Z. Wahrscheinlichkeitstheorie verw. Gebiete
48, 17–34 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Growth Rates in the Branching Random Walk

J.D. Biggins

Department of Probability and Statistics, University of Sheffield,
Sheffield S3 7RH, England

1. Introduction

We will consider the branching random walk on the real line. An initial ancestor is at the origin. He has children, the first generation, and these have positions which form a point process on the line. These children in their turn have children, independently of each other. The positions of the children of a first generation person, relative to his own position form a point process; this point process has the same distributions as the one giving the positions of the initial ancestor's children. This gives the second generation. Subsequent generations are formed in the same way. Branching random walks are essentially the same as spatially homogeneous branching processes and are closely related to cluster fields, both of which are discussed in [14].

Let $\{z_r^{(n)}\}$ be the positions of the nth generation people and let

$$Z^{(n)}(t) = \# \{r: z_r^{(n)} \leqq t\},$$

the number of nth generation people to the left of t. The increasing function F defined by

$$F(t) = \mathscr{E}[Z^{(1)}(t)]$$

(which is assumed to be finite for all t) satisfies

$$F^{n*}(t) = \mathscr{E}[Z^{(n)}(t)]$$

where F^{n*} is the n-fold Stieltjes convolution of F. Throughout this paper we will assume that $F(\infty) > 1$ so that the Galton-Watson process formed by the generation sizes survives with positive probability. We will also assume that F is non-degenerate, that is that F has more than one point-of-increase. The Laplace-Stieltjes transform of F will be denoted by m, that is

$$m(\theta) = \int_{-\infty}^{\infty} e^{-\theta t} F(dt).$$

In [5] it was established that, under certain conditions, $(Z^{(n)}(na))^{1/n}$ and $(F^{n*}(na))^{1/n}$ behave similarly for large n; in fact an analogue of Chernoff's

0044-3719/79/0048/0017/$03.60

theorem on the asymptotic behaviour of $(F^{n*}(na))^{1/n}$ was established for $(Z^{(n)}(na))^{1/n}$. This is a rather crude result, the exponent n^{-1} smothers much potential variation between the two sequences. Bahadur and Ranga Rao [3] have given a more precise result on the size of $F^{n*}(na)$ (their proof is valid even when $F(\infty)>1$ as was noted by Kingman [12]) and an analogue of this result for $Z^{(n)}(na)$ is a consequence of the result established here. Let ϕ be in the interior of $\{\theta: m(\theta)<\infty\}$, which is assumed to be non-empty, and let

$$b=-\frac{m'(\phi)}{m(\phi)},$$

then, subject to certain conditions holding,

$$\frac{Z^{(n)}(nb)}{F^{n*}(nb)}\to W(\phi)<\infty \quad \text{a.s.} \tag{1.1}$$

(where $W(\phi)>0$ if the process survives). This tells us how the number of nth generation people to the left of nb grows with n, hence the paper's title.

Notice that (1.1) is formally analogous to

$$\frac{Z^{(n)}(\infty)}{F^{n*}(\infty)}\to W \quad \text{a.s.}$$

when $F(\infty)<\infty$, which is, stated in a slightly unusual form, a classical result in the theory of the Galton-Watson process. In that case $\{Z^{(n)}(\infty)/F^{n*}(\infty)\}$ is a martingale and W is its limit. In fact the limit variable $W(\phi)$ in (1.1) is also the limit of a martingale sequence. To be specific the sequence

$$W^{(n)}(\theta)=\sum_r \frac{\exp(-\theta z_r^{(n)})}{m(\theta)^n}=\frac{\int e^{-\theta t} Z^{(n)}(dt)}{m(\theta)^n}$$

is a non-negative martingale whenever $m(\theta)<\infty$. This martingale is discussed in [4] and the convergence of it to a non-degenerate limit is investigated there. The limit of this martingale, $W(\theta)$, when $\theta=\phi$ is the limit random variable appearing in (1.1).

In the next section we will prove for F^{n*} the result whose analogue for $Z^{(n)}$ we will finally establish. The main result, Theorem B will be stated at the end of that section. The remainder of the paper is devoted to the proof of this main result. (Section four contains results on the relationship between the moments of $W^{(1)}(\phi)$, $W(\phi)$ and $\sup\{W^{(s)}(\phi): s\}$, these may be of some independent interest.)

2. The Tails of F^{n*}

A slight generalization of Theorem 1 in [3] will now be proved. The method of proof is different from theirs.

