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Strong Limit Theorems for Oscillation Moduli of the Uniform Empirical Process

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Summary. Let $U_n(t)=n^{\frac{1}{2}}(\Gamma_n(t)-t)$, $0\leqq t\leqq 1$, denote the uniform empirical process based on the first n of a sequence $\xi_1, \xi_2, \ldots$ of iid uniform $(0,1)$ random variables where $\Gamma_n(t)=n^{-1}\sum_{i=1}^{n} 1_{[0,t]}(\xi_i)$ is the empirical distribution function. The oscillation modulus of U_n is defined by

$$\omega_n(a)=\sup\{|U_n(t+h)-U_n(t)|: 0\leqq t\leqq 1-h, h\leqq a\},$$

and the Lipschitz-$\frac{1}{2}$ modulus of U_n is defined by

$$\tilde{\omega}_n(a)=\sup\{|U_n(t+h)-U_n(t)|/h^{\frac{1}{2}}: 0\leqq t\leqq 1-h, a\leqq h\leqq 1\}.$$

Strong limit theorems are presented for both $\omega_n(a)$ and $\tilde{\omega}_n(a)$ with $a=a_n\to 0$ at various rates. For 'short' intervals with $a_n=cn^{-1}\log n$, $c>0$, the results are related to Erdos-Rényi strong laws of large numbers; at the other extreme, for 'long' intervals with $a_n=1/(\log n)^c$, $c>0$, the results are related to laws of the iterated logarithm for U_n.

1. Introduction

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent uniform $(0,1)$ random variables. For each $n\geqq 1$ and $u\in[0,1]$ let $\Gamma_n(u)$ denote the empirical distribution function based on $\xi_1,\ldots,\xi_n$ and let $U_n(u)=n^{\frac{1}{2}}(\Gamma_n(u)-u)$ denote the empirical process.

It is well known that there exists a sequence of empirical processes U_n and a Brownian bridge U all defined on the same probability space (the Skorokhod (1956) construction) such that

$$\|U_n-U\|\equiv \sup_{0\leqq u\leqq 1}|U_n(u)-U(u)|\to 0 \quad \text{a.s.}$$

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For any half open - half closed interval $C=(s,t]$ with $0 \leqq s \leqq t \leqq 1$ set $|C|=t-s$. Let $\mathscr{C}$ denote the class of all such intervals. For any real valued function f defined on $[0,1]$ let $f(C) \equiv f(s,t) \equiv f(t)-f(s)$ whenever $C=(s,t] \in \mathscr{C}$. Shorack and Wellner [11] recently characterized the class of all nonnegative functions q defined on $[0,1]$ such that for all $\varepsilon>0$

$$\sup\left\{\frac{|U_n(C)-U(C)|}{q(|C|)}: \varepsilon n^{-1}\log n \leqq |C| \leqq 1\right\} \xrightarrow[p]{} 0 \tag{1}$$

for the Skorokhod construction. As demonstration of the potential applicability of their results, they proposed a weighted interval version of the Cramér-von Mises test and derived its limiting distribution as a functional of the Brownian bridge.

In a closely related paper, Stute [13] investigated strong limit theorems for the oscillation modulus $\omega_n(a)$ of U_n which is defined by

$$\omega_n(a)=\sup\{|U_n(C)|: |C| \leqq a\}.$$

Stute [13] showed that for sequences of positive constants $\{a_n\}$ which satisfy

$S1$. $a_n \searrow 0$ and $n a_n \nearrow \infty$;

$S2$. $\log(1/a_n)/\log\log n \to \infty$; and

$S3$. $\log(1/a_n)/(n a_n) \to 0$;

it follows that

$$\lim_{n\to\infty} \frac{\omega_n(a_n)}{(2a_n \log(1/a_n))^{\frac{1}{2}}}=1 \quad \text{a.s.} \tag{2}$$

(With respect to $S1$–$S3$, one should also note Chan's theorem 1.14.2 of [4].) Stute applied (2) along with related results to obtain rates of convergence for various types of density estimators.

Let $x^+=\max(0,x)$ and $x^-=\max(0,-x)$. For $0<a<1$, set

$$\omega_n^+(a)=\sup\{U_n^+(s,t): 0 \leqq t-s \leqq a\},$$

and

$$\omega_n^-(a)=\sup\{U_n^-(s,t): 0 \leqq t-s \leqq a\}.$$

It will be shown in the following section that $\omega_n^+(a)$ and $\omega_n^-(a)$ behave differently, at least for certain choices of $a=a_n$.

Shorack and Wellner [11] and Stute [13] also investigated the behavior of the Lipschitz-$\frac{1}{2}$ modulus of the empirical processes U_n, which is defined, for $0<a \leqq b \leqq 1$, by

$$\tilde{\omega}_n(a,b)=\sup\{|U_n(t,t+h)|/h^{\frac{1}{2}}: a \leqq h \leqq b \text{ and } 0 \leqq t \leqq 1-h\}.$$

(When $b=1$ we will write $\tilde{\omega}_n(a) \equiv \tilde{\omega}_n(a,1)$.) Stute [13] showed that for any sequence of positive constants $\{a_n\}$ that satisfy Conditions $S1$–$S3$ given above that whenever $0 \leqq c \leqq \bar{c} < \infty$

$$\lim_{n\to\infty} \frac{\tilde{\omega}_n(\underline{c}\, a_n, \bar{c}\, a_n)}{(2\log(1/a_n))^{\frac{1}{2}}}=1 \quad \text{a.s.} \tag{3}$$

The choices of $a_n = n^{-1}\log n$ and $a'_n = n^{-1}(\log n)^\alpha$ where $-\infty < \alpha < 1$ do not satisfy Stute's conditions, but Shorack and Wellner were able to show that there exist finite positive constants M and M_α such that

$$\limsup_{n\to\infty} \frac{\tilde{\omega}_n(a_n)}{(\log n)^{\frac{1}{2}}} \leqq M \quad \text{a.s.} \tag{4}$$

and

$$\limsup_{n\to\infty} \frac{\tilde{\omega}_n(a'_n)\log\log n}{(\log n)^{1-\alpha/2}} \leqq M_\alpha \quad \text{a.s.} \tag{5}$$

for these two sequences respectively. (See page 216 of Csörgo and Révész [4] for an Erdos-Rényi type result for the empirical process closely related to these results.)

