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Construction of Strictly Ergodic Systems

III. Bernoulli Systems

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In [2] a K-system M was constructed as the intersection of a decreasing sequence of mixing subshifts of finite type M_i. In each step so many blocks of irregular frequencies were excluded that the intersection became strictly ergodic, but so many blocks were left over that from entropy inequalities and good weak convergence one could conclude that the unique measure on $M=\bigcap M_i$ was K-mixing. In this paper we use the tools of Ornstein's theory of Bernoulli processes to show that the same method with more rapid convergence gives even an M carrying a measure which is isomorphic to a Bernoulli shift. We use the results of Ornstein and Friedman as they can be found in [4].

As far as we know, this is the first application of the fact that the class of Bernoulli systems is closed in the $\bar{d}$-metric.

It is not clear whether the K-systems of [2] might always be Bernoulli, but it seems possible that a combination of the methods of [2] and [5] will constructively determine strictly ergodic K-systems which are not Bernoulli.

Tools

We mainly use the notations of [2]. $\Omega_A=A^{\mathbb{Z}}$ is the shift space over the finite alphabet A, $T\colon \Omega_A\to\Omega_A$ the shift transformation

$$T\colon\ \omega=(\omega_i)_{i\in\mathbb{Z}}\mapsto(\omega_{i+1})_{i\in\mathbb{Z}}.$$

B_l is the set of blocks of length l over the alphabet A, and

$${}_s[P]=\{\omega\,|\,(\omega_s,\ldots,\omega_{s+l-1})=P\}$$

if $P\in B_l$. For a T-invariant subset $N\subset\Omega_A$, $B_l(N)=\{P\in B_l\,|\,{}_0[P]\cap N\neq\emptyset\}$. For a measure μ on Ω_A and a block P we write $\mu(P)=\mu({}_0[P])$. (All measures are supposed to be T-invariant Borel probability measures.) If $P\in B_l$, $Q=(q_1,\ldots,q_r)\in B_r$ and $r\geqq l$, then

$$\mu_Q(P)=(r-l+1)^{-1}\,|\{i\,|\,1\leqq i\leqq r-l+1,\ (q_i,\ldots,q_{i+l-1})=P\}|.$$

For the definition of strictly ergodic sets and mixing subshifts of finite type (m.s.f.t.'s) see e.g. [2, parts I and II]. For a strictly ergodic set $K \subset \Omega_A$, μ_K is the unique invariant measure on K, and for an m.s.f.t. $M \subset \Omega_A$, μ_M is the unique measure of maximal entropy (Parry measure) on M. This μ_M can be computed explicitly for all cylinders ${}_0[P]$ and is positive if $P \in \bigcup_l B_l(M)$. $h(S)$ denotes the topological entropy of the T-invariant closed set $S \subset \Omega_A$, and $h(\mu)$ the measure theoretic entropy of the measure μ.

What we use from [2, part II] is only the simple

Lemma 1. *Let M be an m.s.f.t., $s \in \mathbb{N}$, $\delta > 0$. Then there exists (constructively) an m.s.f.t. $\bar{M} \subset M$ and a $t \in \mathbb{N}$ such that*

1. *if $Q \in B_t(\bar{M})$, $P \in B_s(M)$, then $|\mu_Q(P) - \mu_M(P)| < \delta$, and hence $|\lambda(P) - \mu_M(P)| < \delta$ for every measure λ on $\bar{M}$.*
2. $0 < h(M) - h(\bar{M}) < \delta$.

For a measure μ on Ω_A we have in a natural way a process in Ornstein's sense, namely $(({}_0[a])_{a \in A}, T, \mu)$, and since we only consider processes of this kind on the same shift space, we simply may write μ for the process as well as for the measure.

A Bernoulli process is a measure μ such that (Ω_A, μ, T) is isomorphic to a shift space with a product measure.

The $\bar{d}$-metric between processes (see [3, 4]) is always defined for our processes. We need the following facts about it:

Theorem 2 (Ornstein). *If μ, $(\mu_i)_{i \in \mathbb{N}}$ are Bernoulli processes and $\bar{d}(\mu, \mu_i) \to 0$, then μ is a Bernoulli process.*

This is Theorem 6.1 of [4].

