$ and $G_\omega: L_\omega \to L_\omega$ denotes the uniquely determined bounded linear operator such that $2(G_\omega f_1, f_2)_\omega = \langle f_1, f_2\rangle_\omega$ for $f_1, f_2 \in C_\omega$, then G_ω and G have the same spectrum.*

Theorem (1.9) is a corollary of Theorem (1.7). Notice first that ω-paths which are in the ω-set of (1.3) are dense in M. For paths which are also in the ω-set of (1.8) (i.e. for all $x \in M$, Pr_x-a.a. ω-paths) the mapping $f \to \sqrt{m_0}\,\phi(\omega,f)$ provides an isomorphism between the space $C^\infty(M)$ endowed with the two bilinear forms $(\cdot,\cdot)_{L^2}$ and $2(G\cdot,\cdot)_{L^2}$ and the space C_ω endowed with the two bilinear forms $(\cdot,\cdot)_\omega$ and $\langle\cdot,\cdot\rangle_\omega$. It follows that for such ω, the systems $\{L^2(M), (\cdot,\cdot)_{L^2}, G\}$ and $\{L_\omega, (\cdot,\cdot)_\omega, G_\omega\}$ are isomorphic.

As for the extraction of information about the geometry of (M, g) from the spectrum of Δ or equivalently of G or G_ω we only mention the following well-known approach [8]:

If $\{\lambda_n, n \geqq 0\}$ denote the eigenvalues of $-\Delta$ (including the simple eigenvalue $\lambda_0 = 0$), the function $\psi(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t}$ exists for $t > 0$ and has an asymptotic expansion of the form $(4\pi t)^{-\frac{d}{2}} \sum_{\nu=0}^{\infty} \alpha_\nu t^\nu$ as $t \to 0^+$. Here the α_ν are (in general metric) invariants, to be precise, integrals over M of polynomials in the curvature and their covariant derivatives. In particular $\alpha_0 = \lim_{t \to 0^+} (4\pi t)^{\frac{d}{2}} \psi(t) = m_0$, the volume of M, and if $d=2$, $\alpha_1 = \lim_{t \to 0^+} t\{(4\pi t)^{\frac{d}{2}} \psi(t) - \alpha_0\} = \frac{\pi}{3} \times$ Euler characteristic of M.

It follows from our Theorem (1.9) that a "typical" path can "recognize" the eigenvalues of G, hence the λ_n, thereby the function ψ, hence all α_ν as well as $d = -2 \lim_{t \to 0^+} \frac{\log \psi(t)}{\log t}$, the dimension of M.

§ 2. Green Kernel and Sobolev Spaces

A function p: $(0, \infty) \times M \times M \to \mathbb{R}^1$ is called a fundamental solution of (1.1) if it is a C^1 function in the first, a continuous function in the second and a C^2 function in the third variable, if it satisfies (1.1) and if in addition $\lim_{t \to 0} \int_M p(t,x,y) f(y)\, dm(y) = f(x)$ for all $f \in C(M)$, $x \in M$. A fundamental solution p of (1.1) was constructed in [6] with the method of parametrix (see also [5]). It is well-known that p is the only fundamental solution of (1.1), that $p \in C^\infty((0,\infty) \times M \times M)$, that $p > 0$, that $p(t, \cdot, \cdot)$ is symmetric for all $t > 0$, that $\int_M p(t,x,y)\, dm(y) = 1$ for all $t > 0$, $x \in M$. Moreover p satisfies the Chapman-Kolmogorov equation.

We recall the following estimate for large t from [1]: There exist $\alpha > 0$, $C > 0$ such that

$$\sup_{x,y \in M} |p(t,x,y) - m_0^{-1}| \leqq C e^{-\alpha t}, \quad t \geqq 1. \tag{2.1}$$

For small t we shall use a different estimate. In [6] it is essentially shown that for all $n \geqq 1$, there exists C such that

$$p(t,x,y) \leqq (2\pi t)^{-\frac{d}{2}} e^{-\frac{[r(x,y)]^2}{2t}} + C t^n, \quad x, y \in M,\ t \leqq 1. \tag{2.2}$$

Here $r(x,y)$ denotes the geodesic distance of x and y.

In [1] Baxter and I introduced the Green kernel

$$g(x,y) = \int_0^\infty \{p(t,x,y) - m_0^{-1}\}\, dt, \quad x, y \in M,\ x \neq y. \tag{2.3}$$

Symmetry in x and y for p implies symmetry for g. Obviously g satisfies (1.6b). Moreover $g(x, \cdot)$ is continuous on $M - \{x\}$, since $\int_1^\infty \{p(t,x,y) - m_0^{-1}\}\, dt$ is continuous on M because of (2.1) and $\int_0^1 \{p(t,x,y)-m_0^{-1}\}\,dt$ is continuous on $M-\{x\}$ because of $p(t,x,\cdot)\leqq C\{t^{-\frac{d}{2}}e^{-\frac{\alpha}{t}}+t^n\}$, outside a neighbourhood of x. Also $g(x,\cdot)$ satisfies (1.6a) in distribution sense, which follows from

$$\tfrac{1}{2}\int_M (\Delta\phi)(y)\,dm(y)\int_\varepsilon^T \{p(t,x,y)-m_0^{-1}\}\,dt = \int_M \phi(y)\{p(T,x,y)-p(\varepsilon,x,y)\}\,dm(y),$$

a consequence of (1.1), by letting $\varepsilon\to 0$, $T\to\infty$. We conclude from Weyl's lemma that $g\in C^\infty$ off the diagonal of $M\times M$.

