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Characterizations of the Normal and the Gamma Distributions

Lennart Bondesson

The Normal and the Gamma distribution, properly translated, are characterized by a uniformly minimum variance property of the sample mean. In fact, we answer in particular a question asked by Kagan in 1966. Also an optimality property of the sample variance is proved to characterize the Normal distribution.

1. Introduction

Let $x_1, \dots, x_n$ be independent observations of a random variable X with d.f. $F[(x-\theta)/\sigma]$. The parameters θ and $\sigma(\sigma>0)$ are unknown, but F is supposed to be known.

A function $g(x_1,\dots,x_n)$ is called *translative* if $g(x_1+\lambda,\dots,x_n+\lambda)=\lambda+g(x_1,\dots,x_n)$ for all $\lambda\in R$ and all $(x_1,\dots,x_n)\in R^n$. The meaning of "*g is translation invariant*" is then also obvious. We say that $g(x_1,\dots,x_n)$ is *multiplicative* if, for any $c>0$ and any $(x_1,\dots,x_n)\in R^n$, $g(c x_1,\dots,c x_n)=c\,g(x_1,\dots,x_n)$.

When estimating θ it is natural to consider only translative and multiplicative estimators. We call such estimators proper. Using $E_{(\theta,\sigma)}[(\theta^*-\theta)^2]$ as a measure of the goodness of an estimator θ^*, Pitman [7] showed that there is always a best proper estimator of θ (if $\int x^2\,dF(x)<\infty$). If the sample mean $\bar{x}$ is the best proper estimator, what can be said about the d.f. F? This question was asked by Kagan [4, 6]. Sometimes only the class of unbiased proper estimators of θ is of interest, sometimes also the whole class of unbiased estimators. When is $\bar{x}$ the best unbiased (proper) estimator of θ? In this paper we shall present a solution of these problems.

A function g is called *square multiplicative* if $g(c x_1,\dots,c x_n)=c^2 g(x_1,\dots,x_n)$, $c>0$. We call an estimator of σ^2 proper if it is translation invariant and square multiplicative. We shall show that if a multiple of the sample variance $s^2=\sum(x_i-\bar{x})^2/(n-1)$ is the best proper estimator of σ^2 (such an estimator always exists), then F has to be a Normal d.f.

2. The Main Result

It is well known that if F is a Normal d.f. with mean zero, then the sample mean $\bar{x}$ is the best unbiased or, as we also say, a uniformly minimum variance (UMV) estimator of θ. This is a consequence of the Rao-Blackwell theorem since $(\bar{x}, s^2)$ is a sufficient and complete statistic. Another often used way of establishing this result provides the following UMV-criterion. (Concerning this technique, see e.g. Rao [9], p. 258.)

UMV-Criterion. *An unbiased estimator θ^* of θ with finite variance is UMV if and only if $E_{(\theta,\sigma)}[\theta^* g]=0$ for any statistic g having finite second moment and being an unbiased estimator of zero.*

In this criterion g is usually assumed to be real-valued, but it is readily seen that we can also permit g to be complex-valued.

The UMV-property of $\bar{x}$ does not hold only for the Normal d.f. as Theorem 2.1 below shows. A r.v. Y is Gamma distributed $\Gamma(p,a)$, $p>0$, $a>0$, if it has density function

$$f_Y(y)=\frac{y^{p-1}}{\Gamma(p)\,a^p}\exp\{-y/a\},\qquad y>0.$$

The negative Gamma distribution $\Gamma(p,a)$, $a<0$, we define through: $Y\in\Gamma(p,a)$ if $-Y\in\Gamma(p,-a)$. Clearly, whatever the sign of a is, $E[Y]=p\,a$.

