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Certain Induced Measures and the Fractional Dimensions of their "Supports"*

By

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§ 1

In this paper, I shall discuss the fractional dimensions of the "supports" of certain measures on [0, 1]. The word "support" is used not in the sense of being the smallest closed set carrying the total measure but rather the smallest such set in the sense of fractional dimensions. (See section 3 for a rigorous definition.) These measure are the measures induced by such random variables (r.v.) as $X = \sum_{i=1}^{\infty} X_i k^{-i}$ where $\{X_i\}$ are independent (not necessarily identically distributed) r.v.'s taking values from the set $\{0, 1, 2, \ldots (k-1)\}$. The statements and proofs, however, are given in terms of product measures and convolutions mostly to make the paper readable to the non-probabilistic mathematical fraternity and partly to satisfy my own whims.

In section 2, a slight extension of a well-known theorem of KAKUTANI (see [1]) is given a direct proof using theorems of ANDERSEN and JESSEN (Martingale theory to probabilists). From this a simple set of necessary and sufficient conditions for the absolute continuity etc. of the above-mentioned measures (Corollary 1) is deduced. (A special case, $k = 2$, was treated by elementary methods by the author earlier in [4], in total ignorance of KAKUTANI's result.) In section 3, using BILLINGSLEY's definitions and theorems (see [6], [7]) I calculate the promised dimensions. In the last section, I demonstrate how Corollary 1 and the work in section 3 allow one to decompose, quite effortlessly, absolutely continuous measures of the type referred to above (which includes the Lebesgue measure) into a convolution of two singular measures, both of which "sit" on very small sets (even of dimension zero).

§ 2

In this section, I shall present a very brief proof of a slightly extended version of a theorem of KAKUTANI [1]. Let $(\Omega_n, \mathfrak{R}_n)$ $n \geqq 1$ be a sequence of measurable spaces (for notation and terminology see HALMOS [2]) and let $\mu_n, \sigma_n, n \geqq 1$ be two probability measures on $(\Omega_n, \mathfrak{R}_n)$. Suppose μ_n is absolutely continuous with respect to σ_n, in symbols $\mu_n \ll \sigma_n$. Let

$$\Omega = \mathop{X}_{n=1}^{\infty} \Omega_n, \quad \mathfrak{R} = \mathop{X}_{n=1}^{\infty} \mathfrak{R}_n, \quad \mu = \mathop{X}_{n=1}^{\infty} \mu_n, \quad \sigma = \mathop{X}_{n=1}^{\infty} \sigma_n.$$

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Theorem 1 (Kakutani). *Either $\mu \ll \sigma$ or else $\mu \perp \sigma$ ($\perp$ = singular).*
A necessary and sufficient condition that $\mu \ll \sigma$ is that

$$\prod_{n=1}^{\infty} \varrho(\mu_n, \sigma_n) > 0$$

where

$$\varrho(\mu_n, \sigma_n) = \int \left(\frac{d\mu_n}{d\sigma_n}\right)^{1/2} d\sigma_n .$$

Proof. Let $\mu^{(n)}$, $\sigma^{(n)}$ be restrictions of μ, σ to $\mathfrak{R}^{(n)}$ respectively where

$$\mathfrak{R}^{(n)} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \cdots \times \mathfrak{R}_n \times \Omega_{n+1} \times \cdots .$$

Clearly $\mu^{(n)} \ll \sigma^{(n)}$, $\mathfrak{R}^{(n)} \subset \mathfrak{R}^{(n+1)}$.

Let
$$f_n(\omega) = \frac{d\mu^{(n)}}{d\sigma^{(n)}} .$$

(Then $\{f_n(\omega), \mathfrak{R}^{(n)}, n \geqq 1\}$ is a martingale on $(\Omega, \mathfrak{R}, \sigma)$.)

The following statement, essentially a rephrasing of a general theorem due to Andersen and Jessen [*8*], (see also Doob [*3*], pp. 630—2) will be used:

i) $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ exists a.e. (σ).

ii) $\mu \perp \sigma$ if and only if $f(\omega) = 0$ a.e. (σ).

iii) $\mu \ll \sigma$ if and only if $f_n(\omega)$ is an uniformly integrable sequence (w.r.t. σ).

