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Gaussian Sample Functions and the Hausdorff Dimension of Level Crossings

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Summary. Let X_t be a real Gaussian process with stationary increments, mean 0, $\sigma_t^2 = E[(X_{s+t}-X_s)^2]$. If σ_t^2 behaves like $t^{2\alpha}$ as $t\downarrow 0$, $0<\alpha<1$, the graph of a.e. sample function will have Hausdorff dimension $2-\alpha$. This leads one to feel that the set of zeros of X_t should have Hausdorff dimension $1-\alpha$. This is shown to be true provided the process is stationary and satisfies additional assumptions.

Introduction

Our concern is with real Gaussian processes, and for the most part we assume that they are stationary or at least have stationary increments. The assumptions made will ensure that the processes are continuous with probability one. In Section 1 we obtain a result on the Hausdorff dimension of the graph of the sample functions. The corresponding result for Brownian motion was found by Taylor [13], and indeed we simply apply his method. Taylor also found the dimension of the set of zeros of Brownian motion. Here the treatment of more general Gaussian processes appears much more difficult. For a certain class of stationary processes we obtain a result in Section 4. In Section 2 it is shown that under mild conditions the time that the process $(X_t, 0\leqq t\leqq T)$ spends in a set E, considered as a function of E, is a measure which is, with probability one absolutely continuous with respect to Lebesgue measure.

All processes are assumed to be separable. We visualize our sample curves as point sets in R_2, with the time axis horizontal. We write a.s. for *almost surely*, that is, with probability one.

1. Dimension of the Graph

Let (X_t) be a real Gaussian process with stationary increments, mean zero, covariance $r(s,t)$. Write

$$\sigma_t^2 = E[(X_{s+t}-X_s)^2],$$

and introduce the notations:

$$\alpha^* = \sup\{\alpha\colon\ \sigma_t = o(t^\alpha), t\downarrow 0\}, \qquad \alpha_* = \inf\{\alpha\colon\ t^\alpha = o(\sigma_t), t\downarrow 0\}.$$

Then $0\leqq\alpha^*\leqq\alpha_*\leqq\infty$. When $\alpha^*=\alpha_*=\alpha$ we will say that σ_t has *index* α. We will be interested in the case in which σ has index α, $0<\alpha<1$.

Remark 1. When $0<\alpha^*<1$ almost every sample function obeys a uniform Lipschitz condition of order λ for each $\lambda<\alpha^*$. See Section 9.4 of [2] where much more is proved.

For a linear set Δ define

$$C_\Delta = C_\Delta(\omega) = \{(t, X_t(\omega)):\ t \in \Delta\}.$$

Evidently C_Δ is a (random) set in the Euclidean plane R_2. For $\Gamma \subseteq R_2$ we write $\dim(\Gamma)$ for the Hausdorff dimension of Γ. For the case of Brownian motion $\dim[C_{(0,\infty)}]$ was first calculated by Taylor [13], and his method serves to prove:

Theorem 1. *Suppose σ_t has index $\alpha, 0<\alpha<1$. Then a.s., $\dim[C_\Delta]=2-\alpha$ for every linear set Δ with positive inner measure.*

Remark 2. Note that the exceptional set of probability zero does not depend on Δ.

Proof. According to Remark 1 the sample functions satisfy uniform Lipschitz conditions of order λ, for each $\lambda<\alpha$. For such functions Besicovitch and Ursell [1] showed that the dimension of the graph is at most $2-\lambda$, and so $\dim[C_\Delta] \leqq 2-\alpha$.

For the inequality in the opposite direction one uses the equivalence of Hausdorff dimension and capacity dimension for compact sets in Euclidean space. If Δ is compact so is C_Δ and then, if Δ has positive Lebesgue measure, $\dim[C_\Delta] \geqq 2-\alpha$ will follow from

$$\int_\Delta \int_\Delta ([(X_t-X_s)^2+(t-s)^2]^{\frac{1}{2}})^{-\lambda}\, ds\, dt < \infty, \qquad \lambda < 2-\alpha, \tag{1.1}$$

as is well known; see for instance Theorem B in [13]. It remains to establish (1.1). Let

$$R_t = [(X_t - X_0)^2 + t^2]^{\frac{1}{2}}.$$

One verifies easily that

$$E[R_t^{-\lambda}] = \sqrt{\frac{2}{\pi}}\, \sigma_t^{-\lambda} \cdot \int_0^\infty \left(x^2 + \frac{t^2}{\sigma_t^2}\right)^{-\lambda/2} e^{-x^2/2}\, dx,$$