Theorem 2.1. *Let F be the d.f. of a variable $Y-p\,a$, where $Y\in\Gamma(p,a)$, $a\neq 0$. Then $\bar{x}$ is a UMV-estimator of θ.*

Proof. Let $X=\theta+\sigma(Y-p\,a)$, where $Y\in\Gamma(p,a)$. Then we set $Z=\sigma\,Y$, $\theta'=\theta-\sigma p\,a$, and $\sigma'=\sigma\,a$. Evidently, $Z\in\Gamma(p,\sigma')$. For an unbiased estimator $g(x_1,\ldots,x_n)$ of zero we have

$$E_{(\theta,\sigma)}[g(x_1,\ldots,x_n)]=E_{\sigma'}[g(\theta'+z_1,\ldots,\theta'+z_n)]=0. \tag{2.1}$$

For θ' fixed we see that $g(\theta'+z_1,\ldots,\theta'+z_n)$ is an unbiased estimator of zero based on a sample $z_1,\ldots,z_n$ from a Gamma distribution with unknown scale σ'. As $\bar{z}$ is a complete sufficient statistic, it is also a UMV-estimator of its mean value. Therefore, by the UMV-criterion (adapted for the scale parameter situation),

$$E_{\sigma'}[\bar{z}\,g(\theta'+z_1,\ldots,\theta'+z_n)]=0.$$

Hence and by (2.1)

$$E_{\sigma'}[(\theta'+\bar{z})\,g(\theta'+z_1,\ldots,\theta'+z_n)]=0.$$

But the left hand side is equal to $E_{(\theta,\sigma)}[\bar{x}\,g(x_1,\ldots,x_n)]$. Thus, an application of the UMV-criterion concludes the proof.

For $p\neq 1$ this result provides an example of a case when a non-trivial sufficient statistic does not exist, but still a UMV-estimator of some special parameter may be found.

The author feels that the result obtained must have been noticed earlier, but is not capable of finding it explicitly stated in the literature. However, whatever the case, the main emphasis of this paper is directed to the converse theorem, which we give in the following strong form.

Theorem 2.2. *Let F be continuous. If, for some $n\geqq 7$, the mean $\bar{x}$ is the best unbiased proper estimator of θ, then F is either a Normal d.f. with mean zero or the d.f. of a r.v. $Y-E[Y]$, where Y is Gamma distributed, possibly negatively.*

Proof. To make the ideas clear and to illustrate the various difficulties, we first prove the theorem under the assumptions that all moments are finite and that $n\geqq 9$.

The observations $x_1,\ldots,x_9$ will most often be called $y_1,y_2,y_3,z_1,z_2,z_3,u_1,u_2,u_3$. For a real number $x\neq 0$ and a complex number ξ, we define

$$x^{\xi}=\exp\{\xi\log|x|+i\pi\,\xi\,\varepsilon(-x)\},$$

where $\varepsilon(x)=1$ for $x>0$ and 0 for $x<0$. Since F is continuous, with probability one, $y_1-\bar{y}\neq 0$. Therefore, as all moments are finite, the functions

$$E_{(0,1)}[(y_1-\bar{y})^{\xi}] \quad \text{and} \quad E_{(0,1)}[\bar{y}(y_1-\bar{y})^{\xi}]$$

both become well-defined and continuous for $\operatorname{Re}\xi\geqq 0$ and analytic for $\operatorname{Re}\xi>0$. Because of analyticity, a point $(\xi_1^o, \xi_2^o, \xi_3^o)$, satisfying $\xi_1^o+\xi_2^o+\xi_3^o=1$, can be constructed such that $E_{(0,1)}[(y_1-\bar{y})^{\xi_j}]\neq 0$, $j=1,2,3$, for all points (ξ_1, ξ_2, ξ_3) in some neighborhood of $(\xi_1^o, \xi_2^o, \xi_3^o)$.

Let $g(x_1, \dots, x_n)$ and $h(x_1, \dots, x_n)$ be translation invariant and multiplicative statistics, both with mean values different from zero. Then

$$g/E_{(0,1)}[g]-h/E_{(0,1)}[h]$$

is an unbiased, translation invariant, and multiplicative estimator of zero. As $\bar{x}$ is the best unbiased proper estimator of θ, it follows from an obvious variant of the UMV-criterion that

$$\frac{E_{(0,1)}[\bar{x}\,g]}{E_{(0,1)}[g]}=\frac{E_{(0,1)}[\bar{x}\,h]}{E_{(0,1)}[h]}.$$