For $f\in L^1(M)$, Gf is defined m-a.e. by (1.5), and $\int_M Gf\,dm=0$. For every bounded measurable $f\colon M\to\mathbb{R}^1$, the function Gf is continuous because

$$\sup_{x\in M}\int_M |f(y)|\,dm(y)\int_0^\delta |p(t,x,y)-m_0^{-1}|\,dt$$

is arbitrarily small for sufficiently small $\delta>0$ and

$$\int_M f(y)\,dm(y)\int_\delta^\infty \{p(t,\cdot,y)-m_0^{-1}\}\,dt$$

is continuous by (2.1). Also we have for $f\in L^1(M)$

$$\tfrac{1}{2}\Delta(Gf)=-f+m_0^{-1}\int_M f\,dm \tag{2.4}$$

in distribution sense. By Weyl's lemma we have $Gf\in C^\infty(M)$ for $f\in C^\infty(M)$. Since the only solutions ϕ of $\Delta\phi=0$ are the constant functions, (2.4) implies for $f\in C^\infty(M)$

$$G(\tfrac{1}{2}\Delta f)=-f+m_0^{-1}\int_M f\,dm. \tag{2.5}$$

We introduce for $\alpha>0$ the kernel

$$g_\alpha(x,y)=[\Gamma(\alpha)]^{-1}\int_0^\infty t^{\alpha-1}\{p(t,x,y)-m_0^{-1}\}\,dt, \qquad x,y\in M,\ x\neq y. \tag{2.6}$$

Obviously $g_1(x,y)=g(x,y)$, $g_\alpha(x,y)=g_\alpha(y,x)$. Since

$$\int_M \{p(t_1,x,z)-m_0^{-1}\}\{p(t_2,z,y)-m_0^{-1}\}\,dm(z)=p(t_1+t_2,x,y)-m_0^{-1}$$

and

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt=\Gamma(\alpha)\,\Gamma(\beta)\,[\Gamma(\alpha+\beta)]^{-1} \qquad \text{for } \alpha,\beta>0$$

we conclude

$$\int_M g_\alpha(x,z)\, g_\beta(z,y)\, dm(z) = g_{\alpha+\beta}(x,y) \qquad \text{for } \alpha,\beta>0, \tag{2.7}$$

i.e. the kernels $\{g_\alpha, \alpha>0\}$ form a semigroup. We conclude from (2.1) that for $\alpha>0$

$$\|g_\alpha\| \underset{\mathrm{def}}{=} \sup_{x\in M} \int |g_\alpha(x,y)|\, dm(y) < \infty. \tag{2.8}$$

Notice that for every $\alpha>0$, the kernel g_α is C^∞ off the diagonal. This follows for $\int_0^1 t^{\alpha-1}\{p(t,x,y)-m_0^{-1}\}\,dt$, because $p(\cdot,x,\cdot)\in C^\infty$ $([0,\infty)\times(M-\{x\}))$, and for $\int_1^\infty t^{\alpha-1}\{p(t,x,y)-m_0^{-1}\}\,dt$ because

$$\int_1^\infty t^{\alpha-1}\{p(t,x,y)-m_0^{-1}\}\,dt$$

$$= \int_M dm(z)\, p(1,z,y) \int_0^\infty dt (1+t)^{\alpha-1}\{p(t,x,z)-m_0^{-1}\}$$

(2.9) **Lemma.** *For every dimension $d\geqq 1$, every real $\alpha>0$, there exists c such that for $x,y\in M$*

$$|g_\alpha(x,y)| \leqq \begin{cases} c & \text{if } \alpha>\frac{d}{2} \\ c\{1+\log^- r(x,y)\} & \text{if } \alpha=\frac{d}{2} \\ c[r(x,y)]^{-d+2\alpha} & \text{if } \alpha<\frac{d}{2} \end{cases}$$

where $\log^- t = \max\{0, -\log t\}$.

Proof. We have by (2.2) with $n=0$, after a change of the integration variable,

$$\int_0^1 t^{\alpha-1}\, p(t,x,y)\,dt \leqq c + c[r(x,y)]^{-d+2\alpha} \int_0^{\min\{1,[r(x,y)]^{-2}\}} s^{-\frac{d}{2}-1+\alpha} e^{-\frac{1}{2s}}\,ds$$

$$+ c[r(x,y)]^{-d+2\alpha} \int_1^{\max\{1,[r(x,y)]^{-2}\}} s^{-\frac{d}{2}-1+\alpha}\,ds$$

In this inequality the first integral on the right side is always majorized by the finite integral $\int_0^1 s^{-\frac{d}{2}-1+\alpha} e^{-\frac{1}{2s}}\,ds$, whereas the second integral is majorized by the finite integral $\int_1^\infty s^{-\frac{d}{2}-1+\alpha}\,ds$ if $\alpha<\frac{d}{2}$, by $2\log^- r(x,y)$ if $\alpha=\frac{d}{2}$ and by $c\max\{[r(x,y)]^{d-2\alpha}, 1\}$ if $\alpha>\frac{d}{2}$.

In the following we let

$$L_0^2(M)=\{f\in L^2(M);\ \int_M f\,dm=0\},$$
$$C_0^\infty(M)=\{f\in C^\infty(M);\ \int_M f\,dm=0\},$$
$$C_0(M)=\{f\in C(M),\ \int_M f\,dm=0\},$$
$$B(M)=\{f\colon M\to\mathbb{R}\text{ measurable and bounded}\}.$$

We will usually suppress "M" in the notation.