In this paper, we will give a detailed description of the limiting behavior of $\omega_n(a_n)$ and $\tilde{\omega}_n(a_n)$ for sequences of positive constants $\{a_n\}$ converging to zero at a variety of rates. For instance, we will obtain the exact limit in (4) and a refinement of (5). We also give, in Sect. 4, an approach to theorems for $\omega_n(a_n)$ based upon the strong approximation of Komlós, Major, and Tusnády (1975) and results of Chan (1977) concerning continuity moduli of a Kiefer process.

2. Strong Limit Theorems for the Oscillation Modulus of the Uniform Empirical Process

In this section we will be concerned with the asymptotic behavior of $\omega_n(a_n)$ for sequences $\{a_n\}$ which converge to zero at rates both faster and slower than allowed by Stute's conditions $S1$–$S3$. The sequence $a_n = cn^{-1}\log n$ fails to satisfy $S3$ (it decreases too rapidly), while $a_n = 1/(\log n)^c$, $0 < c < \infty$ fails to satisfy $S2$ (it decreases too slowly). If c is replaced by $c_n \nearrow \infty$ in either case, then $S1$–$S3$ hold.

Let $h(x) = x(\log x - 1) + 1$ for $x > 0$. For any $c > 0$ let $\beta_c^+ > 1$ denote the solution to $h(\beta_c^+) = 1/c$. It can be shown by elementary methods that β_c^+ has the following properties as a function of c:

$B1$. $\beta_c^+ \nearrow$ from 1 to ∞ as $c \searrow 0$.

$B2$. $(c/2)^{\frac{1}{2}}(\beta_c^+ - 1) \searrow 1$ as $c \nearrow \infty$.

Theorem 1. *Let $\{a_n\}$ be a sequence of positive constants less than* 1.

(I) *If $a_n = (c\log n)/n$ with $0 < c < \infty$, then*

$$\lim_{n\to\infty} \frac{\omega_n(a_n)}{(2a_n\log(1/a_n))^{\frac{1}{2}}} = (c/2)^{\frac{1}{2}}(\beta_c^+ - 1) \quad a.s. \tag{6}$$

$$\lim_{n\to\infty} \frac{\omega_n^+(a_n)}{(2a_n\log(1/a_n))^{\frac{1}{2}}} = (c/2)^{\frac{1}{2}}(\beta_c^+ - 1) \quad a.s., \tag{6+}$$

and

$$\lim_{n\to\infty} \frac{\omega_n^-(a_n)}{(2a_n\log(1/a_n))^{\frac{1}{2}}} = 1 \quad a.s. \tag{6−}$$

(II) *If* $a_n=(c_n \log n)/n$ *where* $c_n \to 0$ *at such a rate that*

$$\log(1/c_n)/\log n \to 0, \tag{7}$$

then

$$\limsup_{n\to\infty} \frac{n^{\frac{1}{2}} \log(1/c_n)}{\log n} \omega_n(a_n) \leqq 2 \quad a.s. \tag{8}$$

(III) *If* $a_n=1/(\log n)^c$ *with* $0 \leqq c < \infty$

$$c^{\frac{1}{2}} = \liminf_{n\to\infty} \frac{\omega_n(a_n)}{(2a_n \log_2 n)^{\frac{1}{2}}} \leqq \limsup_{n\to\infty} \frac{\omega_n(a_n)}{(2a_n \log_2 n)^{\frac{1}{2}}} = (1+c)^{\frac{1}{2}} \quad a.s. \tag{9}$$

while

$$\frac{\omega_n(a_n)}{(2a_n \log_2 n)^{\frac{1}{2}}} \xrightarrow[p]{} c^{\frac{1}{2}} \quad as \quad n \to \infty.$$

In (I) and (III), c may be replaced by $c_n \to c$. We will require the following two inequalities.

Inequality 1. Let $0 < a \leqq \delta \leqq \frac{1}{2}$. Then, for every $\lambda > 0$,

$$P(\omega_n(a) \geqq \lambda \sqrt{a}) \leqq \frac{20}{a\delta^3} \exp\left(-(1-\delta)^4 \frac{\lambda^2}{2} \psi\left(\frac{\lambda}{\sqrt{na}}\right)\right)$$

where $\psi(x) = 2h(1+x)/x^2$. Moreover, for $\lambda \geqq \delta^2 (na)^{\frac{1}{2}}$

$$P(\omega_n^+(a) \geqq \lambda \sqrt{a}) \leqq \frac{2}{a\delta^3} \exp\left(-(1-\delta)^4 \frac{\lambda^2}{2} \psi\left(\frac{\lambda}{\sqrt{na}}\right)\right)$$

and

$$P(\omega_n^-(a) \geqq \lambda \sqrt{a}) \leqq \frac{2}{a\delta^3} \exp\left(-(1-\delta)^3 \frac{\lambda^2}{2}\right).$$

The proof of Inequality 1 and properties of ψ are given in the appendix.

Inequality 2. (Stute). Let $r > 0$ and $\{a_n\}$ satisfy (i) $a_n \searrow 0$, (ii) $na_n \nearrow \infty$ and (iii) $\log(1/a_n) = O(na_n)$. Then for any $\varepsilon > 0$ we can choose a $\theta > 0$ depending on ε and the bounding constant in (iii) so small that, for k sufficiently large,

$$P\left(\max_{n_{k-1} \leqq m \leqq n_k} \omega_m(a_m)/\lambda_m \sqrt{a_m} \geqq (r+2\varepsilon)\right)$$
$$\leqq 2P\left(\omega_{n_{k+1}}((1+\theta)a_{n_k})/\sqrt{a_{n_{k+1}}} \geqq (r+\varepsilon)\lambda_{n_k}\right)$$

where $\lambda_m \equiv (2\log(1/a_m))^{\frac{1}{2}}$ and $n_k \equiv [(1+\theta)^k]$. (Here $[x]$ = greatest integer less than or equal to x.)

The proof of Inequality 2 is very much the same as the proof of Lemma 2.6 of Stute [13].