Hence

$$E_{(0,1)}(\bar{x}\,g]/E_{(0,1)}[g]=\text{constant}. \tag{2.2}$$

For an arbitrary point (ξ_1, ξ_2, ξ_3), having coordinate sum equal to 1 and lying in the above mentioned neighbourhood of $(\xi_1^o, \xi_2^o, \xi_3^o)$, we now set

$$g(x_1, \dots, x_n)=(y_1-\bar{y})^{\xi_1}(z_1-\bar{z})^{\xi_2}(u_1-\bar{u})^{\xi_3}.$$

Trivially, g has the required properties. Therefore (2.2) yields

$$\frac{E[\bar{y}(y_1-\bar{y})^{\xi_1}]}{E[(y_1-\bar{y})^{\xi_1}]}+\frac{E[\bar{z}(z_1-\bar{z})^{\xi_2}]}{E[(z_1-\bar{z})^{\xi_2}]}+\frac{E[\bar{u}(u_1-\bar{u})^{\xi_3}]}{E[(u_1-\bar{u})^{\xi_3}]}=\text{constant},$$

where $E[\cdot]$ stands for $E_{(0,1)}[\cdot]$. This is essentially Cauchy's functional equation with solution

$$E[\bar{y}(y_1-\bar{y})^{\xi}]=(A\,\xi+B)\cdot E[(y_1-\bar{y})^{\xi}], \tag{2.3}$$

A and B being constants. The equality holds for all ξ close enough to ξ_1^o (or ξ_2^o or ξ_3^o), but therefore, on account of analyticity and continuity, for all ξ, $\operatorname{Re}\xi\geqq 0$. Especially we have

$$E[\bar{y}(y_1-\bar{y})^{k}]=(A\,k+B)\cdot E[(y_1-\bar{y})^{k}], \qquad k=0,1,2,\dots.$$

As $E_{(0,1)}[\bar{x}]=0$, $E[\bar{y}]=0$. Setting $k=0$, we find $B=0$. Hence (cp. Rao [10])

$$E[\bar{y}\cdot\exp\{i\,t(y_1-\bar{y})\}]\underset{D}{=}A\,i\,t\cdot E[(y_1-\bar{y})\exp\{i\,t(y_1-\bar{y})\}], \tag{2.4}$$

where $\underset{D}{=}$ means that the left and the right hand members have the same derivatives of all orders at $t=0$. The equality $\underset{D}{=}$ can be handled just as the ordinary one. We set $\psi(t)=\varphi'(t)/\varphi(t)=D\log\varphi(t)$, where φ is the characteristic function of F. The function ψ is well-defined for all t small enough. Of course, $\psi(0)=0$. Observing that $E[y_1\exp\{i\,t\,y_1\}]=-i\,\varphi'(t)$ and manipulating a little, we find that (2.4) can be written as

$$(2At+i)\cdot\psi(2t/3)\underset{D}{=}(2At-i)\cdot\psi(-t/3).$$

We put $\psi(t)\underset{D}{=}\sum_{j=1}^{\infty}a_j t^j$. (This makes sense even though the right hand side diverges.) Identifying coefficients, we easily find

$$a_j=(i3A)^{j-1}a_1, \quad j\geqq 1.$$

As $\psi'(0)=\varphi''(0)$, $a_1=-\mu_2$. Summing up a geometrical series, we obtain

$$\psi(t)\underset{D}{=}-\mu_2 t/(1-i3At). \tag{2.5}$$

Recalling that $\psi(t)=D\log\varphi(t)$, the solution of (2.5) is easily verified to be

$$\varphi(t)\underset{D}{=}\begin{cases}\exp\{-\mu_2 t^2/2\} & \text{if } A=0\\ (1/(1-i3At))^{\mu_2/9A^2}\cdot\exp\{-i\mu_2 t/3A\} & \text{if } A\neq 0.\end{cases} \tag{2.6}$$

Since φ is a c.f. and the right hand members are analytic, ordinary equality holds for all t. Furthermore, the functions on the right are just the c.f.'s for a $N(0,(\mu_2)^{\frac{1}{2}})$-d.f. and a correctly translated $\Gamma(\mu_2/9A^2, 3A)$-d.f., and therefore an application of the inversion theorem for c.f.'s finishes the first part of the proof.

