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An Identification of Ratio Ergodic Limits for Semi-Groups*

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Summary. Let $L_1 = L_1(X, \mathscr{F}, \mu)$, where $(X, \mathscr{F}, \mu)$ is a σ-finite measure space and let $T_t: L_1 \to L_1$, $t \geqq 0$, be a strongly continuous semi-group of positive linear contractions and $U_t: L_\infty \to L_\infty$ be the dual of T_t. The purpose of this paper is to give an identification of the ratio ergodic limit

$$(f/g) = \lim_{s \to \infty} \left(\int_0^s T_t f\, dt \Big/ \int_0^s T_t g\, dt \right)$$

where f and g are in L_1 and $g > 0$. We construct a sub-Banach algebra $\mathscr{A}$ of L_∞ that contains $\mathscr{H} = \{f \in L_\infty | U_t f = f \text{ all } t \geqq 0\}$ and define a transformation π: $\mathscr{A} \to \mathscr{H}$. With multiplication defined by $f g = \pi(f g)$, $\mathscr{H}$ becomes a B^*-algebra which is isometrically * isomorphic under a mapping σ to $C(K)$, the space of complex valued continuous functions on the maximal ideal space K of $\mathscr{H}$. Let $M(K)$ denote the space of finite complex Baire measures on K. Define $\tau: \mathscr{A} \to C(K)$ where $\tau = \sigma\pi$ and $\lambda: L_1 \to M(K)$ where, for f in L_1, $\int f h\, d\mu = \int \sigma h\, d\lambda f$ for every h in $\mathscr{H}$. Then our identification for (f/g) in L_∞ is $\tau(f/g) = d\lambda f / d\lambda g$.

Introduction

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let T_t, $t \geqq 0$ be a strongly continuous semi-group of positive linear contractions on $L_1(X, \mathscr{F}, \mu)$. The existence of the ratio ergodic limits for such a semi-group has recently been proved in [5] and [3]. In [5] Berk has also obtained an identification for these limits on the conservative part of X, which is analogous to a result of Chacon [7], [2] for the discrete case. The purpose of this paper is to obtain a different identification of the ratio ergodic limits for semi-groups over the whole space X. The basic methods used are extensions of the methods in [4] to the continuous case and are different from those used to obtain the identification on the conservative part only [5, 7, 2]. The discrete analogue of our identification, although not stated, would follow from the results in [4].

Definitions and Preliminaries

Let $T = \{T_t | t \geqq 0\}$ denote a semi-group of positive linear contractions from L_1 to L_1 where L_1 is the usual real Banach space of a σ-finite measure space $(X, \mathscr{F}, \mu)$. Hence for all $s, t \geqq 0$, $T_t: L_1 \to L_1$, $\|T_t\| \leqq 1$, $T_0 = I$, $T_{t+s} = T_t T_s$ and $T_t L_1^+ \subset L_1^+$ where L_1^+ denotes the class of non-negative L_1 functions. We shall further assume that T is strongly continuous which means that, for every $f \in L_1$, $T_{(\cdot)} f$ is continuous on $[0, \infty)$ with respect to L_1 norm and hence is Riemann integrable on every finite interval of $[0, \infty)$.

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For $t \geqq 0$ let U_t: $L_\infty \to L_\infty$ be the dual of T_t and for $\alpha \in L_\infty$ and $t \geqq 0$ define T_t^α: $L_1 \to L_1$ as $T_t^\alpha f = \alpha f + T_t(1-\alpha) f$ and let U_t^α be its dual. If for $E \in \mathscr{F}$, χ_E is the characteristic function we write T_t^E and U_t^E instead of $T_t^{\chi_E}$ and $U_t^{\chi_E}$.

The following partial ordering of L_1^+ is the continuous extension of the one defined in [4].

Definition 1. For $f, g \in L_1^+$, $f \prec g$ if and only if there exists an integer $n \geqq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in L_\infty$ and $t_1, t_2, \ldots, t_n$ such that $0 \leqq \alpha_i \leqq 1$ and $t_i > 0$ for $i = 1, 2, \ldots, n$ and $g = T_{t_n}^{\alpha_n} \ldots T_{t_1}^{\alpha_1} f$.

This reflexive and transitive relation has the property that if $f \prec g$ then $\|f\|_1 \geqq \|g\|_1$ and also, if $f \prec g$, $g = T_{t_n}^{\alpha_n} \ldots T_{t_1}^{\alpha_1} f$ then an induction argument shows that $g \prec T_{t_1+t_2+\cdots+t_n} f$. Hence it follows that $\{g \in L_1^+ \mid f \prec g\}$ is upward directed by $\prec$.

Definition 2. For $f \in L_1^+$, and $E \in \mathscr{F}$ let

$$\Psi_E f = \sup_{g \succ f} \int_E g \, d\mu \quad \text{and} \quad \Theta_E f = \lim_{g \succ f} \Psi_E g.$$

Lemma 1. *For any fixed $t \geqq 0$, $E \in \mathscr{F}$ and any integer $n \geqq 1$, if $\alpha_i \in L_\infty, 0 \leqq \alpha_i \leqq 1$ for $i = 1, 2, \ldots, n$ then*

$$U_t^{\alpha_n} \ldots U_t^{\alpha_1} \chi_E \leqq (U_t^E)^n \chi_E.$$

Proof. Let $g \in L_\infty^+$ satisfy

$$(*) \qquad \begin{aligned} &\chi_E U_t g \leqq \chi_E g \\ &\chi_{E^c} U_t g \geqq \chi_{E^c} g \qquad \text{for all } t \geqq 0. \end{aligned}$$

Then, for all $\alpha \in L_\infty, 0 \leqq \alpha \leqq 1$ and all $t \geqq 0$,

$$U_t^\alpha g = \alpha g + (1-\alpha) U_t g \leqq \chi_E g + \chi_{E^c} U_t g = U_t^E g.$$

