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Radonifying Mappings and Functional Central Limit Theorems

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1. Introduction

The theory of Radonifying mappings, developed by Laurent Schwartz [7, 8], provides a powerful tool for the study of measures on linear spaces. The purpose of this paper is to show how this theory can be applied to obtain functional central limit theorems (invariance principles) concerning the convergence of probability measures to Wiener measure.

The pattern of this paper is as follows. In §2, which is purely descriptive, we recall the basic definitions and results about Radonifying mappings which we shall need. Full details of these results are given in [1, 7] and [8]. §3 contains the fundamental compactness result which we need; this result enables us to avoid the usual tightness arguments. In §4 we show how the operation of integration on the interval [0, 1] can be factorised as the product of a Radonifying mapping and a compact mapping. Throughout this paper we have not hesitated to restrict attention to special cases, and this is particularly true here. Although we go a little further in some directions, the mappings that we consider are special cases of mappings considered in a more sophisticated way by Schwartz ([7] Exposés XIV and XV). The last two sections contain applications of these preceding results to functional central limit theorems. Here a common feature is that rather strong conditions have to be placed on the moments of the random variables (for example, finite fourth moments, which are not too large compared to the second moments). By way of compensation, it is possible to consider fairly weak mixing conditions, and also to consider convergence of the measures on spaces smaller than the space $C_*[0, 1]$ of continuous functions on [0, 1] vanishing at 0. The results obtained here are similar in nature to those obtained recently by Philipp and Webb [5].

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2. Radonifying Mappings

We recall the basic definitions and results about Radonifying mappings. For simplicity, we restrict our attention to Banach spaces. The theory of Radonifying mappings is developed in more generality, and much greater detail, in [7] and [8].

A *Radon measure* μ on a Banach space E is a regular Borel probability measure on E. That is, μ is defined on the Borel sets of E, and $\mu(A) = \sup\{\mu(K)\colon K$ compact,

$K \subseteq A\}$, for any Borel set A. μ is of *order* p (for $p \geqq 1$) if

$$\|\mu\|_p = \left(\int_E \|x\|^p \, d\mu(x)\right)^{1/p} < \infty.$$

A set M of Radon measures on E is *uniformly of order* p if $\sup\{\|\mu\|_p : \mu \in M\} < \infty$. More generally, suppose that r is a lower semi-continuous extended-valued semi-norm on E – i.e. a mapping $r: E \to [0, \infty]$ satisfying

(i) $r(0) = 0$,

(ii) $r(x+y) \leqq r(x) + r(y)$,

(iii) $r(\lambda x) = |\lambda| \, r(x)$ for $\lambda \neq 0$ and

(iv) $C = \{x : r(x) \leqq 1\}$ is closed.

Then a Radon measure μ on E is *of order* (p, r) (for $p \geqq 1$) if

$$\|\mu\|_{(p,r)} = \left(\int_E r(x)^p \, d\mu(x)\right)^{1/p} < \infty$$

and a set M of Radon measures is *uniformly of order* (p, r) if

$$\sup\{\|\mu\|_{(p,r)} : \mu \in M\} < \infty.$$

If f is a continuous linear mapping from a Banach space E into a Banach space F, and if μ is a Radon measure on E, we denote the image measure by $f(\mu)$ (so that $f(\mu)(A) = \mu(f^{-1}(A))$, for each Borel set A in F).

If M and N are closed subspaces of a Banach space E, with $M \subseteq N$, we denote the quotient mapping: $E \to E/N$ by q_N, and the natural mapping: $E/M \to E/N$ by π_{NM}. We denote the collection of closed finite-codimensional subspaces of E by $\mathcal{N}$. A *cylindrical measure* μ on E is then a family $\{\mu_N\}_{N \in \mathcal{N}}$ of Radon measures on the finite dimensional spaces $\{E/N\}_{N \in \mathcal{N}}$ satisfying the consistency condition $\pi_{NM}(\mu_M) = \mu_N$ for $M \subseteq N$. If μ is a Radon measure on E, the family $\{q_N(\mu)\}_{N \in \mathcal{N}}$ is a cylindrical measure on E, and distinct Radon measures define distinct cylindrical measures. We shall therefore use the same symbol for a Radon measure and the cylindrical measure it defines. Not every cylindrical measure is defined by a Radon measure.

If $f \in E'$ (the topological dual of E), we may identify $E/f^{-1}(0)$ with the real line. With this identification, if μ is a cylindrical measure on E, we shall write μ_f for the corresponding measure $f(\mu)$ on the real line. A cylindrical measure μ is *of type* p (for $p \geqq 1$) if there is a constant K such that

$$\left(\int_{-\infty}^{\infty} |t|^p \, d\mu_f(t)\right)^{1/p} \leqq K \, \|f\|,$$

for all $f \in E'$. The least such K is denoted by $\|\mu\|_p^*$. A set M of cylindrical measures is *uniformly of type* p if $\sup\{\|\mu\|_p^* : \mu \in M\} < \infty$. A cylindrical measure defined by a Radon measure μ of order p is of type p, and $\|\mu\|_p^* \leqq \|\mu\|_p$.