Proof of Theorem 1. First consider (I). For any $\varepsilon > 0$ and $r > 0$ define

$$A_m \equiv \{\omega_m(a_m)/\sqrt{a_m} \geqq (r+2\varepsilon)(2\log(1/a_m))^{\frac{1}{2}}\}$$

where we specify r later. We seek to show that $\sum_{m=1}^{\infty} P(A_m)<\infty$, but by the maximal inequality 2 we need only show that $\sum_{k=1}^{\infty} P(D_k)<\infty$ where

$$D_k \equiv \{\omega_{n_{k+1}}((1+\theta)a_{n_k})/\sqrt{a_{n_{k+1}}} \geqq (r+\varepsilon)(2\log(1/a_{n_k}))^{\frac{1}{2}}\}$$

with $n_k=[(1+\theta)^k]$ and θ sufficiently small. Hence by Inequality 1

$$P(D_k) \leqq \frac{20}{\delta^3 a_{n_k}} \exp(-(1-\delta)^4(1+\theta)^{-2}\gamma_k(r+\varepsilon)^2 \log(1/a_{n_k}))$$

$$= \frac{20}{\delta^3} a_{n_k}^{+(1-\delta)^6(r+\varepsilon)^2\gamma_k - 1}$$

for large k and $\theta \leqq \delta$ where

$$\gamma_k = \psi\left(\frac{r+\varepsilon}{(n_{k+1}(1+\theta)a_{n_k})^{\frac{1}{2}}}(2\log(1/a_{n_k}))^{\frac{1}{2}}\right) \sim \psi\left(\frac{2^{\frac{1}{2}}(r+\varepsilon)}{c^{\frac{1}{2}}(1+\theta)}\right) \equiv \gamma. \tag{10}$$

By use of the definition of a_n we may rewrite this inequality as

$$P(D_k) \leqq \frac{20}{\delta^3}\left\{\frac{c\,k\log(1+\theta)}{(1+\theta)^k}\right\}^{(1-\delta)^6(r+\varepsilon)^2\gamma_k - 1}. \tag{11}$$

The series on the right hand side of (11) will be summable for any small $\varepsilon>0$ and sufficiently small choice of $\delta=\delta_\varepsilon$ provided r is at least as large as the solution R of the equation (note (10) and the exponent of (11))

$$1 = R^2\gamma = R^2\psi\left(\frac{2^{\frac{1}{2}}R}{c^{\frac{1}{2}}}\right) = R^2 2\left(\frac{c^{\frac{1}{2}}}{2^{\frac{1}{2}}R}\right)^2 h\left(1+\frac{2^{\frac{1}{2}}R}{c^{\frac{1}{2}}}\right) = c h\left(1+\frac{2^{\frac{1}{2}}R}{c^{\frac{1}{2}}}\right).$$

Thus $1+2^{\frac{1}{2}}R/c^{\frac{1}{2}}=\beta_c^+$, or $R=(c/2)^{\frac{1}{2}}(\beta_c^+-1)$. Since $\varepsilon>0$ can be made arbitrarily small, it follows by use of the Borel-Cantelli lemma that

$$\limsup_{n\to\infty} \frac{\omega_n(a_n)}{(2a_n\log(1/a_n))^{\frac{1}{2}}} \leqq (c/2)^{\frac{1}{2}}(\beta_c^+-1) \quad \text{a.s.}$$

In fact (6) is true; observe that

$$\lim_{n\to\infty} \frac{\omega_n(a_n)}{(2a_n\log(1/a_n))^{\frac{1}{2}}} \geqq \limsup_{n\to\infty}\left(\frac{|U_n(t,t+a_n)|}{(2a_n\log(1/a_n))^{\frac{1}{2}}} : 0 \leqq t \leqq 1-a_n\right).$$

But by the Erdos-Rényi law for the increments of the empirical process due to Komlós et al. [7] the right hand side of (12) is equal to $(c/2)^{\frac{1}{2}}(\beta_c^+-1)$ almost surely. (See page 162 of [7].)

The proofs of (6+) and (6−) are similar using the corresponding parts of Inequality 1 for ω_n^+ and ω_n^- respectively, + and − versions of Inequality 2, and noting that for $a=a_n=cn^{-1}\log n$ and $\lambda_n=(2r\log(1/a_n))^{\frac{1}{2}}$ with $r=(c/2)^{\frac{1}{2}}(\beta_c^+-1)$ or 1, the inequality $\lambda \geqq \delta^2(na)^{\frac{1}{2}}$ holds for δ sufficiently small. This completes the proof of (I).

Now assume that the a_n's are given as in (II). The conditions for the maximal Inequality 2 need not hold in the present case. But by using Inequality 1, it is straightforward to show that

$$\sum_{n=1}^{\infty} P\left(\frac{n^{\frac{1}{2}}\log(1/c_n)}{\log n}\omega_n(a_n)>r\right)<\infty \qquad \text{for any } r>2;$$

the details are left to the reader. Condition (7) can be weakened considerably if the limsup in (8) is increased beyond 2.

Finally, assume that the a_n's are given as in (III). Since in this case $\lambda_n/(n a_n)^{\frac{1}{2}}\to 0$ and the conditions for the maximal inequality hold, we have for all $\varepsilon>0$, $\delta>0$, and k sufficiently large (using Inequality 1)

$$P(D_k)\leqq\frac{20}{\delta^3}\left(\frac{1}{\log[(1+\theta)^k]}\right)^{c(1-\delta)^6(r+\varepsilon)^2-c} \tag{13}$$

where D_k is as above. The series on the right hand side of (13) will be summable for any small $\varepsilon>0$ and sufficiently small choice of $\delta=\delta_\varepsilon$ provided r is at least as large as the solution R of the equation $c(r^2-1)=1$. Thus $R=((1+c)/c)^{\frac{1}{2}}$, and we have shown that the limsup in (9) is $\leqq c^{\frac{1}{2}}R$. Our proof that the limsup in (9) is $\geqq c^{\frac{1}{2}}R$ will be given in Sect. 4. □

Remark 1. Let $\omega_n(a_n,a_n)/(2a_n\log(1/a_n))^{\frac{1}{2}}$ denote the quantity on the right hand side of inequality (12). It is clear from the proof that for a_n's as in (I) that

$$\lim_{n\to\infty}\frac{\omega_n(a_n,a_n)}{(2a_n\log(1/a_n))^{\frac{1}{2}}}=(c/2)^{\frac{1}{2}}(\beta_c^+-1) \quad \text{a.s.} \tag{14}$$

We will make use of this fact in the following section.