Since $\|U_t\| \leqq 1$ for all t then, for $E \in \mathscr{F}$, χ_E satisfies $(*)$ and $U_t^\alpha \chi_E \leqq U_t^E \chi_E$. For a fixed t it follows by induction that $(U_t^E)^n \chi_E$ satisfies $(*)$ for all $n \geqq 0$. Hence we have, again by induction, that

$$U_t^{\alpha_n} \ldots U_t^{\alpha_1} \chi_E \leqq (U_t^E)^n \chi_E \qquad \text{for} \quad \alpha_i \in L_\infty, \; 0 \leqq \alpha_i \leqq 1, \; i = 1, 2, \ldots, n.$$

We note that for any $g \succ f \in L_1^+$, $g = T_{t_n}^{\alpha_n} \ldots T_{t_1}^{\alpha_1} f$ and $\varepsilon > 0$ it follows from the continuity of $T_{(\cdot)} h$ on $(0, \infty)$ for any $h \in L_1$ and from an induction argument that there exists a set of positive rationals $r_1, r_2, \ldots, r_n$ depending on ε with the property that $\|g - T_{r_n}^{\alpha_n} \ldots T_{r_1}^{\alpha_1} f\|_1 < \varepsilon$.

Lemma 2. *For $f \in L_1^+$, $E \in \mathscr{F}$ and $\varepsilon > 0$ there exists a positive rational r and a positive integer n such that*

$$0 \leqq \Psi_E f - \int_E (T_r^E)^n f \, d\mu < \varepsilon.$$

Proof. Choose $\varepsilon > 0$. There exists a $g \succ f$, $g = T_{t_k}^{\alpha_k} \ldots T_{t_1}^{\alpha_1} f$ such that $0 \leqq \Psi_E f - \int_E g \, d\mu < \varepsilon/2$. We can determine by induction a set of positive rationals $r_1, r_2, \ldots, r_k$

with the property that $g'=T_{r_k}^{\alpha_k}\dots T_{r_1}^{\alpha_1}f$ and

$$\left|\int_E g\,d\mu-\int_E g'\,d\mu\right|\leqq\|g-g'\|_1<\varepsilon/2.$$

Hence $0\leqq\Psi_E f-\int_E g'\,d\mu<\varepsilon$.

Let r denote the reciprocal of the least common denominator of $r_1, r_2, \dots, r_k$; then $r_i=n_i r$ where n_i is a positive integer for $i=1,2,\dots,k$. Using the fact that

$$T_{mr}^{\alpha}f=T_r^{\beta_m}\dots T_r^{\beta_1}f \quad \text{where } \alpha, \beta_i\in L_\infty$$

$0\leqq\alpha\leqq 1$, $0\leqq\beta_i\leqq 1, i=1,2,\dots,m$ and $\beta_1=\alpha_1$ and β_j is such that $T_r^{j-1}(1-\alpha)f=(1-\beta_j)\,T_r^{\beta_{j-1}}\dots T_r^{\beta_1}f$ for $j=2,3,\dots,m$ and from Lemma 1 it follows that

$$\int_E g'\,d\mu=\int_E T_r^{\gamma_n}\dots T_r^{\gamma_1}f\,d\mu=\int U_r^{\gamma_1}\dots U_r^{\gamma_n}\chi_E f\,d\mu\leqq\int (U_r^E)^n\chi_E f\,d\mu=\int_E (T_r^E)^n f\,d\mu$$

where $n=\sum_{i=1}^k n_i$ and the γ_i's, $\gamma_i\in L_\infty$, $0\leqq\gamma_i\leqq 1$, $i=1,2,\dots,n$, are determined in a similiar fashion to the above β_i's. Hence

$$\Psi_E f-\varepsilon<\int_E g'\,d\mu<\int_E (T_r^E)^n f\,d\mu\leqq\Psi_E f$$

which completes the proof.

Note that it follows from Lemma 2 that $\Psi_E f=\Omega_E f$ defined in [3].

Definition 3. Let $\tau=\{0,\tau_1,\tau_2,\dots,\tau_n\}$ denote a finite partition of $[0,\infty)$ into intervals such that $\tau_{i-1}<\tau_i$ and τ_i is rational for $1\leqq i\leqq n=n(\tau)$ and define $r_i=\tau_i-\tau_{i-1}$, $i=1,2,\dots,n$. Let $\mathscr{P}$ denote the class of all such finite τ partitions on $[0,\infty)$. Then for $E\in\mathscr{F}$ define

$$\psi_E^\tau=U_{r_n}^E\dots U_{r_1}^E\chi_E \qquad \text{where } n=n(\tau)$$

and

$$\psi_E=\sup_{\tau\in\mathscr{P}}\psi_E^\tau.$$

Note that since $U_{r_n}^E\dots U_{r_1}^E\chi_E\leqq(U_r^E)^N\chi_E$ where r denotes the reciprocal of the least common denominator of $r_r, r_2, \dots, r_n$ and $N=\sum_{i=1}^n n_i$ where n_i is a positive integer such that $r_i=n_i r$ for $i=1,2,\dots,n$ and $(U_r^E)^n\chi_E\uparrow$ as $n\uparrow$ (by induction for any fixed $r>0$) then it follows that $\{\psi_E^\tau|\tau\in\mathscr{P}\}$ is upward directed.

Lemma 3. *Let $E\in\mathscr{F}$; then for $f\in L_1$*

$$\Psi_E f=\int\psi_E f\,d\mu.$$

Proof. It is sufficient to prove the lemma for $f\in L_1^+$. Choose $\varepsilon>0$; by Lemma 2 there exists a rational $r=r(\varepsilon)$ and an integer n such that

$$\Psi_E f-\varepsilon<\int_E (T_r^E)^n f\,d\mu=\int (U_r^E)^n\chi_E f\,d\mu\leqq\int\psi_E f\,d\mu$$

which implies that $\Psi_E f\leqq\int\psi_E f\,d\mu$.