Definition. The process μ is constructively finitely determined (C.F.D.) if, for given $\varepsilon > 0$, there can be computed $n \in \mathbb{N}$ and $\delta > 0$ such that for any process λ with $|h(\lambda) - h(\mu)| < \delta$ and $\sum_{P \in B_n} |\lambda(P) - \mu(P)| < \delta$, $\bar{d}(\lambda, \mu) < \varepsilon$ holds. (See [4], Section 3.)

Theorem 3 (Friedman-Ornstein). *If M is an m.s.f.t., then μ_M is C.F.D.*

Proof. This follows from Theorem 8.1, Lemma 7.1, and Proposition 7.2 of [4]. What we have to check in addition to the assertions there, is

1. μ_M is very weak Bernoulli in a constructive sense, i.e. the number M of Theorem 8.1 is computable.

2. In Lemma 7.1, u and δ are computable.

These points follow directly from the proofs in [4].

Construction

We begin with an m.s.f.t. $M_1 \subset \Omega_A$ and assume for the induction that we have m.s.f.t.'s $M_1 \supset M_2 \supset \cdots \supset M_i$, $\mu_j = \mu_{M_j}$, $1 = t_1 < t_2 < \cdots < t_i \in \mathbb{N}$; $\delta_j < 2^{-j}$ $(1 \leq j < i)$ with $\delta_{j+1} < \frac{1}{2}\delta_j$ and $\delta_j < \frac{1}{2}\min\{\mu_j(P) | P \in B_{t_j}(M_j)\}$ such that for $1 \leq j < i$

a) $\sum_{P \in B_{t_j}} |\mu_Q(P) - \mu_j(P)| < \delta_j$ for every $Q \in B_{t_{j+1}}(M_{j+1})$,

b) $h(\mu_{j+1}) > h(\mu_j) - \frac{1}{2}\delta_j$.

Now we want to construct M_{i+1} which reproduces a) and b). First we apply Theorem 3 to $M = M_i$ $(\mu = \mu_i)$ with $\varepsilon = 2^{-i}$. Enlarging if necessary t_i we obtain

$$\delta_i < \tfrac{1}{2}(\delta_{i-1} \wedge \min\{\mu_i(P) \mid P \in B_{t_i}(M_i)\})$$

such that

$$\left.\begin{array}{l} \text{c) } |h(\lambda) - h(\mu_i)| < \delta_i \\ \text{d) } \sum\limits_{P \in B_{t_i}} |\lambda(P) - \mu_i(P)| < \delta_i \end{array}\right\} \Rightarrow \bar{d}(\lambda, \mu_i) < 2^{-i}.$$

When δ_i is found, Lemma 1 gives us $M_{i+1} \subset M_i$ and $t_{i+1} > t_i$ such that $Q \in B_{t_{i+1}}(M_{i+1})$, a) and b) hold with $j = i$, hence

d) is valid for any λ on M_{i+1}, and in particular,

e) for any measure λ on M_{i+1} and $P \in B_{t_i}(M_i)$, $\lambda(P) > 0$.

When all the M_i are constructed, we put $K = \bigcap_i M_i$. The strict ergodicity of K follows immediately from condition a) for all j. We note that $B_{t_i}(K) = B_{t_i}(M_i)$ for all i because of e). Since $h(\mu_i) = h(M_i) \searrow h(K) = h(\mu_K)$, b) yields $|h(\mu_K) - h(\mu_i)| < \delta_i$, so we have c) for $\lambda = \mu_K$, and d) also holds for μ_K because μ_K is supported by M_{i+1}. Therefore $\bar{d}(\mu_K, \mu_i) < 2^{-i}$. Theorem 2 now implies that μ_K is a Bernoulli process.

References

1. Friedman, N.A., Ornstein, D.S.: On isomorphism of weak Bernoulli transformations. Advances in Math. **5**, 365–394 (1970)
2. Grillenberger, Ch.: Construction of strictly ergodic systems. I. Given entropy. Z. Wahrscheinlichkeitstheorie verw. Gebiete **25**, 323–334 (1973). II. K-systems. Z. Wahrscheinlichkeitstheorie verw. Gebiete **25**, 335–342 (1973)
3. Ornstein, D.S.: An application of ergodic theory to probability. Ann. Probability **1**, 43–58 (1973)
4. Ornstein, D.S.: Ergodic theory, randomness, and dynamical systems. Yale Math. Monographs 5. Yale Univ. Press, 1974
5. Ornstein, D.S., Shields, P.: An uncountable family of K-automorphisms. Advances in Math. **10**, 63–88 (1973)

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