Consider the measure

$$\bar{F}(dx)=\frac{e^{-\phi(x+b)}}{m(\phi)}F(dx+b).$$

It is easy to check that $\bar{F}$ is a distribution function centred on zero with a finite variance which will be denoted by σ^2. In fact

$$\sigma^2 = \frac{m''(\phi)}{m(\phi)} - \left(\frac{m'(\phi)}{m(\phi)}\right)^2$$

and is strictly positive because F is non-degenerate. Furthermore the Laplace-Stieltjes transform of $\bar{F}$,

$$\int e^{-\theta x} \bar{F}(dx) = \frac{e^{\theta b} m(\theta+\phi)}{m(\phi)}, \tag{2.1}$$

converges in a neighbourhood of the origin. The important property of $\bar{F}$, which can be proved quite easily by induction is that

$$\bar{F}^{n*}(dx) = e^{-\phi x} \frac{e^{-\phi n b}}{m(\phi)^n} F^{n*}(dx+nb). \tag{2.2}$$

The concave function ζ is defined by

$$\zeta(a) = \inf\{\theta a + \log m(\theta) : \theta\}$$

so that, by calculus,

$$\zeta(b) = \phi b + \log m(\phi)$$

and hence (2.2) can be rewritten as

$$\bar{F}^{n*}(dx) = e^{-\phi x} e^{-n\zeta(b)} F^{n*}(dx+nb). \tag{2.3}$$

The local version of the central limit theorem, which I will now describe, gives very precise information about the measure $\sigma\sqrt{2\pi n}\bar{F}^{n*}(dx)$ as n tends to infinity. By using the formula (2.3) this can be converted into information about $F^{n*}(dx)$.

For $d>0$ let $L_d = \{nd : n \in \mathbb{Z}\}$ then the distribution function $\bar{F}$ is either lattice of span d, in which case all of the points of increase of $\bar{F}$ lie in $c+L_d$ for some $0 \leq c < d$ and this holds for no larger d, or it is non-lattice, in which case we will describe it as lattice with span $d=0$. We will let $L_0 = \mathbb{R}$. Notice that F and $\bar{F}$ have the same span.

Let $\ell_d(dx)$ attach weight d to each point of L_d and let $\ell_0(dx)$ be Lebesgue measure on the real line. A function f will be called directly integrable with respect to ℓ_d if, when $d>0$. $\int |f(x)| \ell_d(dx) < \infty$ and if, when $d=0$, f is directly Riemann integrable ([9], Chapter XI, using the obvious extension of this definition to functions defined on $(-\infty, \infty)$). Notice that in either case f must be bounded on L_d.

When $d>0$ define

$$c_n = nc - d(nc/d)_\wedge$$

where $(x)_\wedge$ is the largest integer not greater than x, and when $d=0$ let $c_n=0$. In the former case the points of increase of $\bar{F}^{n*}$ are contained in $c_n + L_d$ where

$0 \leqq c_n < d$. The version of the local limit theorem that we will use is the following one; $\xi(x)$ is the standard normal density.

If $\bar{F}$ is lattice with span d and f is directly integrable with respect to ℓ_d then

$$\sup_{y \in L_d} \left| \sigma \sqrt{n} \int f(x+y-c_n) \bar{F}^{n*}(dx) - \int f(x) \xi\left(\frac{x-y+c_n}{\sigma\sqrt{n}}\right) \ell_d(dx) \right| = A_n \to 0 \qquad (2.4)$$

as $n \to \infty$.

This result follows easily from the results of [10] (Sect. 49), which covers the case $d>0$, and [15] (Lemma 2), which covers the case $d=0$.

This local limit theorem together with the formula (2.3) yield the following theorem. We assume from now on that F has span $d \geqq 0$.

Theorem A. *If $e^{\phi x} g(x)$ is directly integrable with respect to ℓ_d then*

$$\sup_{y \in L_d} \left| \frac{\sigma \sqrt{n} e^{\phi(y-c_n)}}{e^{n\zeta(b)}} \int g(x+y-c_n-nb) F^{n*}(dx) - \int e^{\phi x} g(x) \xi\left(\frac{x-y+c_n}{\sigma\sqrt{n}}\right) \ell_d(dx) \right| \to 0$$

as $n \to \infty$.

Let us write B_n for $(\sigma\sqrt{2\pi n}\, e^{-n\zeta(b)} e^{-\phi c_n})$ then the following corollary follows easily from Theorem A.

Corollary I. $B_n \int g(x-c_n-nb) F^{n*}(dx) \to \int e^{\phi x} g(x) \ell_d(dx)$.

If we assume that $\phi<0$ and let g be the indicator function of $[0, \infty)$ then, since F^{n*} has all of its points of increase in c_n+nb+L_d

$$\int g(x-c_n-nb) F^{n*}(dx) = F^{n*}([nb+c_n, \infty)) = F^{n*}([nb, \infty))$$

and so

$$B_n F^{n*}([nb, \infty)) \to \begin{cases} -\phi^{-1} & \text{if } d=0 \\ d(1-e^{\phi d})^{-1} & \text{if } d>0. \end{cases}$$

This is Theorem I of [3].