Since $\{\psi_E^\tau | \tau \in \mathscr{P}\}$ is upward directed and countable then there exists a sequence $\{\psi_E^{\tau_m}\}$ such that $\psi_E^{\tau_m} \uparrow \psi_E$ a.e. as $m\uparrow$ and hence for any $f \in L_1^+$ $\psi_E^{\tau_m} f \uparrow \psi_E f \leqq f$ a.e. By the Monotone Convergence Theorem we have $\sup_m \int \psi_E^{\tau_m} f\,d\mu = \int \psi_E f\,d\mu$ but since $\int \psi_E^{\tau_m} f\,d\mu \leqq \Psi_E f$ for all m then $\int \psi_E f\,d\mu \leqq \Psi_E f$ for any $f \in L_1^+$. The lemma follows.

Remark. If $g \succ f \in L_1^+$ then it follows from the definition that $\Psi_E g \leqq \Psi_E f$ for any $E \in \mathscr{F}$. Since, for fixed $t>0$ and all $n \geqq 1$, $T_{nt} f = T_t^0\, T_{(n-1)t} f \succ T_{(n-1)t} f$ where 0 denotes the zero L_∞ function, then $\Psi_E(T_{nt} f)\downarrow$ as $n\uparrow$. Using this fact and Lemma 3 we have that for $E \in \mathscr{F}$ and fixed $t>0$, $U_t^n \psi_E \downarrow$.

Definition 4. Let $\tau \in \mathscr{P}$, $\tau = \{0, \tau_1, \tau_2, \ldots, \tau_n\}$ and $r_i = \tau_i - \tau_{i-1}$, $i = 1, 2, \ldots, n = n(\tau)$ and $E \in \mathscr{F}$ then define

$$\theta_E^\tau = U_{r_n} \ldots U_{r_1} \psi_E \qquad n = n(\tau)$$

and

$$\theta_E = \inf_{\tau \in \mathscr{P}} \theta_E^\tau.$$

Note that since $U_t^n \psi_E \downarrow$ for fixed $t>0$ then $\{\theta_E^\tau | \tau \in \mathscr{P}\}$ is downward directed. Also, using the previous remark and the properties of $\prec$, it can be shown that $\{\Psi_E g | g \succ f, f \in L_1^+\}$ is downward directed which then yields $\Theta_E f = \lim_{g \succ f} \Psi_E g = \inf_{g \succ f} \Psi_E g = \lim_{g \succ f} \sup \int_E g\,d\mu$.

Lemma 4. *For $f \in L_1^+$, and $E \in \mathscr{F}$*

$$\Theta_E f = \int \theta_E f\,d\mu.$$

Proof. Choose $\varepsilon > 0$. There exists a $g \succ f$ such that $|\Theta_E f - \Psi_E g| < \varepsilon$ where $g = T_{t_n}^{\alpha_n} \ldots T_{t_1}^{\alpha_1} f$. Let $t = t_1 + t_2 + \cdots + t_n$ and choose any rational $r \geqq t$. Then we have $T_t f \succ g$ and $T_r f = T_{r-t}^0\, T_t f \succ T_t f$. Hence $\Theta f \leqq \Psi_E(T_r f) < \Psi_E(T_t f) \leqq \Psi_E g$ and $\Theta f + \varepsilon > \Psi_E(T_r f) = \int \psi_E T_r f\,d\mu = \int U_r \psi_E f\,d\mu > \int \theta_E f\,d\mu$ which implies $\Theta_E f \geqq \int \theta f\,d\mu$.

Since $\{\theta_E^\tau | \tau \in \mathscr{P}\}$ is downward directed and $\int \theta_E^\tau f\,d\mu \geqq \Theta_E f$ for all $\tau \in \mathscr{P}$ then using the Monotone Convergence Theorem we have $\int \theta_E f\,d \geqq \Theta_E f$ which completes the proof.

Note. Using the first part of the above proof it can be shown that $\Theta_E f = \lim_{r \to \infty} \Psi_E(T_r f)$.

Definition 5. For $E \in \mathscr{F}$ and a fixed positive rational r since $(U_r^E)^n \chi_E \uparrow$ then the limit exists and we define

$$\psi_E^r = \lim_{n \to \infty} (U_r^E)^n \chi_E.$$

Note that $\psi_E = \sup_{r \in R_r^+} \psi_E^r$ a.e. where R_r^+ is the set of positive rationals.

Lemma 5. *Let $F, E \in \mathscr{F}$, $F \supset E$, $f \in L_1^+$ and $r \in R_r^+$ then*

$$\int \psi_E^r f\,d\mu = \int \psi_E^r (T_r^F)^n f\,d\mu$$

and

$$\int \psi_E^r f\,d\mu = \lim_{n \to \infty} \int_F \psi_E^r (T_r^F)^n f\,d\mu.$$

Proof. Identical to the first part of proof of Lemma 2 [4].

Lemma 6. *Let $F, E \in \mathscr{F}$, $F \supset E$. For any $f \in L_1^+$ there exists a sequence of L_1^+ functions $\{g_n\}$ with the following properties:*

(i) $g_n \succ f$ *all* $n \geqq 1$.

(ii) $\lim_{n\to\infty} \int_F \psi_E\, g_n\, d\mu = \int \psi_E\, f\, d\mu$.

(iii) $\lim_{n\to\infty} \int_{E^c} \psi_E\, g_n\, d\mu = 0$.