Let $\pi_n(\omega) = \omega_n$ where $\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots)$. Clearly

$$f_n(\omega) = \prod_{i=1}^{n} \frac{d\mu_i}{d\sigma_i} (\pi_i(\omega)) .$$

Hence
$$\int f_n^{1/2}(\omega)\, d\sigma(\omega) = \prod_{i=1}^{n} \int \left(\frac{d\mu_i}{d\sigma_i}\right)^{1/2} d\sigma_i = \prod_{i=1}^{n} \varrho(\mu_i, \sigma_i) > 0 .$$

Since
$$(f_n(\omega))^{1/2} \to (f(\omega))^{1/2} \quad \text{a.e.} \quad (\sigma),$$

by Fatou's lemma one has

$$\prod_{i=1}^{\infty} \varrho(\mu_i \sigma_i) = \lim_{n\to\infty} \int (f_n(\omega))^{1/2} d\sigma \geqq \int f^{1/2}(\omega)\, d\sigma .$$

Hence the vanishing of $\prod_{i=1}^{\infty} \varrho(\mu_i, \sigma_i)$ will imply that $f(\omega) = 0$ a.e. (σ). If on the other hand $\prod_{i=1}^{\infty} \varrho(\mu_i, \sigma_i) > 0$ then $\{f_n(\omega)\}$ is an uniformly integrable sequence. Indeed, if $m < n$

$$\begin{aligned}
&\int |f_m(\omega) - f_n(\omega)|\, d\sigma(\omega) \\
&= \int |f_n^{1/2} - f_m^{1/2}| \cdot |f_n^{1/2} + f_m^{1/2}|\, d\sigma \\
&\leqq (\int |f_n^{1/2} - f_m^{1/2}|^2 d\sigma)^{1/2} (\int |f_n^{1/2} + f_m^{1/2}|\, d\sigma)^{1/2} \\
&\leqq 8^{1/2} (1 - \prod_{m+1}^{n} \varrho(\mu_i, \sigma_i))^{1/2} \qquad \text{(by Schwarz inequality)}
\end{aligned}$$

whence the L_1-mean-fundamental nature and consequently uniform integrability of $\{f_n\}$ may be inferred. The last inequality follows from the following computations:

$$\int |f_n^{1/2} - f_m^{1/2}|^2 \, d\sigma = \int \{f_n + f_m - 2(f_n f_m)^{1/2}\} \, d\sigma$$
$$= 2\Big(1 - \prod_{m+1}^{n} \varrho(\mu_i, \sigma_i)\Big) \quad \text{since} \quad \int f_n \, d\sigma = 1.$$

and similarly

$$\int |f_n^{1/2} + f_m^{1/2}|^2 \, d\sigma = 2\Big(1 + \prod_{m+1}^{n} \varrho(\mu_i, \sigma_i)\Big) \leqq 4.$$

A reference to the above-mentioned paraphrase of a theorem of Andersen and Jessen completes the proof of theorem 1.

A simple special case of interest in the theory of real variables is obtained by specializing to $\Omega_i = \{0, 1, 2, \ldots (k-1)\}$. Consider

$$\Phi(\omega) = \sum_{i=1}^{\infty} \pi_i(\omega) \, k^{-i}$$

and let

$$\mu_i(\{j\}) = \mu_{ij} \geqq 0, \quad \sum_{0}^{k-1} \mu_{ij} = 1; \quad \sigma_i(\{j\}) = 1/k \quad \begin{matrix} i \geqq 1 \\ 0 \leqq j \leqq k-1. \end{matrix}$$

$\Phi(\omega)$ is a mapping of Ω onto [0, 1]. Clearly $\sigma\Phi^{-1}$ is the Lebesgue measure λ on [0, 1] (see Halmos [2], pp. 159) and let $\mu\Phi^{-1} = \nu$. The following then follows immediately from theorem 1:

Corollary 1. a) $\nu \ll \lambda$ *if and only if* $\sum_{j=0}^{k-1} \sum_{i=1}^{\infty} (\mu_{ij} - 1/k)^2 < +\infty$.

b) $\nu \perp \lambda$ *if and only if* $\sum_{j=0}^{k-1} \sum_{i=1}^{\infty} (\mu_{ij} - 1/k)^2 = +\infty$.

c) ν *is purely nonatomic if and if*

$$\prod_{i=1}^{\infty} \max\{\mu_{ij} \mid 0 \leqq j \leqq k-1\} = 0$$

otherwise ν *is purely atomic.*

Proof of Corollary 1. Clearly here $\varrho(\mu_i, \sigma_i) = k^{-1/2} \sum_{j=0}^{k-1} \mu_{ij}^{1/2}$. I shall show that $\sum_{i=1}^{\infty} \{1 - \varrho(\mu_i, \sigma_i)\} < +\infty$ if and only if $\sum_{j=0}^{k-1} \sum_{k-1}^{\infty} (\mu_{ij} - k^{-1})^2 < +\infty$.