Similarly if $d>0$ and $y \in L_d$ then

$$B_n F^{n*}(\{y+c_n+nb\}) \to d e^{\phi y}$$

whilst if $d=0$ and $I=[y, x+y)$ then

$$B_n F^{n*}(nb+I) \to \phi^{-1} e^{\phi y}(e^{\phi x}-1).$$

We can now state the main result of this paper, which is an analogue for $Z^{(n)}$ of Corollary I. In all that follows $e^{\phi x} g(x)$ is assumed to be directly integrable with respect to ℓ_d.

Theorem B. *If*

$$\zeta(b)>0 \tag{2.5}$$

and for some $\varepsilon>0$

$$\mathscr{E}[W^{(1)}(\phi)(\log_+ W^{(1)}(\phi))^{\frac{d}{2}+\varepsilon}]<\infty \tag{2.6}$$

then

$$B_n \int g(x-c_n-nb)\, Z^{(n)}(dx) \to W(\phi) \int e^{\phi x} g(x)\, \ell_d(dx) \quad \text{a.s.} \tag{2.7}$$

as $n\to\infty$.

It can be seen from the discussion just before Theorem A in [4] that there is an open interval (Θ_1,Θ_2) such that $\zeta(b)=\zeta\left(-\frac{m'(\phi)}{m(\phi)}\right)>0$ if and only if $\phi\in(\Theta_1,\Theta_2)$. Also, by Theorem A of that paper $\mathscr{E}[W(\phi)]=1$ if and only if

$$\phi\in(\Theta_1,\Theta_2) \quad \text{and} \quad \mathscr{E}[W^{(1)}(\phi)\log_+ W^{(1)}(\phi)]<\infty.$$

Hence the conditions of this theorem suffice to ensure that $\mathscr{E}[W(\phi)]=1$. (When this holds it is easy to check, by letting $u\to\infty$ in (2.1) of [4] that if the process survives $W(\phi)>0$ a.s.)

The condition that $\zeta(b)>0$ ensures that $B_n\to 0$; otherwise $B_n\to\infty$. Since $Z^{(n)}(t)$ is an integer valued funtion its similarity with $F^{n*}(t)$ will be less marked when $F^{n*}(t)$ is small. To illustrate this if $\phi>0$ and $\zeta(b)<0$ then, as is shown in [5], $Z^{(n)}(nb)=0$ for all but finitely many n.

As for F we can deduce that (2.7) implies that

$$B_n Z^{(n)}(\{nb+y+c_n\}) \to W(\phi)\, e^{\phi y} d \quad \text{a.s.}$$

when $d>0$ and $y\in L_d$, and that a.s.

$$B_n Z^{(n)}(nb+I) \to \phi^{-1} e^{\phi y}(e^{\phi x}-1)\, W(\phi) \quad \text{a.s.}$$

when $d=0$ and $I=[y,x+y)$.

If we take $\phi=0$ then $b=-m'(0)/m(0)$ and is the 'centre' of the first generation. In this case Theorem B is very similar to Theorem 2 of [2] and the proof of Theorem B given here follows theirs quite closely. In the next section we will establish a decomposition of the left hand side in (2.7) which will form the basis of the proof.

3. The Basic Decomposition

We will assume from now on that $b=0$, so that

$$m'(\phi)=0 \quad \text{and} \quad \zeta(b)=\zeta(0)=\log m(\phi).$$

Then the Condition (2.5) is equivalent to

$$m(\phi)>1.$$

This assumption does not involve any loss in generality. To see this suppose that $b \neq 0$. From the original branching random walk a new one can be constructed with the nth generation people at $\{z_r^{(n)} - nb\}$. Quantities in this new branching random walk are denoted by a subscript, T; then $Z_T^{(n)}(t) = Z^{(n)}(t+nb)$, $W_T^{(n)}(\theta) = W^{(n)}(\theta)$, and $\zeta_T(a-b) = \zeta(a)$. Hence Theorem B holds for $Z^{(n)}$ if it holds for $Z_T^{(n)}$ with $b=0$.