(iv) $\lim_{n\to\infty} \int_{E^c} \theta_E\, g_n\, d\mu = 0$.

Proof. Since, for any $h \in L_1^+$, $\chi_E h = h$, $\int \psi_E\, h\, d\mu = \int_E h\, d\mu$ then $\psi_E = 1$ a.e. on E. By definition, for each $n \geqq 1$, there exists a $g_n \succ f$, g_n depending on $1/n$, such that $0 \leqq \Psi_E f - \int_E g_n\, d\mu < 1/n$. Hence, by construction, we have that

$$\lim_{n\to\infty} \int_E \psi_E\, g_n\, d\mu = \lim_{n\to\infty} \int_E g_n\, d\mu = \Psi_E f$$

and since $\int_E \psi_E\, g_n\, d\mu \leqq \int_F \psi_E\, g_n\, d\mu \leqq \Psi_E f$ then (i) and (ii) are satisfied.

Also

$$0 \leqq \int_{F^c} \psi_E\, g_n\, d\mu \leqq \int_{E^c} \psi_E\, g_n\, d\mu = \int \psi_E\, g_n\, d\mu - \int_E \psi_E\, g_n\, d\mu \leqq \Psi_E f - \int_E \psi_E\, g_n\, d\mu \to 0$$

as $n \to \infty$ which satisfies (iii).

Since $0 \leqq \Theta(\chi_{E^c} g_n) \leqq \Psi_E(\chi_{E^c} g_n)$ for all $n \geqq 1$ then (iv) follows.

Lemma 7. *Let $E_i \in \mathscr{F}$, $E = \bigcup_{i=1}^{m} E_i$ for $i = 1, 2, \ldots, m$ and $f \in L_1^+$. Then there exists a sequence of L_1^+ functions $\{g_n\}$ such that $g_n \succ f$ all $n \geqq 1$ and $\lim_{n\to\infty} \int_E \psi_{E_i}\, g_n\, d\mu = \int \psi_{E_i}\, f\, d\mu$ for $i = 1, 2, \ldots, m$.*

Proof. It is sufficient to prove the lemma for $m = 2$. We shall show that for any $\varepsilon > 0$ there exists a $g = g(\varepsilon)$, $g \succ f$ with the property that $0 \leqq \int \psi_{E_i} f\, d\mu - \int_E \psi_{E_i}\, g\, d\mu < \varepsilon$ for $i = 1, 2$ where $E = E_1 \cup E_2$.

Choose $\varepsilon > 0$. There exist positive rationals r_1, r_2 and positive integers n_1, n_2 such that

$$0 \leqq \int \psi_{E_i} f\, d\mu - \int_{E_i} (T_{r_i}^{E_i})^{n_i} f\, d\mu < \varepsilon/2 \qquad i = 1, 2.$$

Let r denote the reciprocal of the least common denominator of r_1 and r_2. Then it follows that $(U_{r_i}^{E_i})^{n_i} \chi_{E_i} \leqq (U_r^{E_i})^n \chi_{E_i}$ for $i = 1, 2$ where $n = \max\{n_1, m_1, n_2\, m_2\}$ where $r_i = m_i r$, $i = 1, 2$. Hence

$$\int \psi_{E_i} f\, d\mu - \varepsilon/2 < \int_{E_i} (T_{r_i}^{E_i})^{n_i} f\, d\mu \leqq \int_{E_i} (T_r^{E_i})^n f\, d\mu \leqq \int \psi_{E_i}^r f\, d\mu \leqq \int \psi_{E_i} f\, d\mu$$

for $i = 1, 2$.

By Lemma 5 there exist integers p_1 and p_2 such that

$$0 \leqq \int \psi_{E_i}^r f\, d\mu - \int_E \psi_{E_i}^r (T_r^E)^{p_i} f\, d\mu < \varepsilon/2 \qquad \text{for } i = 1, 2.$$

Letting $p=\max\{p_1,p_2\}$ we have

$$\int \psi_{E_i} f\,d\mu \geqq \int \psi_{E_i}(T_r^E)^p f\,d\mu \geqq \int_E \psi_{E_i}(T_r^E)^p f\,d\mu \geqq \int_E \psi_E^r(T_r^E)^p f\,d\mu > \int \psi_{E_i}^r f\,d\mu - \varepsilon/2$$
$$> \int_{E_i} f\,d\mu - \varepsilon \quad \text{for } i=1,2.$$

Defining $g(\varepsilon)=g=(T_r^E)^p f$ we have $g \succ f$ and

$$0 \leqq \int \psi_{E_i} f\,d\mu - \int_E \psi_{E_i} g\,d\mu < \varepsilon \quad \text{for } i=1,2.$$

Define $g_n=g(1/n)$, $n\geqq 1$. Then by construction we have $g_n \succ f$, $n\geqq 1$ and $\lim_{n\to\infty} \int_E \psi_{E_i} g_n\,d\mu = \int \psi_{E_i} f\,d\mu$ for $i=1,2$ which completes the proof.

Lemma 8. *If a_i is real and $E_i\in\mathscr{F}$ for $i=1,2,\ldots,m$ and $E=\bigcup_{i=1}^m E_i$ then $\chi_E \sum_{i=1}^m a_i \psi_{E_i} \geqq 0$ implies $\sum_{i=1}^m a_i \psi_{E_i} \geqq 0$ and $\sum_{i=1}^m a_i \theta_{E_i} \geqq 0$ a.e.*

Proof. Choose any $f\in L_1^+$. By Lemma 7 there exists a sequence $\{g_n\}$ $g_n \succ f$ all $n\geqq 1$ and $\lim_{n\to\infty} \int_E \psi_{E_i} g_n\,d\mu = \int \psi_{E_i} f\,d\mu$ for $i=1,2,\ldots,m$. Hence $0\leqq \sum_{i=1}^m a_i \int \psi_{E_i} g_n\,d\mu \to \sum_{i=1}^m a_i \int \psi_{E_i} f\,d\mu$ as $n\to\infty$. Since $\sum_{i=1}^m a_i \int \psi_{E_i} f\,d\mu \geqq 0$ for all $f\in L_1^+$ then $\sum_{i=1}^m a_i \psi_{E_i} \geqq 0$ a.e.