We are now in a position to define Radonifying mappings. A continuous linear mapping T from a Banach space E into a Banach space F is *p-Radonifying* if whenever μ is a cylindrical measure of type p on E, $T(\mu)$ is (defined by) a Radon measure of order p on F. Note that the composition of a p-Radonifying mapping and a continuous linear mapping is p-Radonifying. T is *absolutely p-summing* if there exists a constant M such that for any finite set $x_1, \ldots, x_n$ of elements of E

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leqq M \sup_{\substack{f \in E' \\ \|f\| \leqq 1}} \left(\sum_{i=1}^{n} |f(x_i)|^p\right)^{1/p}.$$

The least such M is denoted by $\pi_p(T)$. We can now quote a simplified form of the fundamental theorem about p-Radonifying mappings (cf. [7] Exposés 11 and 12; [8] Théorème 3.4 and Proposition 3.9).

Theorem A. *If T is a continuous linear mapping from a Banach space E into a Banach space F and if $p > 1$, the following are equivalent:*

(i) *T is p-Radonifying;*

(ii) *T is p-Radonifying, and there exists $M > 0$ such that $\|T(\mu)\|_p \leqq M \|\mu\|_p^*$ for each cylindrical measure of type p on E;*

(iii) *T is absolutely p-summing.*

If so, $\pi_p(T)$ is the best possible constant in (ii).

A similar result holds for $p = 1$, if F is the separable strong dual of a Banach space or if F is reflexive. This theorem can be generalised a great deal.

The importance of Theorem A is that absolutely p-summing operators have been studied intensively, and can be characterised quite simply.

Theorem B (*Grothendieck-Pietsch: cf.* [6], *Theorem* 2). *If T is a continuous linear mapping from a Banach space E into a Banach space F, T is absolutely p-summing if and only if there is a regular Borel probability measure μ on the unit ball B of E', with the weak*-topology, and a constant M such that*

$$\|Tx\| \leqq M\Big(\int_B |f(x)|^p \, d\mu(f)\Big)^{1/p}. \tag{1}$$

If so, $\pi_p(T)$ is the best possible constant in (1).

If μ is a regular Borel probability measure on a compact Hausdorff space X, there is a natural norm-decreasing map J_p of $C(X)$ into $\mathscr{L}^p(\mu)$ which sends a function into its μ equivalence class.

Corollary C. *The map J_p: $C(X) \to \mathscr{L}^p(\mu)$ is absolutely p-summing.*

We shall be concerned with the convergence of Radon measures and cylindrical measures. We give $P(E)$, the space of Radon measures on a Banach space E, the *narrow topology* (or topology of weak convergence) – a net μ_ν converges narrowly to μ if $\int_E g \, d\mu_\nu \to \int_E g \, d\mu$ for each bounded continuous function g on E. We give $\check{P}(E)$, the space of cylindrical measures on E, the *cylindrical topology* – a net μ_ν converges cylindrically to μ if $(\mu_\nu)_N \to \mu_N$ narrowly, for each N in $\mathscr{N}$. It follows from the Cramér-Wold theorem that $\mu_\nu \to \mu$ cylindrically if and only if $(\mu_\nu)_f \to \mu_f$ narrowly, for each $f \in E'$.

3. The Composition of a p-Radonifying Mapping and a Compact Mapping

The following important theorem has useful consequences.

Theorem D ([7] *Proposition* (IV; 6, 1); [8] *Proposition* 1.16). *Suppose that r is an extended-valued seminorm on a Banach space E such that* $C=\{x\colon r(x)\leqq 1\}$ *is compact. Then* $Q=\{\mu\in P(E)\colon \|\mu\|_{(p,r)}\leqq 1\}$ *is compact in* $P(E)$ *in the narrow topology, for any* $p\geqq 1$.

As a first consequence we have

Theorem 1. *Suppose that E, F and G are Banach spaces, that* $T\colon E\to F$ *is p-Radonifying for some* $p>1$, *and that* $S\colon F\to G$ *is a compact linear mapping. If* $A_p=\{\mu\in\check{P}(E)\colon \|\mu\|_p^*\leqq 1\}$, $ST(A_p)$ *is relatively compact in* $P(G)$ *in the narrow topology.*

Let B be the unit ball in F and let $D=\pi_p(T)\,\overline{S(B)}$. D is compact; let r be the gauge of D. If $f\in B$, $\pi_p(T)\,S(f)\in D$, so that $r(Sf)\leqq \pi_p(T)^{-1}$. Thus by homogeneity $r(Sf)\leqq \pi_p(T)^{-1}\,\|f\|_F$ for all f in F. Hence if $\mu\in A_p$

$$\begin{aligned}\|ST(\mu)\|_{(p,r)}^p &= \int_G r(g)^p\, d\big(ST(\mu)\big)(g)\\ &= \int_F r(Sf)^p\, d\big(T(\mu)\big)(f)\\ &\leqq \pi_p(T)^{-p}\int_F \|f\|_F^p\, d\big(T(\mu)\big)(f)\leqq 1,\end{aligned}$$

and so we can apply Theorem D.