Remark. In order to justify changing the order of integration in some of the arguments to follow, it will be helpful to notice that for $0\leqq\sigma<d$, $0\leqq\tau<d$

$$\int_M [r(x,z)]^{-\sigma}[r(z,y)]^{-\tau}\,dm(z)\leqq\begin{cases} c & \text{if } \sigma+\tau<d\\ c\{1+\log^- r(x,y)\} & \text{if } \sigma+\tau=d\\ c[r(x,y)]^{d-\sigma-\tau} & \text{if } \sigma+\tau>d.\end{cases}$$

The proof is straightforward (though somewhat tedious) in normal coordinates and follows along the same lines as in the case of bounded Euclidean regions. (It uses the decomposition

$$\begin{aligned}M=\{z;r(x,z)<\tfrac{1}{2}r(x,y)\}\cup\{z;r(y,z)<\tfrac{1}{2}r(x,y)\}\\ \cup\{z;\tfrac{1}{2}r(x,y)\leqq r(x,z)\leqq r(y,z)\}\\ \cup\{z;\tfrac{1}{2}r(x,y)\leqq r(y,z)\leqq r(x,z)\}).\end{aligned}$$

If we let for $\alpha>0$

$$(G_\alpha f)(x)=\int_M g_\alpha(x,y)\,f(y)\,dm(y), \tag{2.10}$$

then (2.10) defines a semigroup of bounded symmetric linear operators $G_\alpha\colon L_0^2\to L_0^2$. Notice that (using the preceding remark)

$$\|G_\alpha f\|_{L^2}\leqq\|f\|_{L^2}\cdot\|g_{2\alpha}\|^{1/2}. \tag{2.11}$$

Obviously $G_1=G$. The operators G_α are invertible because of (2.4). Invertibility and the semigroup property imply that the G_α are positive definite. Just as in the case $\alpha=1$ we have for $\alpha>0$ that $G_\alpha f\in C(M)$ if $f\colon M\to\mathbb{R}$ is bounded and measurable. We note incidentally that for $\alpha>\frac{d}{4}$, $G_\alpha\colon L_0^2\to L_0^2$ is Hilbert-Schmidt, since in this case $\iint[g_\alpha(x,y)]^2\,dm(x)\,dm(y)<\infty$. We could define the G_α by standard functional analytic methods, but it seems easier and faster to use the probabilistic approach we have taken.

(2.12) *Definition.* For real $\alpha>0$ let $H_0^\alpha=G_{\alpha/2}(L_0^2)$, endowed with pointwise addition and pointwise multiplication by scalars and with the inner product

$$\langle G_{\alpha/2}f_1,G_{\alpha/2}f_2\rangle_{H_0^\alpha}=2^\alpha(f_1,f_2)_{L^2}. \tag{2.13}$$

We write $\| \ \|_{H_0^\alpha}$ for the norm induced by $\langle \ , \ \rangle_{H_0^\alpha}$.

Obviously the spaces H_0^α are complete; moreover $H_0^\alpha \subseteq L_0^2$ and

$$\|f\|_{L^2} \leqq 2^{-\alpha/2} \|g_\alpha\|^{1/2} \|f\|_{H_0^\alpha} \quad \text{for } f \in H_0^\alpha.$$

From definition (2.12) and from (2.11) we conclude that for $\alpha_1 < \alpha_2$ we have $H_0^{\alpha_2} \subseteq H_0^{\alpha_1}$ and

$$\|f\|_{H_0^{\alpha_1}} \leqq 2^{-\frac{\alpha_2-\alpha_1}{2}} \|g_{\alpha_2-\alpha_1}\|^{1/2} \cdot \|f\|_{H_0^{\alpha_2}} \quad \text{for } f \in H_0^{\alpha_2}.$$

Since for every integer $k \geqq 1$, every $\phi \in C_0^\infty$, we have $\phi = G_k\{(-1)^k 2^{-k} \Delta^k \phi\}$, it follows that $C_0^\infty \subseteq H_0^{2k}$ and hence $C_0^\infty \subseteq H_0^\alpha$ for all real $\alpha > 0$. Since $G_k \phi \in C_0^\infty$ and since C_0^∞ is dense in L_0^2, we have that C_0^∞ is dense in H_0^{2k}, $k \geqq 1$; hence C_0^∞ is dense in all H_0^α, $\alpha > 0$. In other words, the spaces H_0^α are the completions of C_0^∞ with the norm $\| \ \|_{H_0^\alpha}$. It can be shown that they are the Sobolev spaces of [7] restricted by the metric condition $\int_M f \, dm = 0$. To this end one has to show that on C_0^∞ the norms H_0^k are equivalent to the admissible norms in [7] which are defined in terms of (non canonical) inner products on the k-jets. The rest is interpolation.

From (2.11) we also conclude that (2.10) defines a semigroup of bounded linear operators which are symmetric and positive definite, on each H_0^β for $\beta > 0$. The following lemma follows from (2.10) by application of the Cauchy-Schwarz inequality and of Lemma (2.9).

(2.14) **Lemma.** *If* $\alpha > \frac{d}{4}$, *then* (2.10) *defines a bounded linear operator* $G_\alpha : L^2 \to B$.

As one would expect one has the following Sobolev theorem.

(2.15) **Theorem.** *If* $\alpha > \frac{d}{2}$, *then the set* H_0^α *is contained in* C_0 *and*

$$\|f\|_\infty \leqq 2^{-\alpha/2} \sup_{x \in M} |g_\alpha(x,x)|^{1/2} \|f\|_{H_0^\alpha} \quad \textit{for } f \in H_0^\alpha.$$

Proof. If $\alpha > \frac{d}{2}$, then $f = G_{\alpha/2} \bar{f}$ is bounded for $\bar{f} \in L_0^2$ by Lemma (2.14), to be precise

$$\|f\|_\infty \leqq \sup_{x \in M} |g_\alpha(x,x)|^{1/2} \|\bar{f}\|_{L^2} = 2^{-\alpha/2} \sup_{x \in M} |g_\alpha(x,x)|^{1/2} \|f\|_{H_0^\alpha}.$$

Now let $\bar{\phi}_n \in C_0^\infty$ be such that $\|\bar{\phi}_n - \bar{f}\|_{L^2} \to 0$ and let $\phi_n = G_{\alpha/2} \bar{\phi}_n$. Then $\|\phi_n - f\|_\infty = \|G_{\alpha/2}(\bar{\phi}_n - \bar{f})\|_\infty \leqq C \|\bar{\phi}_n - \bar{f}\|_{L^2}$; and since $\phi_n \in C_0$, it follows that $f \in C_0$.