Let us now define the measure $\bar{Z}^{(n)}$ by

$$\bar{Z}^{(n)}(dx) = \frac{e^{-\phi x}}{m(\phi)^n} Z^{(n)}(dx);$$

thus for any interval I

$$\bar{Z}^{(n)}(I) = \sum \frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n} \tag{3.1}$$

where the summation is taken over those r with $z_r^{(n)} \in I$; in particular

$$\bar{Z}^{(n)}(\mathbb{R}) = W^{(n)}(\phi).$$

This transformation of $Z^{(n)}$ is of course like that given in (2.3) for F^{n*}, and by taking expectations in (3.1) we see that

$$\mathscr{E}[\bar{Z}^{(n)}(I)] = \bar{F}^{n*}(I)$$

and so

$$\mathscr{E}\left[\sum_r f(z_r^{(n)}) \frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n}\right] = \mathscr{E}[\int f(x)\, \bar{Z}^{(n)}(dx)]$$
$$= \int f(x)\, \bar{F}^{n*}(dx). \tag{3.2}$$

As with F and $\bar{F}$ properties of $Z^{(n)}$ can be deduced from those of $\bar{Z}^{(n)}$. We can now rewrite (2.7) as

$$\sigma\sqrt{2\pi n} \int f(x-c_n)\, \bar{Z}^{(n)}(dx) \to W(\phi) \int f(x)\, \ell_d(dx) \quad \text{a.s.} \tag{3.3}$$

where $f(x) = e^{\phi x} g(x)$, and this is what we will prove.

Let $\bar{Z}^{(n)}(dx; z_r^{(s)})$ be defined on the branching random walk which has $z_r^{(s)}$ as its initial ancestor in the same way that $\bar{Z}^{(n)}(dx)$ is defined on the original branching random walk; thus

$$\bar{Z}^{(n)}(I; z_r^{(s)}) = \sum \frac{\exp(-\phi(z_q^{(n+s)} - z_r^{(s)}))}{m(\phi)^n}$$

where the summation is taken over those q for which $z_q^{(n+s)}$ is a descendant of $z_r^{(s)}$ and $(z_q^{(n+s)} - z_r^{(s)}) \in I$. Given $\mathfrak{F}^{(s)}$, the σ-field generated by the first s generations, $\{\bar{Z}^{(n)}(I; z_r^{(s)}) : r\}$ are independent copies of $\bar{Z}^{(n)}(I)$. Now

$$\int h(x)\, \bar{Z}^{(n)}(dx) = \sum_q h(z_q^{(n)}) \frac{\exp(-\phi z_q^{(n)})}{m(\phi)^n}$$
$$= \sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s} \int h(x + z_r^{(s)})\, \bar{Z}^{(n-s)}(dx : z_r^{(s)}) \tag{3.4}$$

and it is upon this decomposition that the proof is based. The basic idea, which occurs in a number of recent papers on branching processes, is that the right side of this equality is close to its expected value conditional on $\mathfrak{F}^{(s)}$. We will be considering

$$\sqrt{n}\int f(x-c_n)\,\bar{Z}^{(n)}(dx)-\sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}\sqrt{n}\int f(x+z_r^{(s)}-c_n)\,\bar{F}^{(n-s)*}(dx). \tag{3.5}$$

which can be rewritten as

$$\sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}\sqrt{n}\,\{\int f(x+z_r^{(s)}-c_n)(\bar{Z}^{(n-s)}(dx\colon z_r^{(s)})-\bar{F}^{(n-s)*}(dx))\}. \tag{3.6}$$

We will show that as n tends to infinity, with s related to n in an appropriate way, this tends to zero almost surely. Then, when $(n-s)$ is large we can use (2.4) to estimate the second term in (3.5).

To explain how s and n are related let v and η be two positive integers with $v<\eta$ (later we will require that $\eta-v>2$ and that η/v should be near one). Then for any n satisfying

$$j^\eta \leqq n<(j+1)^\eta \tag{3.7}$$

where j is a positive integer, let

$$s=j^v; \tag{3.8}$$

thus $s=((n^{1/\eta})_\wedge)^v$ and so s grows like $n^{v/\eta}$ but is confined to a subsequence of the positive integers. If K is sufficiently large then

$$\#\{n\colon s=j^v\}\leqq K j^{\eta-1}, \tag{3.9}$$

an estimate which will be useful.

The expression (3.6) can be written as

$$\sum_r a_r(s)\sqrt{n}\{Y_r\} \tag{3.10}$$

where given $\mathfrak{F}^{(s)}$ $\{a_r(s)\}$ are constants and Y_r are independent random variables with zero mean. The variables $\{Y_r\}$ do not all have the same distribution and so we will need a bound on their tail behaviour. For large K $|f(x)|\leqq K$ and then

$$|Y_r|\leqq K(\bar{Z}^{(n-s)}(\mathbb{R}\colon z_r^{(s)})+1)$$

and $\bar{Z}^{(n-s)}(\mathbb{R}\colon z_r^{(s)})$ has the same distribution as $W^{(s)}(\phi)$. Let

$$G(t)=\mathscr{P}[(\max\{K,1\})(1+\sup\{W^{(n)}(\phi)\colon n\})\leqq t]$$

then for each r

$$\mathscr{P}[|Y_r|\leqq t]\geqq G(t) \qquad \text{for all } t \tag{3.11}$$

(no matter what values n and s take). Notice that $G(t)=0$ for $t<1$. In proving the main result we will need to impose a moment condition on $G(t)$, that is on

$\sup\{W^{(n)}(\phi)\colon n\}$. The relationship between moment conditions on $W^{(1)}(\phi)$ and on $\sup\{W^{(n)}(\phi)\colon n\}$ is the subject of the next section.