It is easy to prove that the functional equation

$$H_1(\xi_1)+H_2(\xi_2)+H_3(\xi_3)=0, \quad \xi_1+\xi_2+\xi_3=1,$$

where the continuous functions H_1, H_2, and H_3 are not supposed to be equal, only admits linear solutions. So if we instead set

$$g(x_1,\dots,x_n)=(y_1-\bar{y})^{\xi_1}(x_4-x_5)^{\xi_2}(x_6-x_7)^{\xi_3},$$

(2.3) still follows. This shows that $n\geqq 9$ can be replaced by $n\geqq 7$.

The general proof, not assuming the existence of all moments, will now be given. (A more detailed version will appear in a forthcoming paper.)

That the variance is finite is obvious. We set

$$\begin{aligned}g(x_1,\dots,x_n)=&\left(|y_1-\bar{y}|+\tfrac{1}{2}(y_1-\bar{y})\right)^{\xi_1}\cdot(|y_1-\bar{y}|)^{\eta_1}\\&\cdot\left(|x_4-x_5|+\tfrac{1}{2}(x_4-x_5)\right)^{\xi_2}\cdot(|x_4-x_5|)^{\eta_2}\cdot\left(|x_6-x_7|+\tfrac{1}{2}(x_6-x_7)\right)^{\xi_3}\cdot(|x_6-x_7|)^{\eta_3},\end{aligned}$$

where $\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3$ are positive and have total sum 1. Then g is translation invariant and multiplicative. Therefore (2.2) holds. Manipulating a little we get a functional equation of a new but easily solvable type. We find

$$\begin{aligned}&E\left[\bar{y}\left(|y_1-\bar{y}|+\tfrac{1}{2}(y_1-\bar{y})\right)^{\xi}(|y_1-\bar{y}|)^{\eta}\right]\\&\quad=A(\xi+\eta)\cdot E\left[\left(|y_1-\bar{y}|+\tfrac{1}{2}(y_1-\bar{y})\right)^{\xi}(|y_1-\bar{y}|)^{\eta}\right], \quad \xi+\eta\leqq 1.\end{aligned}$$

Here is $E[\cdot]=E_{(0,1)}[\cdot]$. Because of analyticity and continuity, this relation also holds for all purely imaginary ξ and η. We set

$$\xi=it_1, \quad \eta=it_2, \quad v_1=\log\left(|y_1-\bar{y}|+\tfrac{1}{2}(y_1-\bar{y})\right), \quad v_2=\log(|y_1-\bar{y}|)$$

and get

$$E[\bar{y}\cdot\exp\{it_1v_1+it_2v_2\}]=A(it_1+it_2)\cdot E[\exp\{it_1v_1+it_2v_2\}].$$

Equivalently,

$$E[\bar{y}\cdot\exp\{it_1v_1+it_2v_2\}]=A\cdot E\left[\left(\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_2}\right)\exp\{it_1v_1+it_2v_2\}\right].$$

The idea is now that in this relation $\exp\{it_1v_1+it_2v_2\}$ can be replaced by any sufficiently smooth function $h(v_1,v_2)$. The technical argumentation is omitted. Using this fact for

$$h(v_1,v_2)=\exp\{2it(e^{v_1}-e^{v_2})\}=\exp\{it(y_1-\bar{y})\},$$

we obtain

$$E[\bar{y}\cdot\exp\{it(y_1-\bar{y})\}]=Ait\cdot E[(y_1-\bar{y})\cdot\exp\{it(y_1-\bar{y})\}].$$

Equivalently,

$$(2At+i)\cdot\psi(2t/3)=(2At-i)\cdot\psi(-t/3)$$

if t is small enough. The transformation $f(t)=(1-i3At)\cdot\psi(t)$ gives

$$f(2t/3)=-2f(-t/3).$$

We differentiate and obtain

$$f'(2t/3)=f'(-t/3).$$

Hence f' is constant. It then easily follows that (2.6) holds in the ordinary sense for all t in some neighborhood of the origin. This ends the proof.

Remark. Employing Hilbert space theory, one can prove that there always exists a best unbiassed proper estimator of θ (at least if $\int x^2\,dF(x)<\infty$). However, an explicit expression of this estimator seems hard to obtain (cp. the next section).