Remark. A refinement of the preceding argument shows that $\bigcap_{\alpha > 0} H_0^\alpha = C_0^\infty$. This implies in particular that for $\alpha > 0$, we have $G_\alpha f \in C_0^\infty$ if $f \in C_0^\infty$.

We shall now give a characterization of the functions in H_0^α in terms of their Fourier coefficients. This characterization is quite standard when M is a flat d-dimensional torus.

Denote by $0<\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \dots$ the nonzero eigenvalues of $-\Delta$ and by $\phi_1, \phi_2, \dots$ an orthonormal sequence of corresponding eigenfunctions. Thus $\Delta \phi_n = -\lambda_n \phi_n$, $\int_M \phi_{n_1} \phi_{n_2} dm = \delta_{n_1 n_2}$. Moreover $\phi_n \in C_0^\infty(M)$ and the ϕ_n are complete in L_0^2.

For $f \in L_0^2$, let $f_n = (f, \phi_n)_{L^2}$. We have $f = \sum f_n \phi_n$ in L^2. We note for later use that $G\phi_n = 2\lambda_n^{-1} \phi_n$, hence $G_\alpha \phi_n = 2^\alpha \lambda_n^{-\alpha} \phi_n$.

(2.16) *Definition.* $\phi_n^\alpha = \lambda_n^{-\alpha/2} \phi_n$, $n \geqq 1$, $\alpha > 0$.

From $\langle G_{\alpha/2} f, \phi_n^\alpha \rangle_{H_0^\alpha} = 2^{\alpha/2} f_n$ for $f \in L_0^2$, we conclude the following

(2.17) **Theorem.**

(1) *For all* $\alpha > 0$, *the functions* $\{\phi_n^\alpha, n \geqq 1\}$ *form a complete orthonormal system in* H_0^α.

(2) *A function* $f \in L_0^2$ *belongs to* H_0^α *iff* $\sum_{n=1}^{\infty} \lambda_n^\alpha f_n^2 < \infty$.

(3) *For* $f \in H_0^\alpha$, $\|f\|_{H_0^\alpha}^2 = \sum_{n=1}^{\infty} \lambda_n^\alpha f_n^2$.

This theorem implies immediately

(2.18) **Corollary.** *For* $f \in H_0^1$, *the vectorfield* grad f *exists weakly and* $\|\operatorname{grad} f\| \in L^2$. *Moreover for* $f_1, f_2 \in H_0^1$

$$\langle f_1, f_2 \rangle_{H_0^1} = \int_M \operatorname{grad} f_1 \cdot \operatorname{grad} f_2 \, dm \tag{2.19}$$

Here $\| \ \|$ denotes the g-norm of a tangent vector. (2.19) implies that for $f_1 \in H_0^1$, $f_2 \in L^2$, $\int_M \operatorname{grad} f_1 \cdot \operatorname{grad} G f_2 \, dm = 2 \int_M f_1 f_2 \, dm$.

§ 3. A Universal Law of the Iterated Logarithm

In this section we shall use a version of Weyl's celebrated theorem on the distribution of the eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \dots$ of the Laplacian $-\Delta$ on M. According to this theorem there exists $\gamma > 0$ such that

$$\lambda_n \sim \gamma n^{\frac{2}{d}}. \tag{3.1}$$

We will also need Theorem (1.1) of Hörmander [3], by which for corresponding L^2-orthonormal eigenfunctions ϕ_n

$$\Big| \sum_{\lambda_n \leqq \lambda} [\phi_n(x)]^2 - a\lambda^{\frac{d}{2}} \Big| \leqq b \lambda^{\frac{d-1}{2}}, \quad \text{all } x \in M, \ \lambda \geqq 1 \tag{3.2}$$

for some constants a and b. Actually all we need of (3.2) are suitable growth controls for ϕ_n and $\phi_n^\alpha = \lambda_n^{-\frac{\alpha}{2}} \phi_n$, namely

$$\sup_{x \in M} [\phi_n(x)]^2 \leqq A n^{\frac{d-1}{d}} \tag{3.3}$$

and

$$\sup_{x\in M}[\phi_n^\alpha(x)]^2 \leqq A_\alpha n^{-\frac{1}{d}(2\alpha+1)+1} \tag{3.4}$$

which follow from (3.1) and (3.2). Notice that (3.1) follows immediately from (3.2). It can also be derived from the asymptotic expansion of $\sum e^{-\lambda_n t}$, mentioned in §1, via Tauberian theorems.

We note incidentally that (3.1) implies compactness of the operators G_β: $H_0^\alpha \to H_0^\alpha$, $L_0^2 \to L_0^2$, and hence compactness of the embeddings $H_0^{\alpha_1} \subseteq H_0^{\alpha_2} \subseteq L_0^2$, $\alpha_2 < \alpha_1$.