4. Relationships between Moments

We will need to impose the condition that

$$\mathscr{E}[\sup\{W^{(n)}(\phi)\colon n\}\log_+^{\delta}(\sup\{W^{(n)}(\phi)\colon n\})]<\infty \tag{4.1}$$

for some $\delta>\frac{3}{2}$. In the first part of this section we will show that when $\mathscr{E}[W(\phi)]=1$ then $\sup\{W^{(n)}(\phi)\colon n\}$ and $W(\phi)$ have similar tail behaviour so that (4.1) is equivalent to

$$\mathscr{E}[W(\phi)\log_+^{\delta}(W(\phi))]<\infty \tag{4.2}$$

for any $\delta>0$. The result on the similarity of the tail behaviour of $\sup\{W^{(n)}(\phi)\colon n\}$ and $W(\phi)$ is a generalization of a known result for the Galton-Watson process, (i.e. for the case $\phi=0$) which was proved in [11] (Lemma 3.1). In the second part of this section it is shown that (4.2) holds for $\delta>0$ when

$$\mathscr{E}[W^{(1)}(\phi)\log_+^{1+\delta}(W^{(1)}(\phi))]<\infty.$$

Here the technique used is a generalization of one used in [1] when $\phi=0$. Related results have been obtained, using different methods, in [6] and [7].

Let X_i, $i=1,2,\ldots$, be independent identically distributed random variables with zero mean and let P be the set of all probability distributions on the positive integers.

Lemma 1. *For any* $\varepsilon>0$

$$\inf_{\{p_i\}\in P}\mathscr{P}[\textstyle\sum p_i X_i>-\varepsilon]>0.$$

Proof. Let

$$X_i^T=\begin{cases}X_i & \text{if } X_i<T\\ T & \text{if } X_i\geqq T,\end{cases}$$

then $\mathscr{P}[\sum p_i X_i^T>-\varepsilon]\leqq\mathscr{P}[\sum p_i X_i>-\varepsilon]$. Let T be sufficiently large to ensure that

$$\mathscr{E}[\textstyle\sum p_i X_i^T]=\mathscr{E}[X_1^T]>\frac{1}{2}\varepsilon. \tag{4.3}$$

Now suppose that for some $\{p_i\}\in P$

$$\mathscr{P}[\textstyle\sum p_i X_i^T>-\varepsilon]<\dfrac{1}{2}\dfrac{\varepsilon}{\varepsilon+T}$$

which implies that

$$\mathscr{E}[\textstyle\sum p_i X_i^T]\leqq(-\varepsilon)\left(\dfrac{1}{2}\dfrac{\varepsilon}{\varepsilon+T}\right)+(T)\left(\dfrac{1}{2}\dfrac{\varepsilon}{\varepsilon+T}\right)\leqq-\dfrac{1}{2}\varepsilon$$

contradicting (4.3). Therefore

$$\inf_{\{p_i\}\in P} \mathscr{P}[\sum p_i X_i > -\varepsilon] \geqq \frac{1}{2}\frac{\varepsilon}{\varepsilon + T} > 0$$

proving the lemma.

Let

$$W^{(n)}(\phi: z_r^{(s)}) = \bar{Z}^{(n)}(\mathbb{R}: z_r^{(s)})$$

and let $W(\phi: z_r^{(s)})$ be the limit of this martingale (n and s are not related in any way in this section). If $h(x) \equiv 1$ in (3.4) we can see that

$$W^{(n)}(\phi) = \sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s} W^{(n-s)}(\phi: z_r^{(s)}) \tag{4.4}$$

and letting $n \to \infty$ gives

$$W(\phi) = \sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s} W(\phi: z_r^{(s)}) \tag{4.5}$$

where, given $\mathfrak{F}^{(s)}$, $\{W(\phi: z_r^{(s)})\}$ are independent copies of $W(\phi)$.

If $\mathscr{E}[W(\phi)] = 1$ and $0 < a < 1$ then for some $B > 0$ the following lemma holds whenever $t > 1$.

Lemma 2. $\mathscr{P}[W(\phi) \geqq at] \geqq B\mathscr{P}[\sup_n W^{(n)}(\phi) \geqq t] \geqq B\mathscr{P}[W(\phi) \geqq t]$.