Assume now that F is continuous and that, for some $n\geqq 5$, $\bar{x}$ is a UMV-estimator of θ. The relation (2.2) will then hold for all g being translation invariant and homogeneous of a certain degree. The constant will depend on this degree. Using e.g. the statistic

$$g(x_1,\ldots,x_n)=(y_1-\bar{y})^{\xi_1}(x_4-x_5)^{\xi_2},$$

which has degree $\xi_1+\xi_2$, we easily see that the conclusion of Theorem 2.2 remains true.

However, supposing that also all moments are finite and that $\mu_3=0$ but dropping the continuity assumption, then we need only assume the UMV-property of $\bar{x}$ for some $n\geqq 3$ in order to conclude that F is Normal. For, letting $n=3$, if $\mu_3=0$, then the statistic $(x_1-\bar{x})^3$ has mean value zero for all (θ,σ). Therefore $E_{(\theta,\sigma)}[\bar{x}(x_1-\bar{x})^3]=0$. Hence $E_{(\theta,\sigma)}[\bar{x}^2(x_1-\bar{x})^3]=0$, hence $E_{(\theta,\sigma)}[\bar{x}^3(x_1-\bar{x})^3]=0$, and so on. From these relations, for $(\theta,\sigma)=(0,1)$, it is easy to see that all moments must be uniquely determined by μ_2, and thus they are Normal. As the Normal moments uniquely determine the Normal d.f., F is Normal.

3. Kagan's Problem

In this section we present a solution of Kagan's problem mentioned in the introduction.

First we derive an expression for the best proper estimator θ_P^* of θ. Let $s^2=\sum(x_i-\bar{x})^2/(n-1)$ and $U=((x_1-\bar{x})/s,\ldots,(x_n-\bar{x})/s)$. Any proper estimator θ^* can be written $\theta^* = \bar{x} + s\,h(U)$, where h is some function. As $E_{(\theta,\sigma)}[(\theta^* - \theta)^2] = \sigma^2 E_{(0,1)}[(\theta^*)^2]$, we find the best one by minimizing

$$E_{(0,1)}[(\bar{x} + s\,h(U))^2] = E_{(0,1)}[E_{(0,1)}[(\bar{x} + s\,h(U))^2 \mid U]].$$

Hence we easily get

$$\theta_P^* = \bar{x} - s \cdot E_{(0,1)}[\bar{x} \cdot s \mid U] / E_{(0,1)}[s^2 \mid U]$$

(cp. Pitman [7]). Usually θ_P^* is a biassed estimator of θ even though F has mean zero. Notice also

Criterion. *A proper estimator θ^* of θ with finite variance is the best one if and only if $E_{(0,1)}[\theta^* g] = 0$ for every translation invariant and multiplicative statistic g with finite second moment.*

The easy proof is omitted.

If F is a Normal d.f., then a theorem of Basu [1] asserts in particular that $(\bar{x}, s^2)$ and U are independent. Hence, as also $\bar{x}$ and s^2 are independent, $\theta_P^* = \bar{x}$, provided that F has mean zero.

Also for a properly translated Gamma distribution $\bar{x}$ can be the best proper estimator of θ. More exactly

Theorem 3.1. *Let $x_1, \ldots, x_n$ be independent observations of a r.v.*

$$X = \theta + \sigma\left(Y - p\,a - \frac{a}{n}\right),$$

where $Y \in \Gamma(p, a)$. Then $\theta_P^* = \bar{x}$.

For the proof we need a lemma.

Lemma 3.1. *Let $y_1, \ldots, y_n$ be independent $\Gamma(p, a)$-distributed r.v.'s. For any multiplicative statistic $g(y_1, \ldots, y_n)$ we have*

$$E\left[\left(\bar{y} - p\,a - \frac{a}{n}\right) \cdot g(y_1, \ldots, y_n)\right] = 0. \tag{3.1}$$

Proof. It suffices to consider the case $a > 0$. Consider the expression

$$E[g] = \int \cdots \int_{y_i > 0} g(y_1, \ldots, y_n) \frac{y_1^{p-1} \ldots y_n^{p-1}}{[\Gamma(p)]^n \cdot a^{np}} \cdot \exp\{-(y_1 + \cdots + y_n)/a\}\, dy_1 \ldots dy_n.$$

Making the transformation $y_i = c\,z_i$, using the fact that g is multiplicative, then differentiating with respect to c (the derivative is zero), and finally setting $c = 1$, we obtain (3.1).