and noting that

$$\int_0^\infty e^{-x^2/2}(x^2+s)^{-\beta}\, dx \leqq 4 s^{\frac{1}{2}-\beta}, \qquad \beta > \tfrac{1}{2},$$

one obtains for $\lambda > 1$

$$E[R_t^{-\lambda}] \leqq 4 \sigma_t^{-1}\, t^{1-\lambda}.$$

It follows that

$$E\left[\int_0^T \int_0^T ([(X_t-X_s)^2+(t-s)^2]^{\frac{1}{2}})^{-\lambda}\, ds\, dt\right] \leqq 4 \int_0^T \int_0^T (t-s)^{1-\lambda}\, \sigma_{|t-s|}^{-1}\, ds\, dt. \tag{1.2}$$

Since σ_t has index α the last term in (1.2) is finite when $\lambda<2-\alpha$ and then the random variable inside the expectation sign in the first term of (1.2) is finite with probability one. Thus, outside a fixed null set of ω's (1.1) will hold for every measurable Δ included in $[0, T]$, and this suffices to deduce the theorem.

A much more difficult question seems to be the determination of the dimension of the set of zeros of X_t. We shall write

$$L_x = \{(t, x):\ -\infty < t < \infty\},$$

so that $C_\Delta \cap L_x$ represents the set of intersections of the graph of X_t with the level x during the time period Δ. We will investigate $\dim[C_\Delta \cap L_x]$ for stationary Gaussian processes. At least when the process is ergodic it is easy to see that the random variable $\dim[C_{[0,\infty)} \cap L_x]$ is constant with probability one. What does not seem evident is that this constant is independent of x. In Section 3 we will introduce a condition which will ensure that this is so. Using this condition we will solve the indicated problem in Section 4. The condition of Section 3 may well be superfluous: it seems possible that for ergodic processes $\dim[C_{(0,\infty)} \cap L_x]$ never depends on x. There is, in fact, good reason for conjecturing $\dim[C_{[0,\infty)} \cap L_x] = 1-\alpha$ when σ has index α. Such grounds are provided by a result of Marstrand [10] which we cite now for later reference. We will write Λ^s for s-dimensional outer Hausdorff measure and $L(p,\theta)$ for the line through the point p with slope $\tan\theta$, where p is a point in R_2, θ an angle.

Theorem A (Marstrand). *Let $\Gamma \subseteq R_2$ and suppose $0 < \Lambda^s(\Gamma) < \infty$, where $1 < s$. Then for all $p \in \Gamma$ outside some Λ^s-null set,* $\dim[\Gamma \cap L(p,\theta)] = s-1$ *for a.e. (Lebesgue measure) θ.*

We also seem to need some information about the existence of sojourn time densities. These results have independent interest and will now be developed.

2. Sojourn Time Density

Let (X_t) be Gaussian, with mean zero and covariance $r(s,t)$. Let Δ be a linear set of positive Lebesgue measure. Then the sojourn time is defined by

$$\Psi(E) = \Psi_\Delta(E,\omega) = \int_\Delta I_{[X_s \in E]}\, ds,$$

where as usual I_A is the indicator random variable corresponding to the event A.

We will show that under reasonably general conditions $\Psi(\cdot)$ is absolutely continuous with respect to Lebesgue measure. For the case of Brownian motion this was first shown by Levy [8]; in that case the sojourn time density is known as the *local time*.

The Lebesgue decomposition theorem allows us to write

$$\Psi(E) = \Psi_c(E) + \Psi_s(E),$$

where Ψ_c and Ψ_s denote the absolutely continuous and singular component of Ψ. We will write $\varphi_t(z)$ for the density of X_t, that is the normal density with variance $r(t,t)$, and we let $D_{s,t}$ be the determinant of the covariance matrix of X_s and X_t, that is $D_{s,t} = r(s,s)\,r(t,t) - r^2(s,t)$.

Finally, if z is a real number $E_n(z)$ is to denote the binary interval of length 2^{-n} containing z; this will be uniquely determined if we agree to take the intervals closed on the left and open on the right.