For $f\in L_1^+$

$$0 \leqq \lim_{g\succ f} \sum_{i=1}^m a_i \int \psi_{E_i} g\,d\mu = \sum_{i=1}^m a_i \lim_{g\succ f} \int \psi_{E_i} g\,d\mu = \sum_{i=1}^m a_i \int \theta_{E_i} f\,d\mu$$

which implies that $\sum_{i=1}^m a_i \theta_{E_i} \geqq 0$ a.e.

Definition 6. For $E, F\in\mathscr{F}$ let

$$\psi_{EF}=\psi_E+\psi_F-\psi_{E\cup F} \quad \text{and} \quad \theta_{EF}=\theta_E+\theta_F-\theta_{E\cup F}.$$

Ψ_{EF} and Θ_{EF}, the functionals on L_1 defined by the L_∞ functions ψ_{EF} and θ_{EF} are monotone and subadditive in each index.

Lemma 9. *If $\chi_E \theta_E \geqq a\chi_E$ a.e. where $E\in\mathscr{F}$ and a a real number then $\theta_E \geqq a\psi_E$ a.e.*

Proof. It is sufficient to assume $a>0$. Choose $f\in L_1^+$ and $\varepsilon>0$. There exist $g\succ f$ such that $\int_E g\,d\mu > \Psi_E f - \varepsilon/a$. Since $\Theta_E f = \lim_{g'\succ f} \Psi_E g' = \Theta_E g$ for any $g\succ f$ then we have

$$\Theta_E f = \Theta_E g = \int \theta_E g\,d\mu \geqq \int_E \theta_E g\,d\mu \geqq a\int_E g\,d\mu > a(\Psi_E f - \varepsilon/a)$$

or $\int \theta_E f\,d\mu - a\int \psi_E f\,d\mu > -\varepsilon$ which implies that $\theta_E \geqq a\psi_E$ a.e. on X.

Lemma 10. *For $E, F\in\mathscr{F}$, $\|\theta_E\|_\infty = \|\chi_E\theta_E\|_\infty = 0$ or 1 and $\|\theta_{Ef}\|_\infty = \|\chi_E\theta_{EF}\|_\infty = \|\chi_F\theta_{EF}\|_\infty = 0$ or 1.*

Proof. For $f \in L_1^+$ there exists a sequence $\{g_n\}$ of L_1^+ functions such that $g_n \succ f$, $n \geqq 1$ and $\theta(\chi_{E^c} g_n) \to 0$ as $n \to \infty$. Then $\Theta_E(f) = \Theta_E(g_n) = \Theta_E(\chi_E g_n) + \Theta_E(\chi_{E^c} g_n)$ shows that $\|\theta_E\|_\infty = \|\chi_E \theta_E\|_\infty$.

For $f \in L_1^+$ and a rational $r > 0$ there exists a sequence $\{g_n(r)\}$ of L_1^+ functions such that $g_n(r) \succ T_r f$, $n \geqq 1$ and satisfying Lemma 6. Hence

$$\begin{aligned} \Theta_E(f) &= \Theta_E(T_r f) = \Theta_E(g_n(r)) = \lim_{r\to\infty} \lim_{n\to\infty} \Theta_E(\chi_E g_n(r)) \\ &\leqq \lim_{r\to\infty} \lim_{n\to\infty} \|\theta_E\|_\infty \|\chi_E g_n(r)\|_1 = \lim_{r\to\infty} \|\theta_E\|_\infty \Psi_E(T_r f) = \|\theta_E\|_\infty \Theta_E(f) \end{aligned}$$

this completes the first part of the proof since $\|\theta_E\|_\infty \leqq 1$.

For the second part let $\{g_n\}$ and $\{g_n(r)\}$ be the same as above; since

$$0 \leqq \Theta_{EF}(\chi_{E^c} g_n) \leqq \Theta_E(\chi_{E^c} g_n) \to 0 \qquad \text{as } n \to \infty$$

then it follows that $\|\theta_{EF}\|_\infty = \|\chi_E \theta_{EF}\|_\infty$.

Now

$$\Theta_E(f) - \Theta_E(\chi_{E^c} g_n(r)) = \Theta_E(\chi_E g_n(r)) \leqq \Theta_{E\cup F}(\chi_E g_n(r)) \leqq \|\chi_E g_n(r)\|_1 \leqq \Psi_E(T_r f)$$

and

$$\Theta_E(f) \leqq \lim_{n\to\infty} \theta_{E\cup F}(\chi_E g_n(r)) \leqq \Psi_E(T_r f).$$

Letting $r \to \infty$ we have $\Theta_E(f) = \lim_{r\to\infty} \lim_{n\to\infty} \Theta_{E\cup F}(\chi_E g_n(r))$. Since

$$\Theta_{EF}(\chi_E g_n(r)) = (\Theta_E + \Theta_F - \Theta_{E\cup F})(\chi_E g_n(r))$$

then, first letting $n \to \infty$ then $r \to \infty$, we have

$$\Theta_{EF}(f) = \Theta_E(f) + \lim_{n\to\infty} \Theta_F(\chi_E g_n(r)) - \lim_{n\to\infty} \Theta_{E\cup F}(\chi_E g_n(r))$$

and

$$\Theta_{EF}(f) = \lim_{r\to\infty} \lim_{n\to\infty} \Theta_F(\chi_E g_n(r)).$$

also

$$\begin{aligned} \Theta_{EF}(f) &= \lim_{r\to\infty} \lim_{n\to\infty} \Theta_{EF}(\chi_E g_n(r)) \\ &\leqq \|\theta_{EF}\|_\infty \lim_{r\to\infty} \lim_{n\to\infty} \|\chi_E g_n(r)\|_1 \\ &\leqq \|\theta_{EF}\|_\infty \lim_{r\to\infty} \Psi_E(T_r f) \\ &\leqq \|\theta_{EF}\|_\infty \Theta_E(f). \end{aligned}$$