Proof of Theorem 3.1. Let $g(x_1, \ldots, x_n)$ be any translation invariant and multiplicative statistic. According to the criterion we have to prove that

$$E_{(0,1)}[\bar{x} \cdot g(x_1, \ldots, x_n)] = 0.$$

However, this relation follows immediately from (3.1) and the fact that g is translation invariant.

The solution of Kagan's problem does not come as a surprise.

Theorem 3.2. *Let F be continuous. If, for some $n \geqq 7$, $\bar{x}$ is the best proper estimator of θ, then F is either a Normal d.f. with mean zero or the d.f. of some variable $Y - pa - \frac{a}{n}$, where $Y \in \Gamma(p, a)$. In particular, if F has mean zero, then F is Normal.*

Proof. Using the criterion, we see that (2.2) still holds with the constant being equal to zero. A slightly modified proof of Theorem 2.2 shows that F is either Normal or the d.f. of some translated Gamma variable. In the first case F must have mean zero, for otherwise $\theta_P^* = \bar{x} + k \cdot s$, $k \neq 0$. In the second case Lemma 3.1 and the criterion show that only the translation $-\left(pa + \frac{a}{n}\right)$ is possible. This ends the proof.

Observe that if one supposes that $\theta_P^* = \bar{x}$ for two different sample sizes, then there is only the possibility of a Normal d.f.

Remark. Kagan has informed that he has also very recently obtained a solution of his own problem. However, his method is different from the one used here.

4. Variance Estimation

The best proper estimator of σ^2 can be proved to be

$$(\sigma^2)_P^* = s^2 \cdot E_{(0,1)}[s^2 | U] / E_{(0,1)}[s^4 | U].$$

Generally $(\sigma^2)_P^*$ is biased. If a sufficient statistic $(S_1, \ldots, S_k)$ exists, some calculations will show that $(\sigma^2)_P^*$ only depends on this statistic. From the explicit expression of $(\sigma^2)_P^*$ above it follows that, for any translation invariant and square multiplicative statistic g,

$$E_{(0,1)}[(\sigma^2)_P^* \cdot g] = E_{(0,1)}[g]. \tag{4.1}$$

It is also true that an estimator $(\sigma^2)^*$ which satisfies this condition must coincide with $(\sigma^2)_P^*$. A consequence of (4.1) and the UMV-criterion for proper σ^2-estimators, the formulation of which is easily realized, is that a constant c_n may be found such that $c_n(\sigma^2)_P^*$ is the best unbiased proper estimator of σ^2 (cp. [2]).

Let σ_0^2 be the variance of F. If F is Normal, s^2/σ_0^2 is the UMV-estimator of σ^2 and hence also the best unbiased proper estimator of σ^2. Therefore $(\sigma^2)_P^* = s^2/c_n \sigma_0^2$. Another way to see this provides the result by Basu [1], asserting in particular that s^2 and U are independent. We then find

$$(\sigma^2)_P^* = s^2 \cdot E_{(0,1)}[s^2] / E_{(0,1)}[s^4] = \frac{n-1}{n+1} s^2/\sigma_0^2.$$

Now we give a converse result.

Theorem 4.1. *Let F be continuous and have moments of all orders. If, for some $n \geqq 12$, a constant k_n exists such that $k_n s^2$ is the best proper estimator of σ^2, then F is a Normal d.f.*

Proof. For the sake of simplicity we give the proof only for the case of twelve observations, here most often labelled x_{ij}, $i = 1, 2, 3, 4$, $j = 1, 2, 3$. With slight modifications the proof will also work for all $n \geqq 12$.