First suppose that $\sum_{i=1}^{\infty} \{1 - \varrho(\mu_i, \sigma_i)\} < +\infty$. Put $\mu_{ij} = k^{-1} + \varepsilon_{ij}$. Then

$$\sum_{i=1}^{k-1} \varepsilon_{ij} = 0 \quad \text{and} \quad -k^{-1} \leqq \varepsilon_{ij} \leqq 1 - k^{-1}.$$

By using Taylor's expansion one easily obtains the inequality

$$(k^{-1}+t)^{1/2} \leqq k^{-1/2} + \frac{t\,k^{1/2}}{2} - \frac{t^2}{8} \quad \text{for} \quad -k^{-1} \leqq t \leqq 1-k^{-1}$$

whence

$$1-\varrho(\mu_i,\sigma_i) \geqq \frac{k^{-1/2}}{8} \sum_{j=0}^{k-1} \varepsilon_{ij}^2 .$$

Thus the convergence of $\sum_{j=0}^{\infty} \{1-\varrho(\mu_i,\sigma_i)\}$ implies the convergence of

$$\sum_{j=0}^{k-1} \sum_{i=1}^{\infty} \varepsilon_{ij}^2 = \sum_{j=0}^{k-1} \sum_{i=1}^{\infty} (\mu_{ij}-k^{-1})^2 .$$

Suppose now that

$$\sum_{j=0}^{k-1} \sum_{i=1}^{\infty} (\mu_{ij}-k^{-1})^2 < +\infty .$$

Then in particular $\lim_{i\to\infty} \mu_{ij} = k^{-1}$ $0 \leqq j \leqq k-1$ and hence for i sufficiently large μ_{ij}'s are all bounded away (uniformly) from zero; i.e. ε_{ij} above are such that there exists a $\delta > 0$ for which $\delta - k^{-1} \leqq \varepsilon_{ij} \leqq 1-k^{-1}$ for $i \geqq N$. By using Taylor's expansion one can again show that $(k^{-1}+t)^{1/2} \geqq k^{-1/2} + \frac{t\,k^{1/2}}{2} - c\,t^2$ for $\delta - k^{-1} \leqq t \leqq 1-k^{-1}$ for some $c > 0$ so that now for

$$i \geqq N, \quad 1-\varrho(\mu_i,\sigma_i) \leqq c\,k^{-1/2} \sum_{j=0}^{k-1} \varepsilon_{ij}^2 = c\,k^{-1/2} \sum_{j=0}^{k-1} (\mu_{ij}-k^{-1})^2 .$$

Hence the convergence of $\sum_{j=0}^{k-1} \sum_{i=1}^{\infty} (\mu_{ij}-k^{-1})^2$ implies the convergence of

$$\sum_{i=1}^{\infty} \{1-\varrho(\mu_i,\sigma_i)\} .$$

To complete the proof of Corollary 1, I first point out that Φ is a one-to-one mapping of Ω onto $[0, 1]$ except for k-ray rationals x in $[0, 1]$, for which there are exactly two points of Ω which are mapped into x (the terminating and non-terminating expansions). Now if ν is purely non-atomic then and only then $\nu(\{x\}) = 0$ for every x in $[0, 1]$. This clearly implies the validity of the equation in (c). On the other hand, let

$$\prod_{i=1}^{\infty} \max_{0\leqq j\leqq k-1} \mu_{ij} > 0 .$$

Then there exists a point $\bar\omega = (\bar\omega_1, \bar\omega_2, \ldots, \bar\omega_n, \ldots) \in \Omega$ such that $\mu(\{\omega\}) > 0$ (e.g. take $\bar\omega_n$ such that $\mu_{n\bar\omega_n} = \max_{0\leqq j\leqq k-1} \mu_{nj}$). The denumerable set

$$A = \bigcup_{N=1}^{\infty} \{\omega \mid \pi_n(\omega) = \pi_n(\bar\omega);\ n \geqq N\}$$

is such that $\mu(A) = 1$. This can be proved either by alluding to the zero-one law or by an amusing direct computation as in Chatterji [*4*]. Hence μ and so ν will be purely atomic.