Hence

$$\begin{aligned} \Theta_{EF}(f) &= \lim_{r\to\infty} \lim_{n\to\infty} \Theta_{EF}(\chi_E g_n(r)) \\ &\leqq \|\theta_{EF}\|_\infty \lim_{r\to\infty} \lim_{n\to\infty} \Theta_F(\chi_E g_n(r)) \\ &\leqq \|\theta_{EF}\|_\infty \Theta_{EF}(f) \end{aligned}$$

which completes the proof since $\|\theta_{EF}\|_\infty \leqq 1$.

Definition 7. $\Sigma = \{E \in \mathscr{F} \mid \theta_{EE^c} = 0\}$.

Note that since $0 \leqq \theta_{GG^c} \leqq \theta_{G(E^c \cup F^c)} \leqq \theta_{GE^c} + \theta_{GF^c} \leqq \theta_{EE^c} + \theta_{FF^c} = 0$ for $E, F \in \Sigma$ and $G = E \cap F$ then Σ is a field.

Definition 8. $\mathscr{A}$ is the L_∞ closure of Σ-simple functions.

$\mathscr{A}$ is a sub-Banach space of L_∞.

Theorem 1. *For $f \in L_\infty$ the following conditions are equivalent:*

(i) $f \in \mathscr{A}$.

(ii) $\lim_{g \succ g_0} \int f g \, d\mu$ *exists for all* $g_0 \in L_1^+$.

(iii) *for all real numbers α and $\varepsilon > 0$, $\theta_{EF} = 0$ where $E = \{x \mid f(x) \leqq \alpha\}$ and $F = \{x \mid f(x) \geqq \alpha + \varepsilon\}$.*

Proof. The proof for (i) → (ii) and (iii) → (i) is identical to the proof of Theorem 1 [4].

For (ii) → (iii) suppose that E and F are as in (iii) but that $\theta_{EF} \neq 0$ then $\|\theta_{EF}\|_\infty = 1$ and for all $\delta > 0$ there exists a $g_0 \in L_1^+$ with $\|g_0\|_1 = 1$ and $\int \theta_{EF} \, g_0 \, d\mu > 1 - \delta$. Hence $\Theta_E \, g_0 > 1 - \delta$ and $\Theta_F \, g_0 > 1 - \delta$ and $\limsup_{g \succ g_0} \int f g \, d\mu > (1-\delta)(\alpha + \varepsilon)$ and $\liminf_{g \succ g_0} \int f g \, d\mu < \alpha + \|f\|_\infty \, \delta$. If δ is chosen sufficiently small then $\lim_{g \succ g_0} f g \, d\mu$ does not exist.

Identification of a Ratio Ergodic Limit

For any $f, g \in L_1$ with $g > 0$ it was proved in [5] and [3] that the limit

$$(f/g) = \lim_{s \to \infty} \frac{\int_0^s T_t f \, dt}{\int_0^s T_t g \, dt}$$

exists a.e. We shall identify this limit function (f/g).

Using the fact that $\Psi_E f = \Omega_E(f)$ for any $f \in L_1$ and Theorem 1 in [3] it follows that if $\alpha \leqq (f/g) \leqq \beta$ a.e. on $E \in \mathscr{F}$ then $\alpha \leqq \Psi_E f / \Psi_E g \leqq \beta$ (cf. [6, 1]).

Theorem 2. *If $f, g \in L_1^+$ with $g > 0$ and $E = \{x \mid (f/g)(x) \leqq a\}$, $F = \{x \mid (f/g)(x) \geqq a + \varepsilon\}$ then $\theta_{EF} = 0$ for all $a \geqq 0$ and $\varepsilon > 0$.*

Proof. Identical to the proof of Theorem 3 [4].

Corollary. *If $f, g \in L_1$, $g > 0$ and $(f/g) \in L_\infty$ then $(f/g) \in \mathscr{A}$.*

Theorem 3. *If $(f/g) \in L_\infty$, $f, g \in L_1$, $g > 0$ and $h \in \mathscr{A}$ then $\lim_{t \to \infty} \int h \, T_t f \, d\mu = \lim_{t \to \infty} \int h (f/g) \, T_t \, g \, d\mu$.*

Proof. Recall that $(f/g) \in \mathscr{A}$. Choose $\varepsilon > 0$. There exists a Σ partition of $X \{E_{ij}\}$, $1 \leqq i$, $j \leqq k$ such that $\|h - \sum_{ij} h_i \chi_{E_{ij}}\|_\infty < \varepsilon$ and $\|(f/g) - \sum_{ij} \alpha_j \chi_{E_{ij}}\|_\infty < \varepsilon$ for suitable real h_i, α_j with $|h_i| \leqq \|h\|_\infty$ $|\alpha_j| \leqq \|(f/g)\|_\infty$. Since for $E \in \Sigma$ $\lim_{t \to \infty} \int_E T_t \, g \, d\mu = \Theta_E \, g$ then

$$\begin{aligned} &\left|\lim_{t \to \infty} \int h (f/g) \, T_t \, g \, d\mu - \sum_{ij} h_i \, \alpha_j \lim_{t \to \infty} \int_{E_{ij}} T_t \, g \, d\mu\right| \\ &= \left|\lim_{t \to \infty} \int h (f/g) \, T_t \, g \, d\mu - \sum_{ij} h_i \, \alpha_i \, \Theta_{E_{ij}} \, g\right| \leqq \varepsilon \|g\|_1 (\|h\|_\infty + \|(f/g)\|_\infty). \end{aligned}$$