We now turn to Brownian motion on (M, g) as introduced in §1, and define for bounded measurable f: $M \to \mathbb{R}^1$

$$L_t(f,\omega) = \int_0^t f(X_s(\omega))\,ds, \qquad t \geqq 0. \tag{3.5}$$

If $\alpha > \frac{d}{2}$, then for fixed $\omega \in \Omega$, $t \geqq 0$, $\int_0^t g_{\alpha/2}(\cdot, X_s)\,ds \in L_0^2$ since

$$\int dm(x)\left(\int_0^t g_{\alpha/2}(x, X_s)\,ds\right)^2 \leqq t^2 \sup_{y,z\in M}|g_\alpha(y,z)|.$$

If we define for $\alpha > \frac{d}{2}$ the H_0^α-valued process $L^\alpha(t,\omega)$ by

$$\begin{aligned} L^\alpha(t,\omega)(x) &= 2^{-\alpha}\, G_{\alpha/2}\left\{\int_0^t g_{\alpha/2}(\cdot, X_s)\,ds\right\} \\ &= 2^{-\alpha}\int_0^t g_\alpha(x, X_s)\,ds \end{aligned} \tag{3.6}$$

then

$$\langle L^\alpha(t,\omega), f\rangle_{H_0^\alpha} = L_t(f,\omega) \qquad \text{for } f \in H_0^\alpha. \tag{3.7}$$

Such L^α were introduced in [2] without the kernel g_α for Brownian motion on the flat torus. For every $\omega \in \Omega$, the H_0^α-valued process $L^\alpha(t,\omega)$ is strongly continuous in t, since

$$\|L^\alpha(t,\omega) - L^\alpha(t_0,\omega)\|_{H_0^\alpha}^2 = 2^{-\alpha}(t-t_0)^2 \sup_{y,z\in M}|g_\alpha(y,z)|.$$

If we use in H_0^α the σ-field of its Borel sets, then the function L^α: $[0,\infty) \times \Omega \to H_0^\alpha$ is progressively measurable. In [1] we studied the Central Limit Theorem and the Law of the Iterated Logarithm for the $\mathbb{R}^1$-valued variables in (3.5). In those two theorems the asymptotic variance was given by the form $2m_0^{-1}(f, Gf)_{L^2}$ which equals $2^{-\alpha+1} m_0^{-1}\langle G_{\alpha+1} f_1, f_2\rangle_{H_0^\alpha}$ for $f_1, f_2 \in H_0^\alpha$. In §5 we shall give a $\log_2$-law for the process $L^\alpha(t)$. We start now with

(3.8) **Theorem.** *For any compact C^∞ Riemannian manifold (M, g) of dimension $d \geqq 1$ and associated Brownian motion X, let the H_0^α-valued process $L^\alpha(t)$ be* *defined by* (3.6) *for* $\alpha > \frac{d}{2}$. *If* $\alpha > \max\left(d - \frac{3}{2}, \frac{d}{2}\right)$, *then for all* $x \in M$, Pr_x*-a.a.* ω *the random set* $\left\{\frac{L^\alpha(t,\omega)}{\sqrt{2t\log\log t}}, t \geqq 3\right\}$ *in* H_0^α *is conditionally norm-compact.*

Proof. First we notice that a version of Ito's formula holds. For all $x \in M$, $f \in H_0^\alpha$

$$M_t(f,\omega) \underset{\text{def}}{=} L_t(f,\omega) + (Gf)(X_t(\omega)) - (Gf)(X_0(\omega)) \tag{3.9}$$

is a Pr_x-martingale. Its increasing process is

$$\begin{aligned}\langle M_t(f), M_t(f)\rangle &= \int_0^t |\text{grad}\, Gf|^2 (X_s)\, ds \\ &= 2\int_0^t f(X_s)(Gf)(X_s)\, ds.\end{aligned} \tag{3.10}$$

Furthermore if we let $g^\alpha(x) = 2^{-\alpha} g_{\alpha+1}(x, \cdot)$, then by (2.9), $g^\alpha(x) \in H_0^\alpha$,

$$\sup_{x \in M} \|g^\alpha(x)\|^2_{H_0^\alpha} = 2^{-\alpha} \sup_{x \in M} g_{\alpha+2}(x,x) < \infty,$$

and

$$(Gf)(x) = \langle g^\alpha(x), f\rangle_{H_0^\alpha} \quad \text{for } f \in H_0^\alpha.$$

Obviously $Pr_x\left\{\omega;\ \lim_{t\to\infty} \frac{g^\alpha(X_t)}{\sqrt{2t\log\log t}} = 0\right\} = 1.$

If we define the H_0^α-valued process $M^\alpha(t,\omega)$ by

$$M^\alpha(t,\omega) = L^\alpha(t,\omega) + g^\alpha(X_t(\omega)) - g^\alpha(X_0(\omega)), \tag{3.11}$$

the theorem is proved if we prove Pr_x-almost sure conditional compactness in H_0^α of $\left\{\frac{M^\alpha(t,\omega)}{\sqrt{2t\log\log t}}, t \geqq 3\right\}$.

In view of (3.9) we have $\langle M^\alpha(t,\omega), f\rangle_{H_0^\alpha} = M_t(f,\omega)$ for all $f \in H_0^\alpha$. We let $M_t^{n,\alpha}(\omega) = M_t(\phi_n^\alpha, \omega)$. From (3.1), (3.4) and (3.10) we conclude

$$\langle M_t^{n,\alpha}, M_t^{n,\alpha}\rangle \leqq c t n^{-\frac{1}{d}(2\alpha+3)+1}. \tag{3.12}$$

If $\alpha > d - \frac{3}{2}$, $\beta \underset{\text{def}}{=} \frac{1}{d}(2\alpha+3) - 1$, then $\beta > 1$.