The rest of the proof of Corollary 1 is easy in view of the preceding observations and Theorem 1. (The case $k=2$ of the Corollary is also discussed in KAKUTANI [1]. See CHATTERJI [4] for an elementary proof.)

The construction of ν on [0, 1] can be described geometrically. I shall indicate it for the case $k = 2$. Subdivide the interval [0, 1] into halves $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ and define

$$\nu[0, \tfrac{1}{2}] = q_1, \quad p_1 + q_1 = 1,$$
$$\nu[\tfrac{1}{2}, 1] = p_1, \quad p_1 \geqq 0, \quad q_1 \geqq 0.$$

Subdivide the subintervals and split the masses in the ratio $p_2 : q_2$ i.e. define

$$\nu[0, \tfrac{1}{4}] = q_1 q_2, \quad \nu[\tfrac{1}{4}, \tfrac{1}{2}] = q_1 p_2,$$
$$\nu[\tfrac{1}{2}, \tfrac{3}{4}] = p_1 q_2, \quad \nu[\tfrac{3}{4}, 1] = p_1 p_2.$$

If one continues this process indefinitely with a sequence $p_i \geqq 0$, $q_i \geqq 0$, $p_i + q_i = 1$, a measure ν is obtained as in Corollary 1 with $k=2$. (To avoid atomicity assume $\prod_{i=1}^{\infty} \max(p_i, q_i) = 0$.) Then $\nu \ll \lambda$ of $\nu \perp \lambda$ according as $\sum_{i=1}^{\infty} (\frac{1}{2} - p_i)^2 < +\infty$ or equals $+\infty$ (λ = Lebesgue measure). The cumulative distribution function $F(x) = \nu[0, x]$ in case $\nu \perp \lambda$ provides example of a strictly increasing, continuous function $F(x)$ such that $F'(x) = 0$ a.e. (λ). A slightly special case of this construction is to be found in SALEM [5]. Notice incidentally that $k = 3$, $\mu_{i0} = \mu_{i2} = 1/2$ and $\mu_{i1} = 0$ gives the Cantor measure on [0, 1].

§ 3

In this section I shall study the fractional dimensions $d(\nu)$ (defined rigorously a little later) of the "supports" of the measures "ν" refered to in Corollary 1. I shall first introduce the following definitions due to BILLINGSLEY [6], [7].

Let $\mathfrak{K}$ be the class of intervals of the type $(j k^{-n}, (j+1) k^{-n})$, $0 \leqq j \leqq k^n - 1$, $n \geqq 1$. Let ν be a probability measure on the Borel sets of [0, 1]. For any set M in [0, 1] define

$$\nu_\alpha(M, \varrho) = \inf\{\sum_i \nu^\alpha(A_i) \mid A_i \in \mathfrak{K}, \bigcup_{i=1}^{\infty} A_i \supset M, \ \nu(A_i) < \varrho\},$$
$$\nu_\alpha(M) = \lim_{\varrho \to 0} \nu_\alpha(M, \varrho),$$
$$\dim_\nu(M) = \sup\{\alpha \mid \nu_\alpha(M) = +\infty\} = \inf\{\alpha \mid \nu_\alpha(M) = 0\}.$$

It can be shown (see BILLINGSLEY [6]) that if $\nu = \lambda$ where λ is the Lebesgue measure on [0, 1] then $\dim_\lambda(M)$ equals the classical Hausdorff fractional dimension of M. I shall further define: (λ in the sequel shall always stand for Lebesgue measure.)

$$d(\nu) = \inf\{\alpha \mid \alpha = \dim_\lambda(M), \ \nu(M) = 1, \ \lambda(M) = 0\}$$
$$= \inf\{\alpha \mid \alpha = \dim_\lambda(M), \ \nu(M) = 1\}.$$

I can then prove

Theorem 2. *Let ν be a measure* on $[0, 1]$ *induced by the process referred to in Corollary 1. Then*

$$d(\nu) = \delta = \underline{\lim}_{n \to \infty} \frac{-1}{n \ln k} \sum_{i=1}^{n} \delta_i$$

where

$$\delta_i = \sum_{j=0}^{k-1} \mu_{ij} \ln \mu_{ij} \quad (\text{with } x \ln x = 0 \text{ if } x = 0)\,.$$