Corollary 2. *If E, F, G, S and T are as above, if* $\{\mu_\alpha\}_{\alpha\in A}$ *is a net in* $\check{P}(E)$ *which converges cylindrically to* μ *and if* $\{\mu_\alpha\}_{\alpha\in A}$ *is uniformly of type p, then* $ST(\mu)$ *is a Radon measure of order p, and* $ST(\mu_\alpha)\to ST(\mu)$ *narrowly.*

4. The Operators T_α

We shall apply the results of §§2 and 3 to the operator T_1 of integration. If $f\in\mathscr{L}^1(0,1)$, let $(T_1 f)(x)=\int_0^x f(t)\,dt$. T_1 is a norm-decreasing linear mapping of $\mathscr{L}^1(0,1)$ into the space $C_*[0,1]$ of continuous functions on $[0,1]$ which vanish at 0. We can express T_1 as a convolution operator, and as a product of convolution operators. For $0<\alpha\leqq 1$ (this range of values can clearly be extended, but is sufficient for our purpose) let $\phi_\alpha(t)=\Gamma(\alpha)^{-1}\,t^{\alpha-1}$ for $0<t\leqq 1$. Then $\phi_\alpha\in\mathscr{L}^1(0,1)$, and, if $\alpha>0$, $\beta>0$ and $\alpha+\beta\leqq 1$,

$$\int_0^t \phi_\alpha(s)\,\phi_\beta(t-s)\,ds=\phi_{\alpha+\beta}(t). \tag{2}$$

If $f\in\mathscr{L}^1(0,1)$, we define

$$(T_\alpha f)(t)=\int_0^t f(s)\,\phi_\alpha(t-s)\,ds. \tag{3}$$

Note that this agrees with the definition of T_1 given earlier. It follows from (2) that $T_\alpha T_\beta = T_{\alpha+\beta}$ for $\alpha>0$, $\beta>0$, $\alpha+\beta\leqq 1$. We shall need the following well-known properties of T_α:

Lemma E. (i) *If* $0<1/p<\alpha\leqq 1$, T_α *maps* $\mathscr{L}^p(0,1)$ *continuously into* $C_*[0,1]$,

(ii) *If* $\alpha>0$, *the mapping* T_α: $C_*[0,1]\to C_*[0,1]$ *is compact.*

If $\alpha>0$, the mapping T_α is one-one on $C_*[0,1]$. We shall denote $T_\alpha(C_*[0,1])$ by $C_*^\alpha[0,1]$, and give it the image norm defined by T_α and the norm on $C_*[0,1]$. Thus it follows from Lemma E(ii) that the inclusion: $C_*^\alpha[0,1]\to C_*[0,1]$ is compact.

Our first result is a special case of much more general results obtained by Schwartz ([7], Exposé XIV). As usual we denote the conjugate index of p by p' (so that $1/p+1/p'=1$).

Theorem 3. *If* $q>p'$ *and* $0<\delta<1/p'-1/q$, T_1 *is a q-Radonifying mapping of* $\mathscr{L}^p(0,1)$ *into* $C_*^\delta[0,1]$.

We can choose α and β such that $\alpha>1/p$, $\beta>1/q$ and $\alpha+\beta+\delta=1$. Consider the following factorisation of T_1:

$$\mathscr{L}^p(0,1)\xrightarrow{T_\alpha} C_*[0,1]\xrightarrow{J_q}\mathscr{L}^q(0,1)\xrightarrow{T_\beta} C_*[0,1]\xrightarrow{T_\delta} C_*^\delta[0,1].$$

J_q is q-Radonifying (Corollary C) and the other mappings exist and are continuous (Lemma E).

Theorem 4. *Suppose that* $q>p'$, *that* $0<\delta<1/p'-1/q$ *and that*

$$A_q=\{\mu\in\check{P}(\mathscr{L}^p(0,1))\colon \|\mu\|_q^*\leqq 1\}.$$

Then $T_1(A_q)$ *is narrowly relatively compact in* $P(C_*^\delta[0,1])$.

We can choose α, β and γ such that $\alpha>1/p$, $\beta>1/q$, $\gamma>0$ and $\alpha+\beta+\gamma+\delta=1$. Consider the following factorisation of T_1:

$$\mathscr{L}^p(0,1)\xrightarrow{T_\alpha} C_*[0,1]\xrightarrow{J_q}\mathscr{L}^q(0,1)$$
$$\xrightarrow{T_\beta} C_*[0,1]\xrightarrow{T_\gamma} C_*[0,1]\xrightarrow{T_\delta} C_*^\delta[0,1].$$

As before $T_\beta J_q T_\alpha$ is q-Radonifying, and $T_\delta T_\gamma$ is compact, by Lemma E, so that we can apply Theorem 1.

Corollary 5. *If* p, q *and* δ *satisfy the conditions of Theorem* 3, *if* $\{\mu_\alpha\}_{\alpha\in A}$ *is a net in* $\check{P}(\mathscr{L}^p(0,1))$ *which converges cylindrically to* μ *and if* $\{\mu_\alpha\}_{\alpha\in A}$ *is uniformly of type* q *then* $T_1\mu_\alpha$ *converges narrowly to* $T_1\mu$, *a Radon measure of order* q, *in* $C_*^\delta[0,1]$.

We now consider the case $p=2$; $\mathscr{L}^2(0,1)$ is a real Hilbert space and the finite-dimensional quotients $\mathscr{L}^2(0,1)/N$ are isometrically isomorphic to finite-dimensional Euclidean spaces. We denote the cylindrical measure obtained by giving these spaces normalised Gaussian measures by w, and call it *white noise.* If $f\in\mathscr{L}^2(0,1)(=(\mathscr{L}^2(0,1))')$, w_f is normally distributed with mean 0, variance $\|f\|^2$. In particular w is of type q, for any q.