Using notations known from analysis of variance, we find

$$\begin{aligned}\sum_{i,j}(x_{ij}-\bar{x}_{..})^2&=\sum_{i,j}x_{ij}^2-12(\bar{x}_{..})^2\\&=\sum_{i,j}x_{ij}^2-0.75\sum_i(\bar{x}_{i.})^2-1.5\sum_{i<i'}\bar{x}_{i.}\cdot\bar{x}_{i'.}.\end{aligned}$$

Assume $(\sigma^2)_P^*=k_n s^2$. Then, according to what have been stated above, for all translation invariant and square multiplicative statistics g,

$$\begin{aligned}&E_{(0,1)}[(\sum_{i,j}x_{ij}^2-0.75\sum_i(\bar{x}_{i.})^2-1.5\sum_{i<i'}\bar{x}_{i.}\cdot\bar{x}_{i'.})\,g(x_{11},\ldots,x_{43})]\\&\quad=k_n' E_{(0,1)}[g(x_{11},\ldots,x_{43})],\end{aligned}\tag{4.2}$$

where k_n' is some new constant. To avoid trouble with zeroes, we set

$$g(x_{11},\ldots,x_{43})=\prod_{i=1}^{4}(|x_{i1}-\bar{x}_{i.}|+\lambda(x_{i1}-\bar{x}_{i.}))^{\xi_i},$$

where $|\lambda|<1$ and the ξ_i's are positive with total sum equal to 2. After simple manipulations finished by square completion we get from (4.2) a functional equation of the form

$$\sum_{i=1}^{4}H(\xi_i)+\left(\sum_{i=1}^{4}G(\xi_i)\right)^2=\text{constant},\qquad \xi_1+\xi_2+\xi_3+\xi_4=2.$$

Here

$$G(\xi)=\frac{E_{(0,1)}[\bar{x}_{1.}(|x_{11}-\bar{x}_{1.}|+\lambda(x_{11}-\bar{x}_{1.}))^{\xi}]}{E_{(0,1)}[(|x_{11}-\bar{x}_{1.}|+\lambda(x_{11}-\bar{x}_{1.}))^{\xi}]}.$$

The exact expression of $H(\xi)$ is uninteresting to know. Clearly, G and H are analytic in some (complex) region containing the open interval $(0,+\infty)$ and continuous on the closed interval $[0,+\infty)$. The solution of this functional equation is

$$\begin{aligned}G(\xi)&=A_\lambda\xi+B_\lambda,\\H(\xi)&=C_\lambda\xi+D_\lambda,\end{aligned}\qquad \xi\geqq 0,$$

A_λ, B_λ, C_λ, and D_λ being constants. To prove this, we set $H_1(\xi)=H(\xi+\frac{1}{2})-H(\frac{1}{2})$, $G_1(\xi)=G(\xi+\frac{1}{2})-G(\frac{1}{2})$ and see that if suffices to show that all solutions being zero at $\xi=0$ of the equation

$$\sum_1^4 H_1(\xi_i)+\left[\sum_1^4 G_1(\xi_i)\right]^2=0,\qquad \xi_1+\xi_2+\xi_3+\xi_4=0\tag{4.3}$$

are linear. We shall prove that

$$H_1^{(k)}(0)=0\quad\text{and}\quad G_1^{(k)}(0)=0\quad\text{when } k\geqq 2.$$

We have the McLaurin expansions

$$\begin{aligned}&G_1(\xi)=\xi\,G_1'(0)+\xi^2\,G_1''(0)/2+o(\xi^2)\\&H_1(\xi)=\xi\,H_1'(0)+\xi^2\,H_1''(0)/2+\xi^3\,H_1^{(3)}(0)/6+\xi^4\,H^{(4)}(0)/24+o(\xi^4).\end{aligned}$$

Substituting these into (4.3), setting $\xi_4 = -\xi_1 - \xi_2 - \xi_3$ and identifying coefficients, we find that $H_1''(0)$, $H_1^{(3)}(0) = 0$, and further that

$$(G_1''(0)/2)^2 (\xi_1^2 + \xi_2^2 + \xi_3^2 + (-\xi_1 - \xi_2 - \xi_3)^2)^2 \\ = (H_1^{(4)}(0)/24)(\xi_1^4 + \xi_2^4 + \xi_3^4 + (-\xi_1 - \xi_2 - \xi_3)^4)$$

holds identically. From this last relation is easily follows that $H_1^{(4)}(0) = 0$ and, which is the most important matter, $G_1''(0) = 0$. Using induction and higher order McLaurin expansions, the remainder of this part of the proof is accomplished in exactly the same way.