Proof. Only the first inequality requires proof. Let

$$E_n = \{W^{(n)}(\phi) \geqq t,\ W^{(s)}(\phi) < t \text{ for } 0 \leqq s < n\}$$

then

$$\mathscr{P}[W(\phi) > at] \geqq \sum_n \mathscr{P}[W(\phi) > at \mid E_n]\,\mathscr{P}[E_n],$$

but, using (4.5)

$$\begin{aligned}
\mathscr{P}[W(\phi) > at \mid E_n] &= \mathscr{P}\left[\sum_r \frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n} W(\phi: z_r^{(n)}) > at \mid E_n\right] \\
&= \mathscr{P}\left[\sum_r \frac{\exp(-\phi z_r^{(n)})}{W^{(n)}(\phi)\, m(\phi)^n} (W(\phi: z_r^{(n)}) - 1) > \frac{at}{W^{(n)}(\phi)} - 1 \mid E_n\right] \\
&\geqq \mathscr{P}\left[\sum_r \frac{\exp(-\phi z_r^{(n)})}{W^{(n)}(\phi)\, m(\phi)^n} (W(\phi: z_r^{(n)}) - 1) > a - 1 \mid E_n\right]
\end{aligned}$$

because $W^{(n)}(\phi) \geqq t$ on E_n. Now, given $\mathfrak{F}^{(n)}$,

$$\mathscr{P}\left[\sum_r \frac{\exp(-\phi z_r^{(n)})}{W^{(n)}(\phi)\, m(\phi)^n} (W(\phi): z_r^{(n)}) - 1) > a - 1\right] \geqq B > 0$$

by Lemma 1, and B is independent of n. Since $E_n \in \mathfrak{F}^{(n)}$ we deduce that

$$\mathscr{P}[W(\phi) > at] \geqq B \sum_n \mathscr{P}[E_n]$$
$$= B\mathscr{P}[\sup_n W^{(n)}(\phi) \geqq t],$$

proving the lemma.

The equivalence of (4.1) and (4.2) for any $\delta > 0$ is an immediate consequence of this lemma.

Let $h: \mathbb{R}^+ \to \mathbb{R}^+$ be a concave (and hence subadditive) function. The following lemma provides an estimate of $\mathscr{E}[W(\phi)\,h(W(\phi))]$ in terms of certain moments of $W^{(1)}(\phi)$.

Lemma 3. $\mathscr{E}[W(\phi)\,h(W(\phi))] \leqq \sum_{n=0}^{\infty} \mathscr{E}\left[W^{(1)}(\phi) \int h\left(\frac{W^{(1)}(\phi)\,e^{-\phi x}}{m(\phi)^n}\right) \bar{F}^{n*}(dx)\right]$.

Proof. Using (4.4) we can rewrite $\mathscr{E}[W^{(n+1)}(\phi)\,h(W^{(n+1)}(\phi)) | \mathfrak{F}^{(n)}]$ as

$$\mathscr{E}\left[\sum_r \frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n} W^{(1)}(\phi\colon z_r^{(n)})\, h\left(\sum_s \frac{\exp(-\phi z_s^{(n)})}{m(\phi)^n} W^{(1)}(\phi\colon z_s^{(n)})\right) \middle| \mathfrak{F}^{(n)}\right]$$
$$\leqq \mathscr{E}\left[\sum_r \frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n} W^{(1)}(\phi\colon z_r^{(n)}) \left\{ h\left(\sum_{s \neq r} \frac{\exp(-\phi z_s^{(n)})}{m(\phi)^n} W^{(1)}(\phi\colon z_s^{(n)})\right)\right.\right.$$
$$\left.\left. + h\left(\frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n} W^{(1)}(\phi\colon z_r^{(n)})\right)\right\} \middle| \mathfrak{F}^{(n)}\right]$$
$$\leqq W^{(n)}(\phi)\,h(W^{(n)}(\phi)) + \mathscr{E}\left[\sum_r \frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n} X\, h\left(\frac{\exp(-\phi z_r^{(n)})}{m(\phi)^n} X\right)\right]$$

where X has the same distribution as $W^{(1)}(\phi)$ and is independent of the whole process. If we take expectations of this inequality we get

$$\mathscr{E}[W^{(n+1)}(\phi)\,h(W^{(n+1)}(\phi))] \leqq \mathscr{E}[W^{(n)}(\phi)\,h(W^{(n)}(\phi))]$$
$$+ \mathscr{E}\left[X \int h\left(\frac{e^{-\phi x} X}{m(\phi)^n}\right) \bar{F}^{n*}(dx)\right].$$

By Fatou's lemma $\mathscr{E}[W(\phi)\,h(W(\phi))] \leqq \liminf \mathscr{E}[W^{(n)}(\phi)\,h(W^{(n)}(\phi))]$ and so

$$\mathscr{E}[W(\phi)\,h(W(\phi))] \leqq \sum_{n=0}^{\infty} \mathscr{E}\left[X \int h\left(\frac{e^{-\phi x} X}{m(\phi)^n}\right) \bar{F}^{n*}(dx)\right],$$

proving the lemma.