Remark 2. It is apparent that (9) is also true for a_n's that satisfy (a) $a_n\searrow 0$, (b) $n a_n/\log\log n\to\infty$, and (c) $c_n=\log(1/a_n)/\log\log n\to c\in(0,\infty)$. Furthermore, Stute's theorem (2) and our (9) can be unified for a_n's in this range as follows: if $S1$ and $S3$ hold, then

$$\limsup_{n\to\infty}\frac{\omega_n(a_n)}{\{2a_n(\log(1/a_n)+\log\log n)\}^{\frac{1}{2}}}=1 \quad \text{a.s.} \tag{15}$$

where $\limsup_{n\to\infty}$ may be replaced by $\lim_{n\to\infty}$ if $S2$ holds. This should be compared with both Theorem 1.2.1 on page 30, and Chan's (1977) Theorem S. 1.15.1 on page 87, of Csörgo and Révész (1981). See Sect. 4 for use of Chan's theorem in combination with the strong approximation of Komlós, Major, and Tusnády (1975) to give another approach to limit theorems for $\omega_n(a_n)$.

Remark 3. A limit theorem for the oscillation modulus of the uniform quantile process similar to Theorem 1 has been given by Mason (1983). For intervals satisfying the Conditions $S1$–$S3$, Mason shows that the oscillation modulus of the quantile process behaves essentially the same as that of the empirical process. For very short intervals, however, the behavior is somewhat different.

3. Strong Limit Theorems for the Lipschitz-$\frac{1}{2}$ Modulus of the Uniform Empirical Process

In this section we will show that $\tilde{\omega}_n(a_n)$ exhibits the same behavior as $\omega_n(a_n)$ for sequences of positive constants $\{a_n\}$ converging to zero at the rates described in the introduction and in Sect. 2. The following theorem summarizes the limiting behavior of the Lipschitz-$\frac{1}{2}$ modulus.

Theorem 2. *Let $\{a_n\}$ be a sequence of positive constants less than* 1.

(Ia) *For a_n's satisfying Conditions S1–S3*

$$\lim_{n\to\infty} \frac{\tilde{\omega}_n(a_n)}{(2\log(1/a_n))^{\frac{1}{2}}} = 1 \quad a.s. \tag{16}$$

(I) *If $a_n = (c\log n)/n$ with $0 < c < \infty$, then*

$$\lim_{n\to\infty} \frac{\tilde{\omega}_n(a_n)}{(2\log(1/a_n))^{\frac{1}{2}}} = (c/2)^{\frac{1}{2}}(\beta_c^+ - 1) \quad a.s. \tag{17}$$

(II) *If $a_n = (c_n \log n)/n$ with $c_n \to 0$ at such a rate that $\log(1/c_n)/\log n \to 0$, then*

$$\limsup_{n\to\infty} \frac{c_n^{\frac{1}{2}}\log(1/c_n)}{(\log n)^{\frac{1}{2}}} \tilde{\omega}_n(a_n) < \infty \quad a.s. \tag{18}$$

(III) *If $a_n = 1/(\log n)^c$ with $0 < c < \infty$, then*

$$\limsup_{n\to\infty} \frac{\tilde{\omega}_n(a_n)}{(2\log(1/a_n))^{\frac{1}{2}}} = \left(\frac{1+c}{c}\right)^{\frac{1}{2}} \quad a.s. \tag{19}$$

As before, our proof will require two inequalities.

Inequality 3 (Shorack and Wellner). For $0 \leqq a \leqq (1-\delta)b < b \leqq \delta \leqq \frac{1}{2}$ and $\lambda > 0$

$$P(\tilde{\omega}_n(a,b) \geqq \lambda) \leqq \frac{24}{a\delta^3} \exp\left(-(1-\delta)^4 \gamma \frac{\lambda^2}{2}\right)$$

where $\gamma \equiv \psi(2^{\frac{1}{2}}\lambda/\delta\sqrt{na})$ satisfies

$$\gamma \geqq \begin{cases} 1-\delta & \text{if } \lambda \leqq (3/2^{\frac{1}{2}})\delta^2(na)^{\frac{1}{2}} \\ \dfrac{3\delta^2(na)^{\frac{1}{2}}(1-\delta)}{2^{\frac{1}{2}}\lambda} & \text{if } \lambda \geqq (3/2^{\frac{1}{2}})\delta^2(na)^{\frac{1}{2}}. \end{cases}$$

(This is Corollary 2 of Shorack and Wellner [11].)

Inequality 4. Let $r > 0$, $0 < b \leqq 1$, and assume that $\{a_n\}$ satisfies the conditions of Inequality 2. Then for any $\varepsilon > 0$ we can choose a $\theta > 0$, depending on ε and the bounding constant in (iii), so small that for all k sufficiently large

$$P\Big(\max_{n_{k-1} \leqq m \leqq n_k} \tilde{\omega}_m(a_m, b)/\lambda_m \geqq (r+2\varepsilon)\Big) \leqq 2P(\tilde{\omega}_{n_{k+1}}(a_{n_{k+1}}, b) \geqq (r+\varepsilon)\lambda_{n_k}) \tag{20}$$

where $\lambda_m = (2\log(1/a_m))^{\frac{1}{2}}$ and $n_k = [(1+\theta)^k]$.

The proof is omitted since it is much like the proof of Lemma 2.6 of [13].

Proof of Theorem 2. We will first prove (Ia). Observe that

$$\liminf_{n\to\infty}\frac{\tilde{\omega}_n(a_n)}{(2\log(1/a_n))^{\frac{1}{2}}}\geqq\lim_{n\to\infty}\frac{\tilde{\omega}_n(a_n,a_n)}{(2\log(1/a_n))^{\frac{1}{2}}} \tag{21}$$

By Theorem 0.1 of Stute [13] the right hand side of (21) is a.s. equal to 1. Hence to complete the proof we will show that

$$\limsup_{n\to\infty}\frac{\tilde{\omega}_n(a_n)}{(2\log(1/a_n))^{\frac{1}{2}}}\leqq 1 \quad \text{a.s.} \tag{22}$$

By Cassels' [2] theorem, for each $0<b<1$

$$\lim_{n\to\infty}\frac{\tilde{\omega}_n(b,1)}{(2\log(1/a_n))^{\frac{1}{2}}}=0 \quad \text{a.s.,}$$

and hence to prove (22) we need only to show that

$$\lim_{b\to 0}\limsup_{n\to\infty}\frac{\tilde{\omega}_n(a_n,b)}{(2\log(1/a_n))^{\frac{1}{2}}}\leqq 1 \quad \text{a.s.} \tag{23}$$

To prove (23), choose any $\varepsilon>0$ and $0<b<\frac{1}{2}$ and let

$$D_k=\{\tilde{\omega}_{n_{k+1}}(a_{n_{k+1}},b)\geqq(1+\varepsilon)\lambda_{n_k}\}$$