Suppose now that $r(t,t)$ is a continuous function of t, positive for $t>0$. For each real number z, introduce the random variables

$$\psi_n(z) = 2^n\, \Psi(E_n(z)), \quad n=1,2, \qquad \psi(z) = \liminf_{n\to\infty} \psi_n(z).$$

According to a basic differentiation theorem (see for example [4], Theorem 2.5 of the supplement) one actually has

$$\psi(z) = \lim_{n\to\infty} \psi_n(z), \qquad z \notin N,$$

where N is a Lebesgue null-set, which may depend on ω, and $\psi(z)$ is a version of the Radon-Nikodym derivative of Ψ_c with respect to Lebesgue measure. Note also that $\psi(z,\omega)$ is measurable in the pair (z,ω). By Fatou's lemma and Fubini's theorem

$$\begin{aligned} E[\psi(z)] &\leqq \lim_{n\to\infty} E[\psi_n(z)] = \lim_{n\to\infty} 2^n E[\Psi(E_n(z))] = \lim_{n\to\infty} 2^n \int_\Delta P[X_s \in E_n(z)]\, ds \\ &= \int_\Delta \varphi_s(z)\, ds. \end{aligned} \tag{2.1}$$

Let the Lebesgue measure of Δ be $\delta > 0$. Suppose now that the inequality in (2.1) were an equality. Then one could integrate with respect to z to obtain

$$\int_{-\infty}^{\infty} E[\psi(z)]\, dz = \int_{-\infty}^{\infty} \left(\int_\Delta \varphi_s(z)\, ds\right) dz = \int_\Delta \left(\int_{-\infty}^{\infty} \varphi_s(z)\, dz\right) ds = \delta,$$

and since the first member is equal to $E[\Psi_c((-\infty,\infty))]$ while evidently $\delta = E[\Psi((-\infty,\infty))]$ one concludes that with probability one $\Psi_c = \Psi$. Now equality will, in fact, obtain in (2.1) if and only if $\psi_n(z)$ is uniformly integrable dP. A sufficient condition for this is that $E[\psi_n^2(z)]$ be bounded uniformly in n, and since

$$E[\psi_n^2(z)] = 2^{2n} \int_\Delta \int_\Delta P[X_s \in E_n(z), X_t \in E_n(z)]\, ds\, dt,$$

one sees easily that

$$\int_\Delta \int_\Delta D_{s,t}^{-\frac{1}{2}}\, ds\, dt < \infty, \tag{2.2}$$

suffices, so we have proved the following result.

Theorem 2. *Suppose $r(t,t)$ continuous and $r(t,t) > 0$ for $t > 0$. Then condition* (2.2) *implies that $\Psi(\cdot)$ is a.s. absolutely continuous with respect to Lebesgue measure, and there exists a version of the Radon-Nikodym derivative $\psi(z,\omega)$ jointly measurable in the pair (z,ω).*

Remark 3. Suppose the process is stationary, with mean zero. Then $r(s,t) = r(|t-s|)$ and $r(t) = r(0) - \frac{1}{2}\sigma_t^2$, where σ_t^2 has the same significance as in Section 1. Observe that if $\alpha_* < 1$ the conditions of Theorem 2 will be satisfied. It seems plausible that for Gaussian process with mean zero the conclusion of Theorem 2 always holds, excepting only the degenerate case $X_t = X_0$ for all t, but we have no proof.

3. Equivalence under Translation

Consider two stochastic processes defined on the same parameter interval, say $(X_t, t \in \Delta)$ and $(Y_t, t \in \Delta)$. There is then a natural isomorphism between the Borel field generated by the (X_t) and that generated by the (Y_t), with $[X_{t_1} < \lambda_1, X_{t_2} < \lambda_2, \ldots,$

$X_{t_n} < \lambda_n]$ corresponding to $[Y_{t_1} < \lambda_1, Y_{t_2} < \lambda_2, \ldots, Y_{t_n} < \lambda_n]$. Call the processes *equivalent* if this isomorphism has the property that the event corresponding to an event A is an event of probability zero if and only if A has probability zero. We will be interested in the case when X_t is stationary Gaussian with mean 0 and $Y_t = X_t + m_t$, where m_t is a non-random function. If we knew for instance that $(X_t, 0 \leqq t \leqq T)$ and $(X_t + x, 0 \leqq t \leqq T)$ are equivalent, it would follow that $P[\dim[C_{[0,T]} \cap L_x] \geqq \lambda] > 0$ if and only if $P[\dim[C_{[0,T]} \cap L_0] \geqq \lambda] > 0$. This will suffice for now as motivation for the lemma of this section.

Let then (X_t) be real, stationary Gaussian, with mean 0 and covariance $r(s, t) = r(|t-s|)$. Then $r(t)$ has the representation

$$r(t) = \int_{-\infty}^{\infty} e^{it\lambda} F(d\lambda),$$

where F is a distribution function, the *spectral distribution* of $r(\cdot)$; since $r(\cdot)$ is real the measure corresponding to F is symmetric around the origin.