(Although we shall not use the result is seems worth noting that when $0 < \delta < 1$ we can show that $\mathscr{E}[W(\phi)^{1+\delta}] < \infty$ when $\mathscr{E}[W^{(1)}(\phi)^{1+\delta}] < \infty$ and $m((1+\delta)\,\phi)/m(\phi)^{1+\delta} < 1$ by taking $h(x) = x^\delta$. Also it is easy to show using (4.5) that when $\mathscr{E}[W(\phi)] > 0$ and $Z^{(1)}$ is non-degenerate these conditions are also necessary to guarantee that $\mathscr{E}[W(\phi)^{1+\delta}] < \infty$.)

We will now take h to be defined by

$$h(x) = \begin{cases} c_0 x & \text{for } x < x_0 \\ c_1 + c_2 \log_+^\delta x & \text{for } x \geqq x_0 > 1, \end{cases} \tag{4.6}$$

where $\delta>0$ and x_0, c_0, c_1 and c_2 are chosen to make h concave. Then

$$h(xy) \leqq K(1+\log_+^\delta x+\log_+^\delta y)$$

when K is large enough. (In the remainder of this paper inequalities involving K hold whenever K is large enough, K does not necessarily represent the same constant in different inequalities.)

Since $m(\phi)>1$, for some $w>0$ and $\beta>1$

$$\inf\{e^{\phi x} m(\phi):\ -w \leqq x \leqq w\} \geqq \beta$$

and so, letting $I_n=(-\infty, -nw)\cup(nw, \infty)$

$$\begin{aligned}\int h\left(\frac{xe^{-\phi y}}{m(\phi)^n}\right)\bar{F}^{n*}(dy) &\leqq h(x\beta^{-n})+\int_{I_n} h\left(\frac{xe^{-\phi y}}{m(\phi)^n}\right)\bar{F}^{n*}(dy)\\ &\leqq h(x\beta^{-n})+\int_{I_n} K\left(1+\log_+^\delta x+\log_+^\delta\left(\frac{e^{-\phi y}}{m(\phi)^n}\right)\right)\bar{F}^{n*}(dy)\\ &\leqq h(x\beta^{-n})+\int_{I_n} K(1+\log_+^\delta x+|\phi y|^\delta)\bar{F}^{n*}(dy). \qquad (4.7)\end{aligned}$$

To apply the preceding lemma we must now consider the sum, over n, of this inequality.

Let $N=((\log\beta)^{-1}\log(x/x_0))_\wedge$ then using (4.6) it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} h(x\beta^{-n}) &= \sum_{n=0}^{N} h(x\beta^{-n})+\sum_{n=N+1}^{\infty} h(x\beta^{-n})\\ &\leqq h(x)N+\sum_{n=N+1}^{\infty} c_0 x\beta^{-n}\\ &\leqq h(x)\frac{\log(x/x_0)}{\log\beta}+\frac{c_0x_0}{\beta-1}\leqq K(1+\log_+^{\delta+1} x). \qquad (4.8)\end{aligned}$$

To deal with the integral term in (4.7), and for similar calculations later, we will need bounds on the tail behaviour of $\bar{F}^{n*}$. The following lemma contains the necessary information.

Lemma 4.

(i) $\bar{F}^{n*}(-y) \leqq \dfrac{e^{-\theta y} m(\theta+\phi)^n}{m(\phi)^n}$ *for* $\theta>0$.

(ii) $\bar{F}^{n*}(I_n) \leqq c^n$ *for some* $c<1$.

Proof. From (2.1)

$$\int e^{-\theta y}\bar{F}^{n*}(dy)=\left(\frac{m(\theta+\phi)}{m(\phi)}\right)^n$$

and (i) follows from this. Let

$$\bar{\zeta}(a)=\inf\left\{\theta a+\log\frac{m(\theta+\phi)}{m(\phi)}:\ \theta\right\},$$

then, since $\bar{F}$ has been assumed to be non-degenerate, $\bar{\zeta}(a)<0$ for all non-zero a. From (i) it follows that

$$\bar{F}^{n*}(-nw)\leqq\exp(n\bar{\zeta}(-w))$$

and that

$$1-\bar{F}^{n*}(nw)\leqq\exp(n\bar{\zeta}(w)),$$

and (ii) follows from these inequalities.