(using the notation of Inequality 4). As in the proof of Theorem 1, it is sufficient to prove, in view of Inequality 4, that $\sum_{k=1}^{\infty}P(D_k)<\infty$. Since $\lambda_n/(na_n)^{\frac{1}{2}}\to 0$ by $S3$, we have for all $b<\delta<\frac{1}{2}$ and k sufficiently large, by Inequality 3,

$$\begin{aligned} P(D_k) &\leqq \frac{24}{\delta^3 a_{n_{k+1}}}\exp(-(1-\delta)^5(1+\varepsilon)^2\log(1/a_{n_k}))\\ &=(24/\delta^3)(a_{n_k}/a_{n_{k+1}})\,a_{n_k}^{(1-\delta)^5(1+\varepsilon)^2-1}\\ &\leqq(48/\delta^3)\,a_{n_k}^{-\nu}. \end{aligned} \tag{24}$$

for some $\nu>0$ by choosing b, and hence δ, sufficiently small. But by Stute's conditions, for all k sufficiently large $a_{n_{k+1}}^{-1}\leqq n_{k+1}$. Hence (24) is

$$\leqq(48/\delta^3)[(1+\theta)^k]^{-\nu}. \tag{25}$$

Since the series in (25) converges, (23) follows by Borel-Cantelli, and this completes the proof of (22).

Now assume that the $\{a_n\}$ are given as in (I). Choose $0<c<\infty$. First notice that by Remark 1

$$\begin{aligned} \liminf_{n\to\infty}\frac{\tilde{\omega}_n(a_n)}{(2\log(1/a_n))^{\frac{1}{2}}} &\geq \lim_{n\to\infty}\frac{\tilde{\omega}_n(a_n,a_n)}{(2\log(1/a_n))^{\frac{1}{2}}}\\ &=\lim_{n\to\infty}\frac{\omega_n(a_n,a_n)}{(2a_n\log(1/a_n))^{\frac{1}{2}}}\\ &=(c/2)^{\frac{1}{2}}(\beta_c^+-1) \quad \text{a.s.} \end{aligned}$$

Hence we need only show that

$$\limsup_{n\to\infty} \frac{\tilde{\omega}_n(a_n)}{(2\log(1/a_n))^{\frac{1}{2}}} \leqq (c/2)^{\frac{1}{2}}(\beta_c^+ - 1) \quad \text{a.s.} \tag{26}$$

By Cassels' [2] theorem and the fact that $(c/2)^{\frac{1}{2}}(\beta_c^+ - 1) > 1$ for $c > 0$ (see $B2$ above), to establish (26) it is enough to show that

$$\lim_{d\to\infty} \limsup_{n\to\infty} \frac{\tilde{\omega}_n(d b_n, d^{-1})}{(2\log(1/a_n))^{\frac{1}{2}}} \leqq 1 \quad \text{a.s.,} \tag{27}$$

and

$$\lim_{d\to\infty} \limsup_{n\to\infty} \frac{\tilde{\omega}_n(a_n, d b_n)}{(2\log(1/a_n))^{\frac{1}{2}}} \leqq (c/2)^{\frac{1}{2}}(\beta_c^+ - 1) \quad \text{a.s.} \tag{28}$$

where $b_n = n^{-1}\log n$. To prove (27) and (28), choose $0 < \varepsilon < 1$ and $d > 0$ so that

$$d^{-1} < \left(\frac{2(1+\varepsilon)^2}{d}\right)^{\frac{1}{4}} \left(\frac{1}{3}\right)^{\frac{1}{2}} < \left(\frac{8}{d}\right)^{\frac{1}{4}} \left(\frac{1}{3}\right)^{\frac{1}{2}} < \frac{1}{2}.$$

Set

$$A_n = \{\tilde{\omega}_n(d b_n, d^{-1}) \geqq (1+2\varepsilon)(2\log(1/a_n))^{\frac{1}{2}}\}$$

and

$$D_k = \{\tilde{\omega}_{n_{k+1}}(d b_{n_{k+1}}, d^{-1}) \geqq (1+\varepsilon)(2\log(1/a_{n_k}))^{\frac{1}{2}}\}.$$

Observe that

$$(n d b_n)^{\frac{1}{2}} / ((1+\varepsilon)(2\log(1/a_n))^{\frac{1}{2}}) \to (d/2)^{\frac{1}{2}}/(1+\varepsilon);$$

hence, for every δ satisfying

$$\left(\frac{2(1+\varepsilon)^2}{d}\right)^{\frac{1}{4}} \left(\frac{1}{3}\right)^{\frac{1}{2}} < \delta < \left(\frac{8}{d}\right)^{\frac{1}{4}} \left(\frac{1}{3}\right)^{\frac{1}{2}},$$

we have, for all n sufficiently large,

$$\delta^2 (n d b_n)^{\frac{1}{2}} (3/2^{\frac{1}{2}}) \geqq (1+\varepsilon)(2\log(1/a_n))^{\frac{1}{2}}.$$

Thus, by the first part of Inequality 3, for all k sufficiently large

$$P(D_k) \leqq 24 \left(\left(\frac{d}{2(1+\varepsilon)^2}\right)^{\frac{1}{4}} 3^{\frac{1}{2}} \right)^3 (a_{n_k}/a_{n_{k+1}})\, a_{n_k}^{\{1-(8/d)^{\frac{1}{4}}(1/3)^{\frac{1}{2}}\}^5 (1+\varepsilon)^2 - 1}. \tag{29}$$

By choosing d sufficiently large, the right hand side of (29) is

$$\leqq C(d, \varepsilon)(1+\theta)^{-k\nu} \tag{30}$$

for some $\nu > 0$ and finite positive constant $C(d, \varepsilon)$ dependent only on d and ε. Since the series in (30) is summable and $\varepsilon > 0$ can be made arbitrarily small, we have (27) by Inequality 4 and the Borel-Cantelli lemma.