Corollary 5 ([7], *Proposition XV* 4; 1). $W=T_1(w)$ *is a Radon measure of order* q *on* $C_*^\delta[0,1]$ *for any* $0\leqq\delta<\frac{1}{2}$ *and any* q.

For we can apply Theorem 3 for any q such that $1/q<\frac{1}{2}-\delta$. Thus W is of order q, for large q, and *a fortiori* for smaller values of q. W is of course Wiener measure.

Corollary 6. *Suppose that $\{\mu_\alpha\}_{\alpha\in A}$ is a net of cylindrical measures, uniformly of type $\theta>2$, on $L^2(0,1)$, which converges cylindrically to w. Then the net $\{T_1\mu_\alpha\}_{\alpha\in A}$ converges narrowly to W on $C_*^\delta[0,1]$ for any $0\leqq\delta<\frac{1}{2}-1/\theta$.*

It is not possible, when $\delta=0$, to take $\theta=2$ in this corollary, for $T_1:\mathscr{L}^2(0,1)\to C_*[0,1]$ is *not* 2-Radonifying. If it were, we would have $\sum_{n=1}^{\infty}\|Te_n\|^2<\infty$ for any orthonormal sequence (e_n) in $\mathscr{L}^2(0,1)$; the Haar system shows that this is not so.

5. Triangular Arrays: Notation and Terminology

In §6 we shall show how these results on Radonifying mappings can be used to obtain functional central limit theorems for triangular arrays. Here we establish the notation which we shall use. We shall suppose that for each $n=1,2,\ldots$ we are given k_n random variables $\xi_{n1},\ldots,\xi_{nk_n}$ on the same probability space (Ω_n,P_n). We shall always suppose that

$$E(\xi_{nj})=0 \qquad \text{for all } n \text{ and } j, \tag{4}$$

and

$$\sigma_{nj}^2=E(\xi_{nj}^2)<\infty \qquad \text{for all } n \text{ and } j\in\{1,k_n\}. \tag{5}$$

We shall write

$$\begin{aligned} S_{nj}&=\sum_{i=1}^{j}\xi_{ni}, \qquad S_n=S_{nk_n}\\ s_{nj}^2&=E(S_{nj}^2), \qquad s_n^2=E(S_n^2). \end{aligned} \tag{6}$$

We shall always suppose that

$$0<s_{n1}^2<s_{n2}^2<\cdots<s_n^2, \qquad \text{for all } n, \tag{7}$$

and

$$s_n^2\to 1 \quad \text{as} \quad n\to\infty. \tag{8}$$

We shall write

$$t_{n0}=0, \qquad t_{nj}=s_n^{-2}s_{nj}^2 \qquad \text{for } 1\leqq j\leqq k_n, \tag{9}$$

and

$$v_{nj}^2=t_{nj}-t_{n,j-1} \qquad \text{for } 1\leqq j\leqq k_n. \tag{10}$$

Note that $v_{nj}^2=s_n^{-2}\sigma_{nj}^2$ if $\xi_{n1},\ldots,\xi_{nk_n}$ are independent.

We now define, for each n, a vector-valued random variable $X_n(t)$ in $\mathscr{L}^2(0,1)$ by setting

$$X_n(t)=v_{nj}^{-2}\xi_{nj} \qquad \text{for } t_{n,j-1}<t\leqq t_{nj}. \tag{11}$$

Each $X_n(t)$ is finite-dimensional, and has a corresponding distribution measure μ_n (which we shall consider as a cylindrical measure). We shall denote T_1X_n by Y_n and $T_1\mu_n$ by ν_n (which we shall consider as a Radon measure). Note that $Y_n(t_{nj})=\sum_{i=1}^{j}\xi_{ni}$ for $j=1,\ldots,k_n$, and that Y_n is linear between these values.

If $f \in \mathscr{L}^2(0, 1)$, we set

$$\beta_{nj} = \int_{t_{n,j-1}}^{t_{nj}} f(t)\, dt. \tag{12}$$

Then $(\mu_n)_f$ is the distribution measure of the random variable

$$\langle X_n, f\rangle = \sum_{j=1}^{k_n} \beta_{nj}\, v_{nj}^{-2}\, \xi_{nj} = \sum_{j=1}^{k_n} \alpha_{nj}\, \eta_{nj}, \tag{13}$$

where $\alpha_{nj} = \beta_{nj}\, v_{nj}^{-1}$ and $\eta_{nj} = v_{nj}^{-1}\, \xi_{nj}$. Note that

$$|\beta_{nj}| \leqq v_{nj} \left(\int_{t_{n,j-1}}^{t_{nj}} (f(t))^2\, dt \right)^{\frac{1}{2}}, \tag{14}$$

by the Cauchy-Schwartz inequality, so that

$$\sum_{j=1}^{k_n} \alpha_{nj}^2 \leqq \sum_{j=1}^{k_n} \int_{t_{n,j-1}}^{t_{nj}} f(t)^2\, dt = \|f\|^2. \tag{15}$$