The proof is a simple application of the following results of BILLINGSLEY [7]. Let $t = \sum_{i=1}^{\infty} x_i(t) k^{-1}$ be the k-radix expansion of t. (For the sake of definiteness consider only the expansions with infinitely many non-zero $x_i(t)$). Let

$$A_n(t) = [\sum_{i=1}^{n} x_i(t)\, k^{-i}, \quad \sum_{i=1}^{n} x_i(t)\, k^{-i} + k^{-n}],$$

$$f_n(t) = \frac{\ln \nu(A_n(t))}{\ln \lambda(A_n(t))}\,.$$

Then

(i) if ν is non-atomic and $\delta \geqq 0$

$$\dim_{\lambda} \{t \,|\, \underline{\lim}_{n \to \infty} f_n(t) \leqq \delta\} \leqq \delta\,.$$

(ii) $M \subset \{t \,|\, \underline{\lim}_{n\to\infty} f_n(t) \geqq \delta\} \Rightarrow \dim_{\lambda}(M) \geqq \delta \dim_{\nu}(M)$.

In the present case

$$f_n(t) = -\sum_{i=1}^{n} \ln y_i(t)/n \ln k$$

where

$$y_i(t) = \mu_{ij} \quad \text{if} \quad x_i(t) = j\,.$$

Clearly $\{y_i(t)\; i \geqq 1\}$ form a sequence of independent functions on $[0, 1]$ under ν-measure. Also

$$\int_0^1 \ln y_i(t)\, d\nu(t) = \sum_{j=0}^{k-1} \mu_{ij} \ln \mu_{ij} = \delta_i\,,$$

$$\int_0^1 \ln^2 y_i(t)\, d\nu(t) = \sum_{j=0}^{k-1} \mu_{ij} \ln^2 \mu_{ij} < C \quad (\text{independent of } i)\,.$$

Hence by a theorem of KOLMOGOROV the set

$$B = \{t \,|\, \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \{\delta_i - \ln y_i(t)\} = 0\}$$

$$= \{t \,|\, \lim_{n\to\infty} (f_n(t) + \frac{1}{n \ln k} \sum_{i=1}^{n} \delta_i) = 0\}$$

has ν-measure 1. Hence $\dim_{\nu}(B) = 1$.

Since

$$B \subset \{t \mid \underline{\lim}_{n\to\infty} f_n(t) = \underline{\delta}\} \subset \{t \mid \underline{\lim}_{n\to\infty} f_n(t) \leqq \underline{\delta}\}$$

by (i) above

$$\dim_\lambda(B) \leqq \dim_\lambda\{t \mid \underline{\lim}_{n\to\infty} f_n(t) \leqq \underline{\delta}\} \leqq \underline{\delta}.$$

Again since

$$B \subset \{t \mid \underline{\lim}_{n\to\infty} f_n(t) \geqq \underline{\delta}\}$$

by (ii) above

$$\dim_\lambda(B) \geqq \underline{\delta}.$$

Therefore

$$\dim_\lambda(B) = \underline{\delta}.$$

If $C \subset B$ and $\nu(C) = 1$ then firstly $\dim_\lambda(C) \leqq \underline{\delta}$ and secondly

$$C \subset B \subset \{t \mid \underline{\lim} f_n(t) \geqq \underline{\delta}\}$$

so that $\dim_\lambda(C) \geqq \underline{\delta}$ (by (ii) above). Therefore $\dim_\lambda(C) = \underline{\delta}$. This completes the proof of theorem 2.

§ 4

Various authors have remarked that it is possible to obtain an absolutely continuous measure by convoluting two singular measures. Very simple examples of such situations may be obtained by considering induced measures ν as in Corollary 1. A good way of looking at such examples is from the point of view of independent random variables (r.v.). Consider e.g.

$$X = \sum_{i=1}^{\infty} X_i 2^{-i}$$

where X_i are independent r.v.'s taking values 1 and 0 with probabilities p_i and $1 - p_i$. Assume that $\sum_{i=1}^{\infty} (\frac{1}{2} - p_i)^2 < +\infty$. Then the measure induced by X is absolutely continuous. However $X = X_1 + X_2$ where

$$X_1 = \sum_{i=1}^{\infty} X_{2i} 2^{-2i} \quad \text{and} \quad X_2 = \sum_{i=0}^{\infty} X_{2i+1} 2^{-2i-1}.$$

X_1 and X_2 obviously induce singular measures. Since X_1 and X_2 are independent, one thus has an example of an absolutely continuous measure which is the convolution of two singular measures.