Now to prove (28), choose $d > c$, and any sequence of partitions $c = p_{1k} < \ldots < p_{kk} = d$ such that

$$r_k = \max_{2 \leqq i \leqq k} (p_{ik}/p_{i-1,k}) \to 1 \quad \text{as} \quad k \to \infty. \tag{31}$$

Note that

$$\frac{\tilde{\omega}_n(a_n, db_n)}{(2\log(1/a_n))^{\frac{1}{2}}} = \max_{2\le i\le k}\left(\frac{\tilde{\omega}_n(p_{i-1,k}n^{-1}\log n, p_{i,k}n^{-1}\log n)}{(2\log(1/a_n))^{\frac{1}{2}}}\right)$$
$$\le r_k^{\frac{1}{2}} \max_{1\le i\le k} \frac{n^{\frac{1}{2}}\omega_n(p_{ik}n^{-1}\log n)}{(p_{ik}\log n)^{\frac{1}{2}}(2\log(1/a_n))^{\frac{1}{2}}}. \tag{32}$$

Part (I) of Theorem 1 implies that (32) converges almost surely to

$$r_k^{\frac{1}{2}} \max_{1\le i\le k} (p_{ik}/2)^{\frac{1}{2}}(\beta^+_{p_{ik}}-1) = r_k^{\frac{1}{2}}(c/2)^{\frac{1}{2}}(\beta_c^+ - 1)$$

by $B2$, and hence (31) completes the proof of (28) and part (I).

Parts (II) and (III) of Theorem 2 are proved much like parts (II) and (III) of Theorem 1, with the additional use of Cassels' theorem at the appropriate steps and noting that $\tilde{\omega}_n(a_n) \ge \tilde{\omega}_n(a_n, a_n)$. □

Remark 4. Although part (Ia) of Theorem 2 is implicit in the results of Stute [13], we have included it here for completeness.

Remark 5. Shorack [10] has used Theorem 2 to give a proof of part of a theorem of Kiefer.

4. An Approach to $\omega_n(a)$ and Stute's Theorem Via Strong Approximation

Let $K(t,s)$, $0\le t\le 1$, $0\le s<\infty$ denote a Kiefer process; i.e. a mean-zero Gaussian process with covariance $E(K(t_1,s_1)K(t_2,s_2)) = (s_1\wedge s_2)(t_1\wedge t_2 - t_1 t_2)$. The strong approximation of Komlós, Major, and Tusnády (1975) yields a single sequence $\xi_1, \xi_2, \ldots$ off *iid* Uniform $(0,1)$ *rv*'s and a Kiefer-process K defined on a common probability space with the property that, if U_n denotes the empirical process of the first n ξ's and $\mathbb{B}_n \equiv K(\cdot, n)/n^{\frac{1}{2}}$,

$$\limsup_{n\to\infty} \|U_n - \mathbb{B}_n\|/\{(\log n)^2/n^{\frac{1}{2}}\} \le \text{some } M<\infty \quad \text{a.s.} \tag{33}$$

Now let

$$\omega_{\mathbb{B}_n}(a) \equiv \sup\{|\mathbb{B}_n(C)|: |C|\le a\}$$
$$= \sup\{|\mathbb{B}_n(t+h) - \mathbb{B}_n(t)|: h\le a, 0\le t\le 1-h\}. \tag{34}$$

It then follows from a theorem of Chan (1977) (see p. 87 of Csörgó and Révész (1981) with their ε_T = our a_n, their T = our n, and their a_T *chosen* to be T = our n) that, if a_n is nonincreasing,

$$\limsup_{n\to\infty} \frac{\omega_{\mathbb{B}_n}(a_n)}{\{2a_n(1-a_n)(\log(1/a_n) + \log\log n)\}^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{35}$$

where the limsup may be replaced by lim if, in addition, $S2$: $\log(1/a_n)/\log\log n \to \infty$ holds.

Combining (33) and (35) yields the following theorem concerning the oscillation modulus $\omega_n(a)$ of U_n.

Theorem 3. *Let $a_n > 0$ be nonincreasing and satisfy*

$$\frac{(\log n)^2}{\{n a_n(\log(1/a_n)+\log\log n)\}^{\frac{1}{2}}} \to 0. \tag{37}$$

Then

$$\limsup_{n\to\infty} \frac{\omega_n(a_n)}{\{2a_n(1-a_n)(\log(1/a_n)+\log\log n)\}^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{38}$$

where the limsup *may be replaced by* lim *if* $S2$: $\log(1/a_n)/\log\log n \to \infty$ *holds.*

Proof. Let $\beta_n \equiv \{2a_n(1-a_n)(\log(1/a_n)+\log\log n)\}^{-\frac{1}{2}}$. Then

$$\begin{aligned}\beta_n \omega_n(a_n) &= \beta_n \sup\{|U_n(C)|: |C| \leqq a_n\} \\ &= \beta_n \sup\{|\mathbb{B}_n(C)|: |C| \leqq a_n\} + O\left(\frac{\beta_n(\log n)^2}{n^{\frac{1}{2}}}\right) \quad \text{a.s.} \quad \text{by (33)} \\ &= \beta_n \omega_{\mathbb{B}_n}(a_n) + o(1) \quad \text{a.s.} \quad \text{by (37).}\end{aligned}$$

Thus Chan's (1977) (35) yields (38). □

Remark 6. Taking $a_n = 1/(\log n)^c$, $c > 0$, in Theorem 3 completes the proof of (9) since, for this a_n, $\log(1/a_n) + \log\log n = \left(1 + \frac{1}{c}\right)\log(1/a_n)$. Theorem 3 goes beyond (III) of Theorem 1 in that even longer intervals than $a_n = 1/(\log n)^c$ are allowed; e.g. $a_n = a > 0$ is possible. When $a_n = a > 0$, note that Theorem 3 agrees with Cassels' (1951) theorem.

Remark 7. While Stute's conditions $S1$–$S3$ are satisfied for $a_n = n^{-1}(\log n)^3$ (and hence (2) holds), (37) fails for this a_n, and hence the approach of this section breaks down in this range of 'short' intervals. However, *if* we could replace $(\log n)^2/n^{\frac{1}{2}}$ by $(\log n)/n^{\frac{1}{2}}$ in (33), then (37) could be replaced by

$$\frac{(\log n)}{\{n a_n(\log(1/a_n)+\log\log n)\}^{\frac{1}{2}}} \to 0, \tag{39}$$

and (39) is implied by $S3$. Thus, if (33) is ever improved to hold with $(\log n)/n^{\frac{1}{2}}$, our proof of Theorem 3 provides a short proof of Stute's theorem (2). There is also another way to look at this: if (33) is ever improved to hold with $(\log n)/n^{\frac{1}{2}}$, then Stute's theorem (2) implies (this case of) Chan's theorem (35) via the improved version of (33). On the other hand, note that even an improved version of (33) will not yield (I) or (II) of Theorem 1 where the intervals a_n are 'very small'.

Let K denote the Kiefer process as defined in [4]. Let $B_n = K(\cdot, n)/n^{\frac{1}{2}}$, so that B_n is distributed as Brownian bridge for each $n \geqq 1$. Let ω_n^* denote the modulus of continuity of B_n.