We shall consider cases in which the random variables (ξ_{nj}) in each row are dependent, and satisfy a mixing condition. For each n and j we denote by $M_j(n)$ the σ-field of Ω_n generated by $\{\xi_{nk}\colon k \leqq j\}$ by $M^j(n)$ the σ-field of Ω_n generated by $\{\xi_{nk}\colon k \geqq j\}$. If $\phi = (\phi_n)$ is a monotonic decreasing sequence of numbers tending to 0, we shall say that the array (ξ_{nj}) is *ϕ-mixing* if

$$|P_n(E_1 \cap E_2) - P_n(E_1)\, P_n(E_2)| \leqq \phi_h\, P_n(E_1) \tag{16}$$

whenever $E_1 \in M_j(n)$ and $E_2 \in M^{j+h}(n)$ for any n and any $1 \leqq j \leqq j+h \leqq k_n$. Similarly if $\psi = (\psi_n)$ is a monotonic decreasing sequence of numbers tending to 0, we shall say that (ξ_{nj}) is *ψ-mixing* if

$$|P_n(E_1 \cap E_2) - P_n(E_1)\, P_n(E_2)| \leqq \psi_h\, P_n(E_1)\, P_n(E_2) \tag{17}$$

whenever $E_1 \in M_j(n)$ and $E_2 \in M^{j+h}(n)$ for any n and for any $1 \leqq j < j+h \leqq k_n$. We need the following facts about ϕ-mixing and ψ-mixing arrays.

Lemma F (cf. [2] pp. 170–171). *Suppose that (ξ_{nj}) is ϕ-mixing. If ξ is measurable $M_j(n)$, if η is measurable $M^{j+h}(n)$, if $\xi \in L^p(\Omega_n)$ and $\eta \in L^{p'}(\Omega_n)$ then*

$$|E(\xi\eta) - E(\xi)\, E(\eta)| \leqq 2\, \phi_h^{1/p}\, \|\xi\|_p\, \|\eta\|_{p'}. \tag{18}$$

In particular, if $\xi, \eta \in L^\infty(\Omega_n)$

$$|E(\xi\eta) - E(\xi)\, E(\eta)| \leqq 2\, \phi_h\, \|\xi\|_\infty\, \|\eta\|_\infty. \tag{19}$$

Lemma G [4]. *Suppose that (ξ_{nj}) is ψ-mixing. If ξ is measurable $M_j(n)$, if η is measurable $M^{(j+h)}(n)$ (with $h > 0$) and if $\xi, \eta \in L^1(\Omega_n)$ then $\xi\eta \in L^1(\Omega_n)$ and*

$$|E(\xi\eta) - E(\xi)\, E(\eta)| \leqq \psi_h\, \|\xi\|_1\, \|\eta\|_1. \tag{20}$$

6. Some Functional Central Limit Theorems

We are now in a position to establish some functional central limit theorems. In each case the procedure will be the same. First we show that the cylindrical measures μ_n are uniformly of type j (for some suitable j) in $\mathscr{L}^2(0,1)$. This implies that the sequence of measures (ν_n) is narrowly relatively compact in $P(C_*^\delta[0,1])$ for some suitable δ (Theorem 4). Next we show that any limit measure of the sequence (ν_n) must be Wiener measure W; this implies that $\nu_n \to W$ narrowly.

In order to show that any limit measure must be W, we use a characterisation due to Rosén (see [2], Theorem 19.1). This says that if μ is a Radon measure on $C_*[0,1]$ for which

$$\int f(t)\,d\mu(f)=0 \qquad \text{for all } 0\leqq t\leqq 1, \tag{21}$$

$$\int (f(t))^2\,d\mu(f)=t \qquad \text{for all } 0\leqq t\leqq 1, \tag{22}$$

and

$$f(t)-f(s) \quad \text{and} \quad f(u)-f(t) \tag{23}$$

are independent random variables whenever $0\leqq s<t<u\leqq 1$, then $\mu=W$. (21) and (22) are usually easy to establish. The next result enables us to deal with (23):

Theorem 7. *Suppose that (ξ_{nj}) is a triangular array which is ϕ-mixing, for some ϕ. Suppose that the corresponding sequence (ν_n) has ν as a limit measure in $P(C_*[0,1])$. If*

$$\max\{\sigma_{nj}\colon 1\leqq j\leqq k_n\}\to 0 \quad \text{as} \quad n\to\infty \tag{24}$$

then the random variables $f(t)-f(s)$ and $f(u)-f(t)$ are ν-independent whenever $0\leqq s<t<u\leqq 1$.

By choosing a subsequence if necessary, we may suppose that $\nu_n\to\nu$ narrowly, and that

$$\max\{\sigma_{nj}\colon 1\leqq j\leqq k_n\}\leqq 1/n^2, \qquad \text{for each } n. \tag{25}$$

Suppose that $0\leqq s<t<u\leqq 1$. Let l_n be the least integer j such that $t_{nj}\geqq t$, and let $m_n=l_n+n$. Note that

$$\begin{aligned} t_{n,j+1}-t_{n,j} &= s_n^{-2}\big(E(S_{n,j+1}^2)-E(S_{nj}^2)\big)\\ &\leqq 2 s_n^{-1}\big((E(S_{n,j+1}^2))^{\frac12}-(E(S_{nj}^2))^{\frac12}\big)\\ &\leqq 2 s_n^{-1}\,\sigma_{n,j+1}, \end{aligned}$$

so that $t_{n,l_n}\to t$. Similarly

$$t_{n,m_n}-t_{n,l_n}\leqq 2 s_n^{-1}\sum_{k=1}^{n}\sigma_{n,l_n+k}\leqq 2 s_n^{-1}\,n^{-1},$$

so that $t_{n,m_n}\to t$, too. In particular, for sufficiently large n, $t_{n,m_n}<u$. For such n let us set $q_n=Y_n(t)-Y_n(s)$, $r_n=Y_n(t_{n,m_n})-Y_n(t)$ and $v_n=Y_n(u)-Y_n(t_{n,m_n})$. Note that

$$(E(r_n^2))^{\frac12}\leqq \sum_{k=0}^{n}\big(E(\xi_{n,l_n+k}^2)\big)^{\frac12}\leqq (n+1)/n^2.$$