Let $\delta_1 \in \left(0, \frac{\beta-1}{2}\right)$, $\varepsilon_n = n^{-\frac{1}{2}-\delta_1}$. Clearly $\delta_2 \underset{\text{def}}{=} \beta - 1 - 2\delta_1 > 0$. For $\nu \geqq 2$ let $t_\nu = 2^\nu$, for $\nu \geqq 2$, $n \geqq 1$ let $\alpha_{n\nu} = \frac{\varepsilon_n n^\beta}{2c}\sqrt{\frac{2\log\log t_\nu}{t_\nu}}$ with c from (3.12). Then for $\nu \geqq 2$

$$
\begin{aligned}
&Pr_x\{\sup_{t_\nu \leqq t < t_{\nu+1}} M_t^{n,\alpha} \geqq \varepsilon_n \sqrt{2t_\nu \log\log t_\nu}\}\\
&\leqq Pr_x\left\{\sup_{t_\nu \leqq t < t_{\nu+1}} \left[\alpha_{n\nu} M_t^{n,\alpha} - \frac{1}{2}\alpha_{n\nu}^2 \langle M_t^{n,\alpha}, M_t^{n,\alpha}\rangle\right] \geqq \frac{\varepsilon_n^2 n^\beta}{2c} \log\log t_\nu\right\}\\
&\leqq Pr_x\left\{\sup_{t_\nu \leqq t < t_{\nu+1}} \exp[\alpha_{n\nu} M_t^{n,\alpha} - \tfrac{1}{2}\alpha_{n\nu}^2 \langle M_t^{n,\alpha}, M_t^{n,\alpha}\rangle] \geqq (\nu \log 2)^{\frac{\varepsilon_n^2 n^\beta}{2c}}\right\}\\
&\leqq (\nu\log 2)^{-\frac{\varepsilon_n^2 n^\beta}{2c}} = (\nu \log 2)^{-\frac{1}{2c} n^{\delta_2}},
\end{aligned}
$$

as $\exp[\dots]$ is a continuous martingale. The last term is majorized by ν^{-n^δ} for $\nu \geqq 2$, $n \geqq n_0 \geqq 2$ with $\delta = \frac{1}{2}\delta_2$.

Hence for $\nu_0 \geqq 2$

$$
\begin{aligned}
&Pr_x\left\{\sup_{t \geqq 2^{\nu_0}} \frac{M_t^{n,\alpha}}{\sqrt{2t\log\log t}} \geqq \varepsilon_n\right\}\\
&\quad\leqq \sum_{\nu=\nu_0}^{\infty} Pr_x\{\sup_{t_\nu \leqq t < t_{\nu+1}} M_t^{n,\alpha} \geqq \varepsilon_n \sqrt{2t_\nu \log\log t_\nu}\} \leqq \sum_{\nu=\nu_0}^{\infty} \nu^{-n^\delta}.
\end{aligned}
$$

Replacing ϕ_n^α by $-\phi_n^\alpha$ we conclude for $\nu_0 \geqq 2$

$$
Pr_x\left\{\sup_{t \geqq 2^{\nu_0}} \frac{|M_t^{n,\alpha}|}{\sqrt{2t\log\log t}} \geqq \varepsilon_n\right\} \leqq 2\sum_{\nu=\nu_0}^{\infty} \nu^{-n^\delta}.
$$

It follows that for $\nu_0 \geqq 2$

$$
Pr_x\left\{\omega;\ \exists n \geqq n_0, \exists t \in [2^{\nu_0}, \infty) \text{ such that } \frac{|M_t^{n,\alpha}|}{\sqrt{2t\log\log t}} > \varepsilon_n\right\} \leqq 2\sum_{n=n_0}^{\infty}\sum_{\nu=\nu_0}^{\infty} \nu^{-n^\delta}
$$

and since $\sum_{n=2}^{\infty}\sum_{\nu=2}^{\infty} \nu^{-n^\delta} < \infty$,

$$
Pr_x\left\{\omega;\ \forall \nu_0 \geqq 2, \exists n \geqq n_0, \exists t \in [2^{\nu_0}, \infty) \text{ such that } \frac{|M_t^{n,\alpha}|}{\sqrt{2t\log\log t}} > \varepsilon_n\right\} = 0 \quad \text{i.e.}
$$

$$
Pr_x\left\{\omega;\ \exists \nu_0 \geqq 2, \text{ such that } \forall n \geqq n_0, \forall t \in [2^{\nu_0}, \infty), \frac{|M_t^{n,\alpha}|}{\sqrt{2t\log\log t}} \leqq \varepsilon_n\right\} = 1. \tag{3.13}
$$

If we set $\varepsilon_n^* = \varepsilon_n$ for $n \geqq n_0$ and $\varepsilon_n^* = 2\sqrt{\frac{2}{m_0}(\phi_n^\alpha, G\phi_n^\alpha)} = 4m_0^{-\frac{1}{2}}\lambda_n^{-\frac{\alpha+1}{2}}$ for $n < n_0$, then (3.13) and (1.4) imply that Pr_x-a.e. there is $\nu_0 \geqq 2$ (depending on ω) such that

$$
\left\{\frac{M^\alpha(t,\omega)}{\sqrt{2t\log\log t}}, t \geqq 2^{\nu_0}\right\} \subseteq \{f \in H_0^\alpha;\ |\langle f, \phi_n^\alpha\rangle_{H_0^\alpha}| \leqq \varepsilon_n^* \quad \text{all} \quad n \geqq 1\}.
$$

Since the last set is compact in H_0^α, the proof is finished.