Theorem 4. *Suppose* $a_r \searrow$, $r a_r \nearrow$ *and* c_r, *defined by*

$$c_r = (\log(1/a_r))/\log\log r \quad or \quad a_r = (\log r)^{-c_r} \tag{40}$$

satisfies

$$c_r \to c \in [0, \infty] \quad \text{as } r \to \infty. \tag{41}$$

If $c \in (0, \infty]$, *then*

$$1 = \liminf_{r\to\infty} \frac{\omega_r^*(a_r)}{(2a_r \log(1/a_r))^{\frac{1}{2}}} \leq \limsup_{r\to\infty} \frac{\omega_r^*(a_r)}{(2a_r \log(1/a_r))^{\frac{1}{2}}} = \left(\frac{1+c}{c}\right)^{\frac{1}{2}} \quad \text{a.s.} \tag{42}$$

If $c \in [0, \infty)$, *then*

$$c^{\frac{1}{2}} = \liminf_{r\to\infty} \frac{\omega_r^*(a_r)}{(2a_r \log\log r)^{\frac{1}{2}}} < \limsup_{r\to\infty} \frac{\omega_r^*(a_r)}{(2a_r \log\log r)^{\frac{1}{2}}} = (1+c)^{\frac{1}{2}} \quad \text{a.s.} \tag{43}$$

Also, if $c \in (0, \infty)$, *then*

$$\liminf_{n\to\infty} \sup_{|C| \geq a_n} \frac{B_n(C)}{(2|C| \log\log n)^{\frac{1}{2}}} = c^{\frac{1}{2}} \quad \text{a.s.} \tag{44}$$

Much of this result, plus analogs for $W(n, \cdot)/n^{\frac{1}{2}}$ of all of this theorem, are discussed in [4]; they are due to Chan, Csörgó and Révész, and Book and Shore. Thus we omit the proof. Theorem 1 (III) is an immediate corollary to (43) and the proof of Theorem 3.

5. Appendix

First we will summarize some properties of the function ψ that appears in Inequality 1 and then provide a proof for the inequality. Our Inequality 1 can also be derived from Lemmas 2.1, 2.2, and 2.4 in Stute (1982); such a proof is no shorter than the proof given below, however.

The function $\psi(x) \equiv 2h(1+x)/x^2$, where $h(x) \equiv x(\log x - 1) + 1$, has the following properties:

C1. $\psi\downarrow$ for $\lambda \geq 0$ with $\psi(0) = 1$,

C2. $\psi(\lambda) \sim (2\log\lambda)/\lambda$ as $\lambda \to \infty$,

C3. $\psi(\lambda) \geq 1 \Big/ \left(1 + \frac{\lambda}{3}\right)$ for all $\lambda > 0$ and this is $\geq 1 - \delta$ for $0 \leq \lambda \leq 3\delta$.

For any $0 < b \leq \frac{1}{2}$ and $\# = +, -$, or $|\ |$, let

$$\|U_n^{\#}/(1-I)\|_0^b = \sup\{|U_n^{\#}(u)/(1-u)| : 0 \leq u \leq b\}.$$

In particular, by James [5] and Shorack [9] the following inequalities hold for any $\lambda > 0$:

(a) $P(\|U_n^+/(1-I)\|_0^b \geq \lambda/(1-b)) \leq \exp(-\lambda^2 \psi(\lambda/bn^{\frac{1}{2}})/2b(1-b))$,

(b) $P(\|U_n^-/(1-I)\|_0^b \geq \lambda/(1-b)) \leq \exp(-\lambda^2/2b(1-b))$,

(c) $P(\|U_n/(1-I)\|_0^b \geq \lambda/(1-b)) \leq 2\exp(-\lambda^2 \psi(\lambda/bn^{\frac{1}{2}})/2b(1-b))$.

Proof of Inequality 1. Choose $0<a\leqq\delta<\frac{1}{2}$. First assume that $\lambda\geqq\delta^2(na)^{\frac{1}{2}}$. Let m denote any positive integer, and observe that for $0\leqq t-s\leqq a$

$$\begin{aligned} U_n(s,t)&=n^{\frac{1}{2}}(\Gamma_n(t)-\Gamma_n(s)-(t-s))\\ &\leqq n^{\frac{1}{2}}\left(\Gamma_n(t)-\Gamma_n\left(\frac{[ms]}{m}\right)-\left(t-\frac{[ms]}{m}\right)\right)+n^{\frac{1}{2}}m^{-1}. \end{aligned}$$

Thus, recalling the definitions of ω_n^+ and ω_n^- given in the introduction,

$$\omega_n^+(a)\leqq\max_{0\leqq j\leqq m-1}\ \sup_{0\leqq r\leqq a+m^{-1}}\left\{U_n\left(\frac{j}{m},\frac{j}{m}+r\right)+n^{\frac{1}{2}}m^{-1}\right\}.$$

Stationarity of the increments of U_n, inequality (a), and C1 yield

$$\begin{aligned} \text{(d)}\qquad P(\omega_n^+(a)\geqq\lambda a^{\frac{1}{2}})&\leqq mP(\|U_n^+\|_0^{a+m^{-1}}\geqq\lambda a^{\frac{1}{2}}-n^{\frac{1}{2}}m^{-1})\\ &\leqq m\exp\left(-\frac{(\lambda a^{\frac{1}{2}}-n^{\frac{1}{2}}m^{-1})^2(1-a-m^{-1})^2}{2(a+m^{-1})(1-a-m^{-1})}\right.\\ &\qquad\left.\cdot\psi\left(\frac{(\lambda a^{\frac{1}{2}}-n^{\frac{1}{2}}m^{-1})(1-a-m^{-1})}{(a+m^{-1})n^{\frac{1}{2}}}\right)\right)\\ &\leqq m\exp\left(-\frac{\lambda^2}{2}\frac{a(1-a-m^{-1})}{(a+m^{-1})}\left(1-\frac{n^{\frac{1}{2}}}{\lambda ma^{\frac{1}{2}}}\right)^2\psi\left(\frac{\lambda}{\sqrt{na}}\right)\right). \end{aligned}$$

Now choose m to be the smallest positive integer for which

$$\text{(e)}\qquad m^{-1}<a\delta^3.$$

This entails that $2/m\geqq 1/(m-1)\geqq a\delta^3$. Also note that if $\lambda\geqq\delta^2(na)^{\frac{1}{2}}$, then (e) implies that

$$\text{(f)}\qquad \lambda\geqq\delta^2(na)^{\frac{1}{2}}\geqq n^{\frac{1}{2}}/(ma^{\frac{1}{2}}\delta).$$