Now let α and β be any real numbers. Then

$$\begin{aligned} &\int e^{i\alpha(f(t)-f(s))}\,e^{i\beta(f(u)-f(t))}\,d\nu_n(f)\\ &\quad = E(e^{i\alpha q_n}\,e^{i\beta(r_n+v_n)}) = E(e^{i\alpha q_n}\,e^{i\beta v_n}) + E\big((e^{i\beta r_n}-1)\,e^{i\alpha q_n}\,e^{i\beta v_n}\big), \end{aligned}$$

and

$$\int e^{i\alpha(f(t)-f(s))}\, d\nu_n(f) \int e^{i\beta(f(u)-f(t))}\, d\nu_n(f)$$
$$= E(e^{i\alpha q_n})\, E(e^{i\beta(r_n+v_n)}) = E(e^{i\alpha q_n})\, E(e^{i\beta v_n}) + E(e^{i\alpha q_n})\, E\big((e^{i\beta r_n}-1)\, e^{i\beta v_n}\big).$$

Now by Lemma F,

$$|E(e^{i\alpha q_n}\, e^{i\beta v_n}) - E(e^{i\alpha q_n})\, E(e^{i\beta v_n})| \leqq 2\phi_n;$$

also

$$\big|E\big((e^{i\beta r_n}-1)\, e^{i\alpha q_n}\, e^{i\beta v_n}\big)\big| \leqq E(|\beta r_n|) \leqq |\beta|\, (E(r_n^2))^{\frac{1}{2}} \leqq \beta(n+1)/n^2$$

and

$$\big|E\big((e^{i\beta r_n}-1)\, e^{i\beta v_n}\big)\big| \leqq \beta(n+1)/n^2.$$

Consequently

$$\begin{aligned}
&\int e^{i\alpha(f(t)-f(s))}\, e^{i\beta(f(u)-f(t))}\, d\nu(f)\\
&\qquad - \int e^{i\alpha(f(t)-f(s))}\, d\nu(f) \int e^{i\beta(f(u)-f(t))}\, d\nu(f)\\
&= \lim_{n\to\infty} \Big[\int e^{i\alpha(f(t)-f(s))}\, e^{i\beta(f(u)-f(t))}\, d\nu_n(f)\\
&\qquad - \int e^{i\alpha f(t)-f(s)}\, d\nu_n(f) \int e^{i\beta f(u)-f(t)}\Big]\, d\nu_n(f)\\
&= 0.
\end{aligned}$$

Since this holds for all α and β, $f(t)-f(s)$ and $f(u)-f(t)$ are independent (cf. [3] Theorem 6.6.1).

First we consider the case where the random variables in each row are independent.

Theorem 8. *Suppose that $\xi_{n1}, \dots, \xi_{nk_n}$ are independent, for each n. If, for some integer $p>1$, there exists $K>0$ such that*

$$\mu_{2p}(n,j) = E(\xi_{nj}^{2p}) \leqq K^{2p}\, \sigma_{nj}^{2p} \quad \textit{for each } n \textit{ and each } j, \tag{26}$$

then the sequence (μ_n) is uniformly of type $2p$ in $\mathscr{L}^2(0,1)$.

If, further, (24) *holds, then $\nu_n \to W$ narrowly in $P(C_*^\delta[0,1])$ for any $0 \leqq \delta < 1/2p$.*

If $f \in \mathscr{L}^2(0,1)$, $(\mu_n)_f$ is the distribution measure of the random variable $\langle X_n, f\rangle$ defined in (13). Since

$$\mu_{2q}(n,j) \leqq (\mu_{2p}(n,j))^{q/p} \leqq K^{2q}\, \sigma_{nj}^{2q} \quad \text{for } q<p,$$

we have

$$\begin{aligned}
E(\langle X_n, f\rangle^{2p}) &= \sum_{\nu_1+\cdots+\nu_{k_n}=p} (2p)! \prod_{j=1}^{k_n} \frac{(\beta_{nj}\, v_{nj}^{-2})^{2\nu_j}\, \mu_{2\nu_j}(n,j)}{(2\nu_j)!}\\
&\leqq \sum_{\nu_1+\cdots+\nu_{k_n}=p} (2p)! \prod_{j=1}^{k_n} \frac{(K s_n\, \alpha_{nj})^{2\nu_j}}{(2\nu_j)!}\\
&\leqq \frac{(2p)!}{2^p\, p!} \sum_{\nu_1+\cdots+\nu_{k_n}=p} p! \prod_{j=1}^{k_n} \frac{(K^2\, s_n^2\, \alpha_{nj}^2)^{\nu_j}}{\nu_j!}\\
&\leqq p^p\, K^{2p}\, s_n^{2p} \left(\sum_{j=1}^{k_n} \alpha_{nj}^2\right)^p\\
&\leqq (p\, s_n^2\, K^2\, \|f\|^2)^p, \quad \text{by (15).}
\end{aligned}$$