(3.14) *Remark.* For any $\omega \in \Omega$, $\left\{\frac{L^\alpha(t,\omega)}{\sqrt{2t\log\log t}}, t \geqq 3\right\}$ is conditionally $\|\ \|_{H_0^\alpha}$-compact iff

$$(1)\ \sup_{t \geqq 3} \frac{\|L^\alpha(t,\omega)\|_{H_0^\alpha}}{\sqrt{2t\log\log t}} < \infty \quad \text{and}$$

$$(2)\ \lim_{N\to\infty} \sup_{t \geqq 3} \frac{\|L^\alpha(t,\omega) - \Pi_N^\alpha L^\alpha(t,\omega)\|_{H_0^\alpha}}{\sqrt{2t\log\log t}} = 0,$$

where Π_N^α: $H_0^\alpha \to H_0^\alpha$ denotes the projection of H_0^α onto the subspace spanned by $\phi_1^\alpha, \ldots, \phi_N^\alpha$. Notice that (2) follows from the conditional compactness of $\left\{\frac{L^\alpha(t,\omega)}{\sqrt{2t\log\log t}}, t \geqq 3\right\}$, because the Π_N^α are uniformly equicontinuous. By Theorem (3.8) we have proved

$$(3.15)\qquad Pr_x\left\{\lim_{N\to\infty} \sup_{t \geqq 3} \frac{\|L^\alpha(t,\omega) - \Pi_N^\alpha L^\alpha(t,\omega)\|_{H_0^\alpha}}{\sqrt{2t\log\log t}} = 0\right\} = 1, \qquad x \in M.$$

We will use this remark in Section 5.

Theorem (3.8) allows us to prove a universal $\log_2$-law.

(3.16) **Theorem.** *For any compact C^∞ Riemannian manifold (M, g) of dimension $d \geqq 1$ and associated Brownian motion X and $L_t(f)$ defined by* (3.5), *we have*

$$(3.17)\quad Pr_x\left\{\omega;\ \text{cluster set}_{t\to\infty} \frac{L_t(f,\omega)}{\sqrt{2t\log\log t}} = \left[-\sqrt{\frac{2}{m_0}(f, Gf)_{L^2}}, \sqrt{\frac{2}{m_0}(f, Gf)_{L^2}}\right]\ \text{all } f \in H_0^\alpha\right\} = 1, \qquad x \in M$$

if $\alpha > \max\left(d - \frac{3}{2}, \frac{d}{2}\right)$.

Proof. By Theorem (3.8) we have for all $x \in M$, Pr_x-a.a. ω

$$C(\omega) = \sup_{t \geqq 3} \frac{\|L^\alpha(t)\|_{H_0^\alpha}}{\sqrt{2t\log\log t}} < \infty.$$

We conclude from $|\langle L^\alpha(t), f\rangle_{H_0^\alpha}| \leqq \|f\|_{H_0^\alpha} \|L^\alpha(t)\|_{H_0^\alpha}$ that for all $x \in M$, Pr_x-a.a. ω

$$\left|\frac{\int_0^t f(X_s)\,ds}{\sqrt{2t\log\log t}}\right| \leqq \|f\|_{H_0^\alpha}\, C(\omega), \qquad t \geqq 3,\ f \in H_0^\alpha.$$

The theorem follows now from our result (1.4) applied to a countable dense set of functions in H_0^α.

§4. The Law of the Iterated Logarithm for Vector Functions

Once we have the $\log_2$-law (1.4) of [1] for a single function, a $\log_2$-law for a single vector function follows at once by the trick of considering arbitrary linear combinations of the components (cf. [4]). We shall give some details.

For $f_1, \dots, f_n \in L_0^2$, the matrix $((f_i, Gf_j)_{L^2}, i,j=1,\dots,n)$ is nonnegative definite. It is positive definite iff $f_1, \dots, f_n$ are linearly independent. For linearly independent $f_1, \dots, f_n \in L_0^2$ we define the ellipsoid $E_{f_1,\dots,f_n}$ by

$$E_{f_1,\dots,f_n} = \left\{(\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n, \sum_{i,j=1}^n a_{ij}\zeta_i\zeta_j \leqq 1\right\},$$

where $\left(\frac{m_0}{2} a_{ij}\right)$ is the inverse matrix of $((f_i, Gf_j)_{L^2}, i,j=1,\dots,n)$.

(4.1) **Theorem.** *For all $n \geqq 1$, all linearly independent bounded measurable functions $f_1, \dots, f_n: M \to \mathbb{R}$ we have for all $x \in M$*

$$\text{(4.2)} \qquad Pr_x\left\{\mathbb{R}^n - \underset{t\to\infty}{\text{cluster set}} \frac{(L_t(f_1), \dots, L_t(f_n))}{\sqrt{2t\log\log t}} = E_{f_1,\dots,f_n}\right\} = 1.$$

Proof. Let $n \geqq 2$. It is sufficient to prove (4.2) for the special case where $(f_i, Gf_j) = \delta_{ij}$ for $i,j=1,\dots,n$. The general case can be reduced to this special case by a linear transformation. In the special case

$$E_{f_1,\dots,f_n} = B_n \underset{\text{def}}{=} \left\{(\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n, \sum_{i=1}^n \zeta_i^2 \leqq \frac{2}{m_0}\right\}.$$

For $\zeta \in \partial B_n$, define $\ell_\zeta: \mathbb{R}^n \to \mathbb{R}$ by $\ell_\zeta(\eta) = \sqrt{\frac{m_0}{2}} \sum_{i=1}^n \zeta_i \eta_i$ and $\bar{f}_\zeta: M \to \mathbb{R}$ by $\bar{f}_\zeta = \ell_\zeta(f_1, \dots, f_n) = \sqrt{\frac{m_0}{2}} \sum_{i=1}^n \zeta_i f_i$.