Thus we have

$$E_{(0,1)}[\bar{x}_{1.}(|x_{11} - \bar{x}_{1.}| + \lambda(x_{11} - \bar{x}_{1.}))^{\xi}] \\ = (A_\lambda \xi + B_\lambda) \cdot E_{(0,1)}[(|x_{11} - \bar{x}_{1.}| + \lambda(x_{11} - \bar{x}_{1.}))^{\xi}], \qquad \xi \geqq 0. \tag{4.4}$$

As we are estimating σ^2, it is certainly no restriction to assume that F has mean value zero. Setting $\xi = 0$, we find that $B_\lambda = 0$. Taking $\xi = 1$, we easily see that A_λ is independent of λ, i.e. $A_\lambda = A_0$. Consider Eq. (4.4) when ξ is an integer k. Binomial expansion of both sides gives, for each k, an equality between two polynomials in λ for $|\lambda| < 1$. The coefficients of λ^k must then be equal, i.e.

$$E_{(0,1)}[\bar{x}_{1.}(x_{11} - \bar{x}_{1.})^k] = A_0 k \cdot E_{(0,1)}[(x_{11} - \bar{x}_{1.})^k], \qquad k = 0, 1, 2, \dots.$$

Therefore (see Section 2) F is either Normal or the d.f. of a translated Gamma variable. However, for the Gamma distribution direct calculations will show that $(\sigma^2)_P^* \neq k_n s^2$. In fact, for the exponential distribution $(\sigma^2)_P^*$ is a function h of the sufficient (and complete) statistic $(\bar{x}, x_{(1)})$, $x_{(1)}$ being the first order statistic, and therefore, since $(\sigma^2)_P^*$ is translation invariant and square multiplicative,

$$(\sigma^2)_P^* = h(\bar{x}, x_{(1)}) = h(\bar{x} - x_{(1)}, 0) = (\bar{x} - x_{(1)})^2 \cdot h(1, 0) \neq k_n s^2.$$

The proof is finished.

No attempt has been made to remove the moment condition, but there is no doubt it can be done.

Corollary. *If s^2/σ_0^2 is the best unbiased proper estimator of σ^2, then, under the conditions of Theorem* 4.1, *the d.f. is Normal.*

Proof. As the best unbiased proper estimator and the best proper estimator of σ^2 only differ by a constant factor, Theorem 4.1 applies.

5. Some Related Results

For a pure location parameter family $F(x - \theta)$ there are the following results. If F is Normal with mean zero, then $\bar{x}$ is the best proper (i.e. translative) estimator of θ. (The best proper estimator is automatically unbiassed.) The converse was proved to hold by Kagan, Linnik, and Rao [3] provided that $n \geqq 3$. For $n = 2$ the author has shown (still unpublished) that $\bar{x}$ is a best unbiassed estimator of θ also only if F is Normal with mean zero.

For a scale parameter family $F(x/\sigma)$, where $\sigma>0$ and $F(0)=0$, Kagan and Rukhin [5] have characterized the Gamma distribution in similar ways. These results will be improved in [2].

Rao [8], and later other authors, have tried to find all classes of distributions admitting UMV-estimators of certain parametric functions.

Note Added in Proof. In Theorem 2.2 and 3.2 the condition $n \geqq 7$ can be replaced by $n \geqq 6$. Hint: Use Eq. (2.2) first for

$$g(x_1, \ldots, x_n) = (|x_1 - x_2|)^{\xi_1} (|x_3 - x_4|)^{\xi_2} (|x_5 - x_6|)^{\xi_3}, \qquad \xi_1 + \xi_2 + \xi_3 = 1,$$

and then for

$$g(x_1, \ldots, x_n) = (|y_1 - \bar{y}| + \tfrac{1}{2}(y_1 - \bar{y}))^{\xi_1} (|y_1 - \bar{y}|)^{\eta_1} (|x_4 - x_5|)^{\xi_2}, \qquad \xi_1 + \eta_1 + \xi_2 = 1.$$

We also mention that the moment condition of Theorem 4.1 really can be removed.

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Lennart Bondesson
University of Lund
Department of Mathematical Statistics
Lund, Sweden
1972/1973 at:
Institut Mittag-Leffler
Auravägen 17
S-182 62 Djursholm
Sweden

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