Thus the sequence (μ_n) is uniformly of type $2p$. If $0 \leqq \delta \leqq \frac{1}{2} - 1/2p$, then (v_n) is uniformly of order $2p$ in $C_*^\delta[0,1]$, by Theorem 3, and narrowly relatively compact in $P(C_*^\delta[0,1]$, by Theorem 4. Suppose now that (24) holds, and that v is a limit measure of the sequence (v_n). Then

$$\int f(t)\,dv_n(f) = E(Y_n(t)) = 0,$$
$$\int (f(t))^2\,dv_n(f) = E(Y_n(t))^2 = t$$

for all n and each $0 \leqq t \leqq 1$. Since the distributions of the random variables $Y_n(t)$ are narrowly relatively compact in $P(R)$, and therefore tight, by Prohorov's theorem (cf. [2], Theorem 6.2), it follows that (21) and (22) hold for v. Finally (23) is satisfied by Theorem 7. Thus $v = W$, and $v_n \to W$ narrowly.

Next we consider a ϕ-mixing triangular array of *bounded* random variables.

Theorem 9. *Suppose that $(\xi_{n,j})$ is a ϕ-mixing triangular array, with $\sum_{j=0}^{\infty} \phi_j^{\frac{1}{2}} = L < \infty$, of bounded random variables, and that there exists a constant C such that*

$$|\xi_{nj}| \leqq C v_{nj} \quad \textit{for all } n \textit{ and all } j. \tag{27}$$

Then the sequence (μ_n) is uniformly of type 4 on $\mathscr{L}^2(0,1)$.

If, further, (24) holds, then $v_n \to W$ narrowly in $P(C_*^\delta[0,1])$, for any $0 \leqq \delta < \frac{1}{4}$.

As before, if $f \in \mathscr{L}^2(0,1)$, $(\mu_n)_f$ is the distribution of $\langle X_n, f\rangle = \sum_{j=1}^{k_n} \alpha_{nj}\eta_{nj}$. Thus, for fixed n,

$$E(\langle X_n, f\rangle^4) \leqq 4! \sum_{r=1}^{k_n} \sum_{\substack{0 \leqq i,j,k \\ i+j+k \leqq k_n - r}} \gamma_r \gamma_{r+i} \gamma_{r+i+j} \gamma_{r+i+j+k} |E(\eta_{nr}\eta_{n,r+i}\eta_{n,r+i+j}\eta_{n,r+i+j+k})|,$$

where $\gamma_t = |\alpha_{nt}|$. Now, arguing as in [2] (p. 173),

$$|E(\eta_{nr}\eta_{n,r+i}\eta_{n,r+i+j}\eta_{n,r+i+j+k})| \leqq 4C^4 \min(\phi_i, \phi_k, \phi_i\phi_k + \phi_j),$$

so that

$$E(\langle X_n, f\rangle^4) \leqq 96\,C^4(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4),$$

where

$$\begin{aligned}
\Sigma_1 &= \sum_{r=1}^{k_n} \sum_{\substack{0 \leqq j,k \leqq i \\ i+j+k \leqq k_n - r}} \gamma_r \gamma_{r+i} \gamma_{r+i+j} \gamma_{r+i+j+k} \phi_i \\
&\leqq \sum_{r=1}^{k_n} \sum_{\substack{0 \leqq j \leqq i \\ i+j \leqq k_n - r}} \gamma_r \gamma_{r+i} \gamma_{r+i+j} \phi_i (i+1)^{\frac{1}{2}} \\
&\leqq \sum_{r=1}^{k_n} \sum_{i=0}^{k_n - r} \gamma_r \gamma_{r+i} \phi_i (i+1) \\
&\leqq \sum_{i=0}^{k_n} \phi_i (i+1) \left(\sum_{r=1}^{k_n - i} \gamma_r^2\right)^{\frac{1}{2}} \left(\sum_{r=1}^{k_n - i} \gamma_{r+1}^2\right)^{\frac{1}{2}} \\
&\leqq \sum_{i=0}^{k_n} \phi_i (i+1) \leqq \left(\sum_{i=0}^{k_n} \phi_i^{\frac{1}{2}}\right)^2 \leqq L^2,
\end{aligned}$$

$$\sum_2 = \sum_{r=1}^{k_n} \sum_{\substack{0 \leqq i,j \leqq k \\ i+j+k \leqq k_n - r}} \gamma_r \gamma_{r+i} \gamma_{r+i+j} \gamma_{r+i+j+k} \phi_k \leqq L^2,$$

$$\sum_3 = \sum_{r=1}^{k_n} \sum_{\substack{0 \leqq i,k \leqq j \\ i+j+k \leqq k_n - r}} \gamma_r \gamma_{r+i} \gamma_{r+i+j} \gamma_{r+i+j+k} \phi_j \leqq L^2,$$

and

$$\begin{aligned} \sum_4 &= \sum_{r=1}^{k_n} \sum_{\substack{0 \leqq i,k \leqq j \\ i+j+k \leqq k_n - r}} \gamma_r \gamma_{r+i} \gamma_{r+i+j} \gamma_{r+i+j+k} \phi_i \phi_k \\ &\leqq \sum_{r=1}^{k_n} \sum_{i+k \leqq k_n - r} \gamma_r \gamma_{r+i} \phi_i \phi_k \\ &\leqq \sum_{k=0}^{\infty} \phi_k \sum_{i=0}^{k_n} \phi_i \left(\sum_{r=1}^{k_n} \gamma_r \gamma_{r+i} \right) \leqq \left(\sum_{k=0}^{\infty} \phi_k \right)^2 \leqq L^2. \end{aligned}$$