Obviously

$$\text{(4.3)} \qquad \ell_\zeta < \sqrt{\frac{2}{m_0}} \quad \text{on} \quad B_n - \{\zeta\}, \ell_\zeta(\zeta) = \sqrt{\frac{2}{m_0}},$$

and for any dense set D on ∂B_n

$$\text{(4.4)} \qquad \bigcap_{\zeta \in D} \left\{\eta;\, |\ell_\zeta(\eta)| \leqq \sqrt{\frac{2}{m_0}}\right\} = B_n,$$

and by our $\log_2$-law (1.4) applied to $\bar{f}_\zeta$, if D_0 is a countable dense set on ∂B,

(4.5)
$$Pr_x\left\{\underset{t\to\infty}{\text{cluster set}}\,\ell_\zeta\left(\frac{(L_t(f_1),\dots,L_t(f_n))}{\sqrt{2t\log\log t}}\right)=\left[-\sqrt{\frac{2}{m_0}},\,+\sqrt{\frac{2}{m_0}}\right],\quad \text{all}\quad \zeta\in D_0\right\}=1,\qquad x\in M.$$

Notice that $(\bar{f}_\zeta, G\bar{f}_\zeta)_{L^2}=1$. If we denote by $A_{f_1,\dots,f_n}(\omega)$ the cluster set in (4.2) and by $\Omega_{f_1,\dots,f_n}$ the ω-set in (4.5), we have for $\omega\in\Omega_{f_1,\dots,f_n}$:

(1) $A_{f_1,\dots,f_n}(\omega)\subseteq B_n$,

(2) $\partial B_n\subseteq A_{f_1,\dots,f_n}(\omega)$.

Notice that (1) follows from $A_{f_1,\dots,f_n}(\omega)\subseteq\left\{\eta;|\ell_\zeta(\eta)|\leqq\sqrt{\frac{2}{m_0}}\right\}$ all $\zeta\in D_0$, for $\omega\in\Omega_{f_1,\dots,f_n}$ and from (4.4). For the proof of (2) we observe that (1) and (4.3) imply $\zeta\in A_{f_1,\dots,f_n}(\omega)$ for $\zeta\in D_0$, $\omega\in\Omega_{f_1,\dots,f_n}$. Using (1) and (2) as well as the projection Π^n: $\mathbb{R}^n\to\mathbb{R}^{n-1}$,

$$\Pi^n(\zeta_1,\dots,\zeta_{n-1},\zeta_n)=(\zeta_1,\dots,\zeta_{n-1}),$$

we conclude that

$$A_{f_1,\dots,f_{n-1}}(\omega)=B_{n-1}\qquad\text{for }\omega\in\Omega_{f_1,\dots,f_n}.$$

In view of our universal $\log_2$-law (3.17), the proof of the preceding theorem also gives the universal version for vector functions.

(4.6) **Theorem.** *If* $\alpha>\max\left(d-\frac{3}{2},\frac{d}{2}\right)$, *then for all* $x\in M$

(4.7)
$$Pr_x\left\{\mathbb{R}^n-\underset{t\to\infty}{\text{cluster set}}\,\frac{(L_t(f_1),\dots,L_t(f_n))}{\sqrt{2t\log\log t}}=E_{f_1,\dots,f_n}\right.$$
$$\left.\textit{all } n\geqq 1,\ \textit{all linearly independent } f_1,\dots,f_n\in H_0^\alpha\right\}=1.$$

§ 5. A Function Space Version of the Law of the Iterated Logarithm

In this section we shall give a $\log_2$-law for the H_0^α-valued process $L^\alpha(t)$ as was done in [2] for the special case of the flat torus. This result can be obtained from the general Theorem (3.1) in Kuelbs [4] and our Theorem (3.8). Such an argument was used in [2] for the flat torus. It is probably simpler for the reader if we restate Kuelbs' argument in the context of our paper.

If we let

$$K_\alpha=\{f\in H_0^{2\alpha+1};\ \tfrac{1}{2}m_0^{1/2}\|f\|_{H_0^{2\alpha+1}}\leqq 1\}$$
$$=2^{-\frac{\alpha}{2}+\frac{1}{2}}m_0^{-\frac{1}{2}}G_{\frac{\alpha+1}{2}}\{f\in H_0^\alpha;\ \|f\|_{H_0^\alpha}\leqq 1\},$$

then K_α is a compact symmetric convex set in H_0^α. $\left(\text{Notice that } G_{\frac{\alpha+1}{2}}: H_0^\alpha \to H_0^\alpha\right.$ is of Hilbert-Schmidt type for $\left.\frac{\alpha+1}{2} > \frac{d}{4}.\right)$

Applying Theorem (4.1) to the functions ϕ_n^α we conclude

(5.1) $$Pr_x\left\{\| \; \|_{H_0^\alpha} - \underset{t\to\infty}{\text{cluster set}} \frac{\Pi_N^\alpha L^\alpha(t)}{\sqrt{2t \log\log t}} = \Pi_N^\alpha(K_\alpha), \text{ all } N \geq 1\right\} = 1, \qquad x \in M,$$

where the Π_N^α are the projections of Remark (3.14).

From (5.1) and (3.15) we have immediately

(5.2) **Theorem.** *Under the assumptions of Theorem* (3.8)

$$Pr_x\left\{\| \; \|_{H_0^\alpha} - \underset{t\to\infty}{\text{cluster set}} \frac{L^\alpha(t)}{\sqrt{2t \log\log t}} = K_\alpha\right\} = 1.$$

Remark. In the original notation of Kuelbs' Theorem (3.1) in [4] we would set his $B = B^* = H_0^{\alpha+1}$, $S = 2^{-\alpha+1} m_0^{-1} G_{\alpha+1}$. His space H_μ can be identified as $H_0^{2\alpha+1}$, except that $\|f\|_\mu = 2^{-1} m_0^{1/2} \|f\|_{H_0^{2\alpha+1}}$.

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Received September 10, 1982; in revised form May 25, 1983