Lemma. *Let $F'(\lambda)$ be the almost everywhere defined derivative of F. Suppose there exist $\varepsilon > 0$ and N such that $F'(\lambda) \geqq \varepsilon(1+|\lambda|)^{-N}$ holds a.e. Then for every finite positive T and every infinitely often differentiable function h_t, $0 \leqq t \leqq T$, the process $(X_t, 0 \leqq t \leqq T)$ is equivalent to the translated process $(X_t + h_t, 0 \leqq t \leqq T)$.*

Proof. We will exploit the fact that a necessary and sufficient condition for the equivalence of $(X_t, -\infty < t < \infty)$ and $(X_t + m_t, -\infty < t < \infty)$ is known; see [6] or [5, 12]. The condition is that m has a representation

$$m_t = \int_{-\infty}^{\infty} g(\lambda)[\cos \lambda t + \sin \lambda t] \, F(d\lambda), \tag{3.1}$$

with g a real valued function satisfying

$$\int_{-\infty}^{\infty} (g(\lambda))^2 \, F(d\lambda) < \infty. \tag{3.2}$$

For given infinitely often differentiable h_t defined on $[0, T]$ there exists an infinitely often differentiable function m_t defined on $(-\infty, \infty)$ which agrees with h on $[0, T]$ and which has compact support. Evidently $(X_t, 0 \leqq t \leqq T)$ will be equivalent to $(X_t + h_t, 0 \leqq t \leqq T)$ if $(X_t, -\infty < t < \infty)$ is equivalent to $(X_t + m_t, -\infty < t < \infty)$. The Fourier inversion formula provides an integrable function $w(\lambda)$ such that

$$m_t = \int_{-\infty}^{\infty} w(\lambda)[\cos \lambda t + \sin \lambda t] \, d\lambda, \tag{3.3}$$

and the assumptions on m_t imply $w(\lambda) = O(\lambda^{-n})$, $n = 1, 2, \ldots$ as $|\lambda| \to \infty$. One may evidently suppose that w vanishes on the null set which is the set of increase of the singular part of F, and then (3.3) implies (3.1) with $g = w/F'$. The assymptotic behaviour of w and the assumptions on F' imply that g satisfies (3.2), which is all that was needed.

4. Dimension of the Level Crossings

In this section X_t is to be a stationary, real Gaussin process with mean zero, covariance $r(s,t)=r(|t-s|)$, and F is to be the corresponding spectral distribution function. The significance of σ_t and L_x is the same as in Section 1. The following result will be established.

Theorem 3. *Let σ_t have index α, $0<\alpha<1$. Suppose F is continuous and that the almost everywhere defined derivative F' satisfies $F'(\lambda)\geqq\varepsilon(1+|\lambda|)^N$ for some $\varepsilon>0$ and some N for a.e. λ. Then, for every x, $\dim[C_{[0,\infty)}\cap L_x]=1-\alpha$ with probability one.*

Remark 3. The continuity of F is equivalent to the ergodicity of X_t; see [7, 11].

Remark 4. The assumption on F' is certainly undesirable and probably unnecessary. Examples where all conditions of the theorem are known to be satisfied are given by $r(t)=e^{-|t|^{2\alpha}}$, $0<\alpha<1$.

Our proof will make use of results of Marstrand. One of these was cited in Section 1 as Theorem A. We will also need the following result from [10], where actually much more is proved; Λ^β denotes β-dimensional Hausdorff outer measure.

Theorem B (Marstrand). *Let $\Gamma\subseteq R_2$. If $\Lambda^\beta(L_x\cap\Gamma)$ is positive for a set of x's of positive Lebesgue measure then $\Lambda^{\beta+1}(\Gamma)>0$.*

Proof of Theorem 3. First it will be shown that

$$P[\dim[L_x\cap C_{[0,\infty)}]\leqq 1-\alpha]=1, \qquad -\infty<x<\infty. \tag{4.1}$$

Evidently it suffices to show that for each fixed T, $T<\infty$, and each x,

$$\dim[L_x\cap C_{[0,T]}]\leqq 1-\alpha$$

a.s. From Theorem 1 we know that a.s. $\dim[C_{[0,T]}]=2-\alpha$, and then Theorem B implies $\dim[L_x\cap C_{[0,T]}]\leqq 1-\alpha$ for a.e. x, where the exceptional null set may depend on ω. Suppose now that for some x, $P[\dim[L_x\cap C_{[0,T]}]>1-\alpha]>0$. The same would then be true for every x, since according to the Lemma of Section 3 $(X_t, 0\leqq t\leqq T)$ and $(X_t+y, 0\leqq t\leqq T)$ are equivalent for every y. It is not hard to justify the applications of Fubini's theorem, which tells us that with positive probability $\{x\colon \dim[L_x\cap C_{[0,T]}]>1-\alpha\}$ has positive Lebesgue measure. This is a contradiction, and (4.1) is established.