Thus (μ_n) is uniformly of type 4, and (ν_n) is uniformly of order in $C_*^\delta[0,1]$, for any $0 \leqq \delta < \frac{1}{4}$. Suppose now that (24) holds. As before, $\int f(t)\,d\nu_n(f) = 0$, for $0 \leqq t \leqq 1$, which implies that (21) is satisfied for any limit measure ν. For fixed t, let $j_n = \sup\{j: t_{nj} \leqq t\}$, so that

$$\int (f(t))^2\,d\nu_n(f) = E\left(\sum_{j=1}^{j_n} \xi_{nj} + \theta_n \xi_{n,j_n+1} \right)^2$$

for some $0 \leqq \theta_n \leqq 1$. Now

$$\begin{aligned} &\left| \left(E\left(\sum_{j=1}^{j_n} \xi_{nj} + \theta_n \xi_{n,j_n+1} \right)^2 \right)^{\frac{1}{2}} - t^{\frac{1}{2}} \right| \\ &\leqq \left| \left(E\left(\sum_{j=1}^{j_n} \xi_{nj} + \theta_n \xi_{n,j_n+1} \right)^2 \right)^{\frac{1}{2}} - \left(E\left(\sum_{j=1}^{j_n} \xi_{nj} \right)^2 \right)^{\frac{1}{2}} \right| + t^{\frac{1}{2}} - t_{n,j_n}^{\frac{1}{2}} \\ &\leqq \theta_n \left(E(\xi_{n,j_n+1}^2) \right)^{\frac{1}{2}} + t^{\frac{1}{2}} - t_{n,j_n}^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus $\int (f(t))^2\,d\nu_n(f) \to t$ as $n \to \infty$, from which (22) follows.

Finally (23) is satisfied, by Theorem 7, so that $\nu = W$ and $\nu_n \to W$ narrowly.

If instead of assuming (27), we make the weaker assumption that there exists a constant C such that

$$E(\xi_{nj}^4) \leqq C^4 v_{nj}^4 \qquad \text{for all } n \text{ and all } j, \tag{28}$$

then using Lemma F, we find that

$$|E(\eta_{nr}\, \eta_{n,r+i}\, \eta_{n,r+i+j}\, \eta_{n,r+i+j+k})| \leqq 4C^4 \min(\phi_i^{\frac{1}{2}}, \phi_k^{\frac{3}{4}}, \phi_i^{\frac{1}{4}} \phi_k^{\frac{3}{4}} + \phi_j^{\frac{1}{2}}),$$

so that at first sight it appears that we must suppose that $\sum_{j=0}^{\infty} \phi_j^{\frac{1}{2}} < \infty$. If, however, we consider sums $\sum_1'$, $\sum_2'$, $\sum_3'$ and $\sum_4'$, where $\sum_1'$ is taken over those suffixes for which $i = \max(i, j^\alpha, k^\beta)$, $\sum_2'$ over those for which $k^\beta = \max(i, j^\alpha, k^\beta)$ and $\sum_3'$ and $\sum_4'$ over those for which $j^\alpha = \max(i, j^\alpha, k^\beta)$, and choose α and β suitably, then we can do rather better. Thus if we take $\alpha = 2.6$, $\beta = 4.2$, it is enough to suppose that $\sum_{j=0}^{\infty} \phi_j^{0.19} < \infty$. Thus we have

Theorem 10. *Suppose that* (ξ_{nj}) *is a* ϕ*-mixing triangular array, with* $\sum_{j=0}^{\infty} \phi_j^{0.19} < \infty$, *for which* (28) *holds. Then the sequence* (μ_n) *is uniformly of type* 4 *on* $\mathscr{L}^2(0, 1)$.

If, further, (24) *holds, then* $\nu_n \to W$ *narrowly in* $P(C_*^\delta[0, 1])$, *for any* $0 \leqq \delta < \frac{1}{4}$.

The details are left to the reader.

Since ψ-mixing is inherently a stronger condition than ϕ-mixing, it is to be expected that we can place weaker conditions on the sequence (ψ_n); the next theorem shows that this is so.

Theorem 11. *Suppose that* (ξ_{nj}) *is a* ψ*-mixing triangular array, with* $\sum_{j=1}^{\infty} \psi_j^{\frac{1}{2}} < \infty$, *for which* (28) *holds. Then the sequence* (μ_n) *is uniformly of type* 4 *on* $\mathscr{L}^2(0, 1)$.

If, further, (24) *holds, then* $\nu_n \to W$ *narrowly in* $P(C_*^\delta[0, 1]$, *for any* $0 \leqq \delta < \frac{1}{4}$.

The proof is similar to that of Theorem 9, using Lemma G in place of Lemma F. Some care is needed with repeated suffixes, but the details are again left to the reader.

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