Using (e) and (f) we have

$$\frac{a(1-a-m^{-1})}{a+m^{-1}}\geqq(1-a)(1-\delta)\geqq(1-\delta)^2,\qquad \left(1-\frac{n^{\frac{1}{2}}}{\lambda ma^{\frac{1}{2}}}\right)\geqq 1-\delta,$$

so that (d) gives the inequality for ω_n^+ when $\lambda\geqq\delta^2(na)^{\frac{1}{2}}$. Likewise, whenever $0\leqq t-s\leqq a$, the inequality

$$\begin{aligned} U_n(s,t)&=n^{\frac{1}{2}}(\Gamma_n(t)-\Gamma_n(s)-(t-s))\\ &\geqq n^{\frac{1}{2}}\left(\Gamma_n\left(\frac{[mt]}{m}\right)-\Gamma_n(s)-\left(\frac{[mt]}{m}-s\right)\right)-n^{\frac{1}{2}}m^{-1} \end{aligned}$$

leads to, using inequality (b) and the same choice of m as in (e) above,

$$\begin{aligned} P(\omega_n^-(a)\geqq\lambda a^{\frac{1}{2}})&\leqq mP(\|U_n^-\|_0^a\geqq\lambda a^{\frac{1}{2}}-n^{\frac{1}{2}}m^{-1})\\ &\leqq m\exp\left(-\frac{\lambda^2 a(1-a)^2}{2a(1-a)}\left(1-\frac{n^{\frac{1}{2}}}{m\lambda a^{\frac{1}{2}}}\right)^2\right) \end{aligned}$$

which yields the inequality for ω_n^- when $\lambda \geqq \delta^2(na)^{\frac{1}{2}}$. Combination of the inequalities for ω_n^+ and ω_n^- yields the inequality for ω_n in the case $\lambda \geqq \delta^2(na)^{\frac{1}{2}}$.

Now assume that $0<\lambda \leqq \delta^2(na)^{\frac{1}{2}}$. Observe that whenever $s \leqq t \leqq [ms]/m + 1/m$

$$|U_n(s,t)| \leqq \left|U_n\left(\frac{[ms]}{m}, s\right)\right| + \left|U_n\left(t, \frac{[ms]}{m}+\frac{1}{m}\right)\right| + \left|U_n\left(\frac{[ms]}{m}, \frac{[ms]}{m}+\frac{1}{m}\right)\right|$$

and whenever $s \leqq [ms]/m + 1/m \leqq t < 1$

$$|U_n(s,t)| \leqq \left|U_n\left(s, \frac{[ms]}{m}+\frac{1}{m}\right)\right| + \left|U_n\left(\frac{[ms]}{m}+\frac{1}{m}, t\right)\right|,$$

so that in either case whenever m is a positive integer such that $m^{-1} \leqq a$ and $0 \leqq t-s \leqq a$

$$\text{(g)} \quad \omega_n(a) \leqq \max_{0 \leqq j \leqq m-1} \sup_{0 \leqq t \leqq a} \left|U_n\left(\frac{j}{m}, \frac{j}{m}+t\right)\right| + 2 \max_{1 \leqq j \leqq m} \sup_{0 \leqq t \leqq m^{-1}} \left|U_n\left(\frac{j}{m}-t, \frac{j}{m}\right)\right|.$$

We now choose m to be the smallest integer such that $m^{-1} \leqq a\delta^2/4$; which entails that

$$\text{(h)} \qquad m < 4/(a\delta^2)+1 \leqq 5/(a\delta^2).$$

Now (g) and the stationary increments of U_n imply that

$$\begin{aligned} P(\omega_n(a) \geqq \lambda \sqrt{a}) &\leqq \sum_{j=0}^{m-1} P\left(\|U_n/(1-I)\|_0^a \geqq \lambda a^{\frac{1}{2}} \frac{(1-a)}{1-a} \frac{1}{1+\delta}\right) \\ &\quad + \sum_{j=0}^{m-1} P\left(\|U_n/(1-I)\|_0^{1/m} \geqq \lambda a^{\frac{1}{2}} \frac{(1-m^{-1})}{(1-m^{-1})} \left(\frac{\delta}{2(1+\delta)}\right)\right) \\ &\equiv b+d. \end{aligned}$$

By inequality (c) we have

$$\begin{aligned} b &\leqq 2m \exp\left(-\frac{\lambda^2 a}{2a(1-a)} \frac{(1-a)^2}{(1+\delta)^2} \psi\left(\frac{\lambda a^{\frac{1}{2}}}{an^{\frac{1}{2}}} \frac{(1-a)}{(1+\delta)}\right)\right) \\ &\leqq \frac{10}{a\delta^2} \exp\left(-(1-\delta)^3 \frac{\lambda^2}{2} \psi\left(\frac{\lambda}{\sqrt{na}}\right)\right). \end{aligned}$$

(Here we are using (h) and C1.) Inequality (c) also gives

$$\begin{aligned} d &\leqq 2m \exp\left(-\frac{\lambda^2 a}{2(1/m)(1-1/m)} \frac{(1-1/m)^2 \delta^2}{4(1+\delta)^2} \psi\left(\frac{\lambda a^{\frac{1}{2}}(1-1/m)\delta}{(1/m) n^{\frac{1}{2}} 2(1+\delta)}\right)\right) \\ &\leqq \frac{10}{a\delta^2} \exp(-(1-\delta)^4 \lambda^2/2) \quad \text{if } \frac{\lambda a^{\frac{1}{2}} m \delta}{2n^{\frac{1}{2}}} \leqq 3\delta. \end{aligned}$$

Thus if $\lambda \leqq 6n^{\frac{1}{2}}/ma^{\frac{1}{2}}$

$$P(\omega_n(a) \geqq \lambda \sqrt{a}) \leqq \frac{20}{a\delta^2} \exp\left(-(1-\delta)^4 \frac{\lambda^2}{2}\right).$$

But this last stipulation on λ holds for $\lambda \leqq \delta^2 (na)^{\frac{1}{2}}$, since (h) implies that $\lambda \leqq \delta^2 (na)^{\frac{1}{2}} \leqq 5 n^{\frac{1}{2}}/m a^{\frac{1}{2}}$. Thus the inequality holds in the case $\lambda \leqq \delta^2 (na)^{\frac{1}{2}}$. □

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