A formal presentation of the above may be given as follows. Consider Corollary 1 for the case $k = 2$ (other cases may also be dealt with similarly). Let ν be the absolutely continuous measure induced by the sequence $\{p_i\}$ where

$$\sum (1 - 2p_i)^2 < +\infty. \quad (p_i = \mu_i(\{1\})).$$

Let $\{p_i'\}$, $\{p_i''\}$ be defined as follows:

$$p' = p_i \quad \text{if } i \text{ is even} \qquad p_i'' = 0 \quad \text{if } i \text{ even}$$
$$= 0 \quad \text{if } i \text{ is odd} \qquad = p_i \quad \text{if } i \text{ odd}.$$

The measures ν', ν'' induced by $\{p_i'\}$, $\{p_i''\}$ respectively are non-atomic singular according to Corollary 1. However quite clearly the absolutely continuous measure ν is a convolution of ν' and ν''.

If ν is taken to be the Lebesgue measure λ i.e. $p_i = 1/2$, $i \geqq 1$ then the above gives a method for representing λ as a convolution of two singular measures ν_1 and ν_2.

Further a little more generally if one defines ν_1, ν_2 be sequences $\{p_i'\}$, $\{p_i''\}$ where

$$p_i' = p_i \quad \text{if} \quad i \in \{i_1, i_2, \ldots\}, \quad 1 \leqq i_1 < i_2 < \cdots$$
$$= 0 \quad \text{otherwise}$$

and

$$p_i'' = p_i \quad \text{if} \quad i \notin \{i_1, i_2, \ldots\}$$
$$= 0 \quad \text{otherwise}$$

then once again $\nu = \nu_1 * \nu_2$ ($*=$ convolution) where ν_1 and ν_2 are always singular. This gives a method for writing down say the Lebesgue measure as the convolution of two singular measures, both of which have supports of arbitrarily small dimension, even of dimension zero. To this end, all one has to do is to choose an infinite set of integers $A = \{i_1, i_2, \ldots\}$ such that the lower number densities of both A and the complementary set of integers are zero i.e. if $n(A)$ equals the number of integers in A less than or equal to n then choose A so that

$$\underline{\lim}_{n\to\infty} \frac{n(A)}{n} = 0$$

and also

$$\underline{\lim}_{n\to\infty} \frac{n(\sim A)}{n} = 0 \quad (\sim A = \text{complement of } A).$$

This can be done by choosing A to be a set of integers as follows:

$$A = \{1\} \cup \{i_2, i_2 + 1, \ldots, i_3\} \cup \{i_4, i_4 + 1, \ldots, i_5\} \cup \cdots$$
$$= \{1\} \bigcup_{r=1}^{\infty} \{i_{2r}, i_{2r} + 1, \ldots, i_{2r+1}\}$$

where i_k's are defined inductively by

$$i_1 = 1;$$

i_2 such that $\quad 2/i_2 < \frac{1}{2}$;

i_3 such that $\quad \frac{(i_3 - i_2) + 2}{i_3} > 1 - \frac{1}{3}$;

i_4 such that $\quad \frac{(i_3 - i_2) + 3}{i_4} < \frac{1}{4}$;

and generally for $s > 2$

$$i_{2s} \text{ such that } \qquad \frac{\sum_{r=2}^{s}(i_{2r-1} - i_{2r-2} + 1) + 2}{i_{2s}} < \frac{1}{2s}$$

and

$$i_{2s+1} \text{ such that } \qquad \frac{\sum_{r=2}^{s}(i_{2r-1} - i_{2r-2} + 1) + (i_{2s+1} - i_{2s}) + 2}{i_{2s+1}} > 1 - \frac{1}{2s+1}.$$

It is clear that the lower number densities of both A and $\sim A$ are zero. Now define

$$\begin{aligned} p_i' &= \tfrac{1}{2} \quad \text{if} \quad i \in A \\ &= 0 \quad \text{if} \quad i \notin A \end{aligned}$$

and

$$\begin{aligned} p_i'' &= \tfrac{1}{2} \quad \text{if} \quad i \notin A \\ &= 0 \quad \text{if} \quad i \in A \end{aligned}$$

Let ν_1, ν_2 be the measures induced by $\{p_i'\}$ and $\{p_i''\}$ respectively (as in Corollary 1). As before the Lebesgue measure λ is a convolution of ν_1 and ν_2. The dimensions of the supports of the measures ν_1, ν_2, according to theorem 2, are simply the lower number densities of the sets A and complement of A respectively and hence equal to zero.

Other interesting examples can also be constructed by using these techniques.

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