It remains to prove

$$P[\dim[L_x\cap C_{[0,T]}]\geqq 1-\alpha]=1, \qquad -\infty<x<\infty. \tag{4.2}$$

Our first concern will be to prove that for $T<\infty$, $\alpha'>\alpha$

$$P[\dim[L_x\cap C_{[0,T]}]\geqq 1-\alpha']>0. \tag{4.3}$$

So let $\alpha'\in(\alpha,1)$ and define

$$C'=\{p\in C_{[0,T]}\colon \dim[L(p,\theta)\cap C_{[0,T]}]\geqq 1-\alpha' \text{ for a.e. } \theta\}.$$

Then, a.s., $\dim[C_{[0,T]} - C'] \leqq 2-\alpha' < 2-\alpha$. For suppose otherwise. Then there exists α'' such that $\alpha<\alpha''<\alpha'$ and $\Lambda^{2-\alpha''}(C_{[0,T]} - C')>0$. Then it is known, see [3], that there exists a set D such that

$$D \subseteq C_{[0,T]} - C', \quad 0<\Lambda^{2-\alpha''}(D)<\infty.$$

To this set D we may apply Marstrand's Theorem A (see Section 1). The conclusion of this theorem is not consistent with $D \subseteq C_{[0,T]} - C'$, and this is the desired contradiction.

We are dealing with sets such as C', which are point sets in R_2. For such sets we will write $\mathrm{Proj}_1(\cdot)$ for the projection onto the (horizontal) time-axis, $\mathrm{Proj}_2(\cdot)$ for the perpendicular projection.

Since a.s. $\dim[C_{[0,T]} - C'] < 2-\alpha$ it follows from Theorem 1 that a.s. $\mathrm{Proj}_1(C_{[0,T]} - C')$ is a null set (Lebesgue measure). Bearing in mind the definition of C' and making a justifiable application of Fubini's theorem one deduces the existence of a θ such that $0 \leqq \theta < \pi/2$ and a.s.

$$\dim[L(X_t, \theta) \cap C_{[0,T]}] \geqq 1-\alpha'' \quad \text{for a.e. } t \in [0,T]. \tag{4.4}$$

Now in (4.4) one may take $\theta=0$. To see this consider what (4.4) tells us about the translated process $(X_t - t\tan\theta,\ 0 \leqq t \leqq T)$ and use the Lemma of Section 3. We now introduce

$$B=\{p \in C_{[0,T]}\colon \dim[L(p,0) \cap C_{[0,T]}] \geqq 1-\alpha'\}.$$

What we have shown is that a.s. $\mathrm{Proj}_1(C_{[0,T]} - B)$ is a null set. That is, if $\Psi_{[0,T]}$ denotes the sojourn time, as in Section 2, then $\Psi_{[0,T]}(B)=T$. Now $\mathrm{Proj}_2(B)$ can be seen to be measurable and it follows now from Theorem 2 that it is a set of positive Lebesgue measure. That is, with probability one $\{x\colon \dim(L_x \cap C_{[0,T]}) \geqq 1-\alpha'\}$ has positive measure. Again it is not hard to justify application of Fubini's theorem to obtain the existence of an x such that (4.3) holds. Using once again that by the Lemma of Section 3 $(X_t, 0 \leqq t \leqq T)$ and $(X_t + y, 0 \leqq t \leqq T)$ are equivalent, it follows that (4.3) holds for all x.

Let E_n be the event $[\dim[L_x \cap C_{[nT,(n+1)T]}] > 1-\alpha']$. Then (4.3) asserts $P[E_0]>0$. Since we have stationarity we may use the ergodic theorem. According to Remark 4 we have ergodicity. So $N^{-1}\sum_1^N E_n \to P[E_0]$ a.s. as $N\to\infty$. This implies (4.2) with α' for α. Since α' is an arbitrary number greater than α, (4.2) follows.

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