Let $\delta > 0$ be fixed and set $E'_{ij} = \{x \mid \theta_{E_{ij}}(x) \geqq 1-\delta\} \cap E_{ij}$. From the proof of Theorem 2 and from Lemma 9 we have that $\theta_{E'_{ij}} = \theta_{E_{ij}}$ and $\theta_{E'_{ij}} \geqq (1-\delta)\,\psi_{E'_{ij}}$. Also, if $|\alpha_j - (f/g)| \leqq \varepsilon$ on E'_{ij} then $|\alpha_j - \Psi_{E'_{ij}} f / \Psi_{E'_{ij}} g| \leqq \varepsilon$. Note that we shall only consider those E_{ij}'s with $\theta_{E_{ij}} \neq 0$. Thus

$$\Big|\sum_{ij} h_i\, \alpha_j\, \Theta_{E_{ij}} g - \sum_{\theta_{E_{ij}} \neq 0} h_i \frac{\Psi_{E'_{ij}} f}{\Psi_{E'_{ij}} g}\, \Theta_{E_{ij}} g\Big| \leqq \varepsilon\, \|h\|_\infty\, \|g\|_1.$$

Also

$$\Big|\sum_{\theta_{E_{ij}} \neq 0} h_i \frac{\Psi_{E'_{ij}} f}{\Psi_{E'_{ij}} g}\, \Theta_{E_{ij}} g - \sum_{ij} h_i\, \Psi_{E'_{ij}} f\Big| \leqq \|h\|_\infty\, \|f\|_1\, \delta k^2.$$

Finally,

$$\Big|\sum_{ij} h_i\, \Psi_{E'_{ij}} f - \sum_{ij} h_i\, \Theta_{E_{ij}} f\Big| \leqq \|h\|_\infty\, \|f\|_1\, k^2\, \delta$$

and

$$\Big|\sum_{ij} h_i\, \Theta_{E_{ij}} f - \lim_{t\to\infty} \int h\, T_t f\, d\mu\Big| \leqq \varepsilon\, \|f\|_1.$$

Combining these inequalities we obtain the result.

Definition 9. $\mathscr{H} = \{f \mid f \in L_\infty, f = U_t f \text{ all } t \geqq 0\}$ is the class of invariant functions of $\{U_t \mid t \geqq 0\}$.

We assume that $\mathscr{H} \neq \{0\}$. Note that $\mathscr{H}$ is a sub-Banach space of L_∞ and if $h \in \mathscr{H}$ and $g' \succ g \in L_1^+$ then by the definition of $\mathscr{H}$ we have $\int h g'\, d\mu = \int h g\, d\mu$ which yields the existence of $\lim_{g' \succ g} \int h g'\, d\mu$. Hence $\mathscr{H} \subset \mathscr{A}$. Also, we note that for $f \in \mathscr{A}$ $\lim_{g' \succ g} \int f g'\, d\mu = \lim_{t\to\infty} \int f\, T_t g\, d\mu = \lim_{r\to\infty} \int f\, T_r g\, d\mu = \lim_{t\to\infty} \int U_t f g\, d\mu$ exists for all $g \in L_1$ and $|U_t f| \leqq \|f\|_\infty$ a.e. on X for all $t \geqq 0$. Hence $\{U_t f\}_{t \geqq 0}$ has a limit $\Pi(f)$ in the ω^* topology of L_∞ and $\Pi(f) \in \mathscr{H}$ so $\Pi: \mathscr{A} \to \mathscr{H}$ is a positive linear contraction.

By the definition of Π and Theorem 3 it follows that

$$\int \Pi(h) \cdot f\, d\mu = \int \Pi\big(h(f/g)\big)\, g\, d\mu \quad \text{if} \quad h \in \mathscr{A} \quad \text{and} \quad (f/g) \in L_\infty.$$

We now introduce complex valued functions in order to apply the Gelfand-Naimark representation theorem which will lead to an identification of (f/g).

Let $L'_1 = L'_1(X, \mathscr{F}, \mu)$ and L'_∞ denote the usual Banach spaces of complex valued μ-integrable functions and complex valued μ-measurable functions bounded a.e. respectively. Defining Σ as in Definition 7, we let $\mathscr{A}'$ denote the L'_∞ closure of the simple complex-valued functions and $\mathscr{H}' = \{f \mid f \in L'_\infty,\ U_t \operatorname{Re}(f) = \operatorname{Re}(f)$ and $U_t \operatorname{Im}(f) = \operatorname{Im}(f)$ for all $t \geqq 0\}$ where $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ denote the real and imaginary parts of f respectively.

Note that $\mathscr{A}'$ and $\mathscr{H}'$ are sub-Banach spaces of L'_∞ and $f \in \mathscr{A}'$ if and only if $\operatorname{Re}(f)$ and $\operatorname{Im}(f) \in \mathscr{A}$. (Similarly for $\mathscr{H}'$ and $\mathscr{H}$.) By the definitions and Theorem 1 we have that $\mathscr{H}' \subset \mathscr{A}'$. The mapping $\Pi: \mathscr{A} \to \mathscr{H}$ induces a positive bounded linear operator $\Pi': \mathscr{A}' \to \mathscr{H}'$ defined by

$$\Pi' f = \Pi\big(\operatorname{Re}(f)\big) + i\,\Pi\big(\operatorname{Im}(f)\big).$$

Definition 10. $\mathscr{A}_0 = \ker \Pi'$ where $\ker \Pi' = \{f \mid f \in \mathscr{A}', \Pi' f = 0\}$.

Hence $\mathscr{A}'/\mathscr{A}_0 \cong \mathscr{H}'$ is a canonical isomorphism.

Note that $\mathscr{A}'$ is a B^*-Algebra with the usual operations. We now show that $\mathscr{A}_0$ is a closed ideal.

Lemma 11. *$\mathscr{A}_0$ is a closed ideal.*

Proof. Clearly $\mathscr{A}_0$ is a subspace of $\mathscr{A}'$ and $\mathscr{A}_0$ is closed since Π' is bounded.

Let $f \in \mathscr{A}_0$ and assume f is real. We shall show that for any $h \in \mathscr{A}'$, $h \neq 0$, $fh \in \mathscr{A}_0$. It is sufficient to assume that h is real.

Choose $\varepsilon > 0$ and set $E = \{x \mid f(x) \geqq \varepsilon\}$. We may assume $E \in \Sigma$. Suppose $\theta_E \neq 0$; then for any $\delta > 0$ there exists a $g \in L_1^+$ such that $\|g\|_1 = 1$ and $\Theta_E(g) > (1-\delta)$.

Thus

$$\begin{aligned} 0 = \lim_{t\to\infty} \int U_t f \cdot g \, d\mu &= \lim_{t\to\infty} \int f \cdot T_t g \, d\mu \\ &\geqq \varepsilon \lim_{t\to\infty} \int_E T_t g \, d\mu - \|f\|_\infty \lim_{t\to\infty} \int_{E^c} T_t g \, d\mu \\ &\geqq \varepsilon(1-\delta) - \|f\|_1 \delta \end{aligned}$$

since $E \in \Sigma$ which yields $\lim_{t\to\infty} \int_E T_t g \, d\mu = \Theta_E(g)$ and $\Theta_E(g) + \Theta_{E^c}(g) = \Theta_X(g) \leqq \|g\|_1$. This inequality is false for small δ so $\Theta_E = 0$. Then if $E = \{x \mid |f(x)| \geqq \varepsilon\}$ it follows that $\theta_E = 0$. Now set $F = \{x \mid |f(x)\, h(x)| \geqq \varepsilon\}$. Then since $F \subset \{x \mid |f(x)| \geqq \varepsilon\} \|h\|_\infty$ we have $\theta_F = 0$. Thus

$$\begin{aligned} \left|\lim_{t\to\infty} \int U_t(fh)\, g \, d\mu\right| &\leqq \|f\|_\infty \|h\|_\infty \lim_{t\to\infty} \int_F T_t g \, d\mu + \varepsilon \|g\|_1 \quad \text{if } g \in L_1^+ \\ &\leqq \varepsilon \|g\|_1 \quad \text{for all } \varepsilon > 0. \end{aligned}$$

Hence $fh \in \mathscr{A}_0$ and then for any $h \in \mathscr{A}'$ $fh \in \mathscr{A}_0$ which completes the proof.

Since $\mathscr{A}_0$ is a closed ideal then $\mathscr{A}'/\mathscr{A}_0$ and hence $\mathscr{H}'$ are B^*-algebras. Note that multiplication on $\mathscr{H}'$ is given by $\Pi'(h_1 \cdot h_2)$ where $h_1, h_2 \in \mathscr{H}'$. Also, since Π' is linear and $\Pi'^2 = \Pi'$ it follows that $\Pi'(h \cdot \Pi' f) = \Pi'(h \cdot f)$ where $h \in \mathscr{H}'$ and $f \in \mathscr{A}'$.

Let K denote the maximal ideal space of $\mathscr{H}'$; we note that K is a compact Hausdorff space. By the Gelfand Naimark Theorem, $C(K)$, the B^* algebra of continuous complex valued functions on K, is isometrically * isomorphic to $\mathscr{H}'$ under the mapping σ: $\mathscr{H}' \to C(K)$. Also, by the Riesz Representation theorem, we have that $M(K)$, the space of finite complex Baire measures on K, is isomorphic to the conjugate space of $C(K)$.

We now define the following mappings:

$$\tau\colon\ \mathscr{A}' \to C(K) \qquad \text{where } \tau = \sigma\,\Pi'$$

and

$$\lambda\colon\ L_1' \to M(K)$$

such that for $f \in L_1'$, $\int_X f \cdot h \, d\mu = \int_K \sigma h \cdot d(\lambda f)$ for any $h \in \mathscr{H}'$.

Theorem 4. (*An identification of* (f/g).)

Let $f, g \in L_1'$, $\operatorname{Im}(f) = \operatorname{Im}(g) = 0$, $g > 0$ *and* $(f/g) \in L_\infty$ *then*

$$\tau(f/g) = \frac{d(\lambda f)}{d(\lambda g)}.$$

Proof. Let $h \in \mathscr{H}'$; then we have to show that

$$\int_K \sigma h \cdot d(\lambda f) = \int_K \sigma h \cdot \tau(f/g)\, d(\lambda g).$$

Hence

$$\begin{aligned}\int_K \sigma h \cdot \tau(f/g)\, d(\lambda g) &= \int_K \sigma h \cdot \sigma \Pi'(f/g)\, d(\lambda g)\\ &= \int_K \sigma\big(\Pi'(h \cdot \Pi'(f/g))\big)\, d\lambda g = \int_X \Pi'\big(h \cdot \Pi'(f/g)\big)\, g\, d\mu = \int_X \Pi'\big(h \cdot (fg)\big)\, g\, d\mu\\ &= \int_X \Pi' h \cdot f\, d\mu = \int_X h \cdot f\, d\mu = \int_K \sigma h \cdot d(\lambda f)\end{aligned}$$

where the fifth equality follows from Theorem 3.

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