For some $c<1$ it is possible to choose $\eta>0$ such that

$$e^{-\eta w}\frac{m(\phi+\eta)}{m(\phi)}\leqq c;$$

this together with (i) of the preceding lemma justifies the following calculation;

$$\begin{aligned}\sum_{n=0}^{\infty}\int_{-\infty}^{-nw}|y|^{\delta}\bar{F}^{n*}(dy)&=\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}\int_{-(r+1)w}^{-rw}|y|^{\delta}\bar{F}^{n*}(dy)\\&\leqq\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}(r+1)^{\delta}w^{\delta}\bar{F}^{n*}(-rw)\\&\leqq\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}(r+1)^{\delta}w^{\delta}e^{-rw\eta}\left(\frac{m(\phi+\eta)}{m(\phi)}\right)^{n}\\&=\sum_{r=0}^{\infty}(r+1)^{\delta}w^{\delta}e^{-rw\eta}\sum_{n=0}^{r}\left(\frac{m(\phi+\eta)}{m(\phi)}\right)^{n}\\&\leqq K\sum_{r=0}^{\infty}(r+1)^{\delta}w^{\delta}e^{-rw\eta}\left(\frac{m(\phi+\eta)}{m(\phi)}\right)^{r}\\&\leqq K\sum_{r=0}^{\infty}(r+1)^{\delta}c^{r}<\infty.\end{aligned}$$

In a similar way

$$\sum_{n=0}^{\infty}\int_{nw}^{\infty}y^{\delta}\bar{F}^{n*}(dy)\quad\text{and}\quad\sum_{n=0}^{\infty}\int_{I_n}\bar{F}^{n*}(dy)$$

can be shown to be finite. Combining these estimates with (4.7) and (4.8) we see that

$$\sum_{n=0}^{\infty}\int h\left(x\frac{e^{-\phi y}}{m(\phi)^{n}}\right)\bar{F}^{n*}(dy)\leqq K(1+\log_{+}^{1+\delta}x),$$

and so by Lemma 3 $\mathscr{E}[W(\phi)\log_{+}^{\delta}W(\phi)]$ is finite whenever

$$\mathscr{E}[W^{(1)}(\phi)\log_{+}^{1+\delta}W^{(1)}(\phi)]$$

is finite (for $\delta>0$).

5. The Proof of Theorem B

The first part of the remainder of the proof consists of showing that (3.6) tends to zero almost surely as n tends to infinity. Using the notation for (3.6) introduced at (3.10) let

$$\tilde{Y}_r = \begin{cases} Y_r & \text{if } |Y_r| < a_r(s)^{-1} \\ 0 & \text{if } |Y_r| \geqq a_r(s)^{-1}; \end{cases}$$

in what follows a_r is often used for $a_r(s)$, suppressing the dependence on s. Now

$$\sqrt{n}\sum_r a_r Y_r = \sqrt{n}\sum_r a_r \mathscr{E}[\tilde{Y}_r] + \sqrt{n}\sum_r a_r(Y_r - \tilde{Y}_r) + \sqrt{n}\sum_r a_r(\tilde{Y}_r - \mathscr{E}[\tilde{Y}_r]),$$

and we will show that each of the three terms on the right of this identity tends to zero as n tends to infinity.

1) It follows from (3.11), the fact that the Y_r's have zero mean and the definitions (3.7) and (3.8) that

$$\begin{aligned} |\sqrt{n}\sum a_r \mathscr{E}[\tilde{Y}_r]| &\leqq \sqrt{n}\sum a_r \int_{a_r^{-1}}^{\infty} t\,G(dt) \\ &< (j+1)^{\eta/2} \sum a_r \int_{a_r^{-1}}^{\infty} t\,G(dt), \end{aligned} \tag{5.1}$$

which only changes when j changes. Let $R = \{r\colon a_r(s) > e^{-ms}\}$ where $0 < m < \log m(\phi)$ then

$$(j+1)^{\eta/2}\sum a_r \int_{a_r^{-1}}^{\infty} t\,G(dt) \leqq (j+1)^{\eta/2}\left\{(\sum_R a_r)\int_0^{\infty} tG(dt) + (\sum a_r)\int_{e^{ms}}^{\infty} tG(dt)\right\}.$$

However, taking $w = |\phi|^{-1}(\log m(\phi) - m)$

$$\begin{aligned} \mathscr{E}[\sum_R a_r] &= \mathscr{E}\left[\sum_R \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}\right]. \\ &\leqq \bar{F}^{s*}(I_s) \end{aligned}$$

and so by Lemma 4(ii)

$$\begin{aligned} \mathscr{E}[\sum_j (j+1)^{\eta/2} \sum_R a_r(s)] &\leqq \sum_j (j+1)^{\eta/2} \bar{F}^{s*}(I_s) \\ &< \sum_j (j+1)^{\eta/2} c^{j^\nu} < \infty. \end{aligned}$$

Furthermore

$$\begin{aligned} (j+1)^{\eta/2}(\sum a_r)\int_{e^{ms}}^{\infty} t\,G(dt) &\leqq K\,s^{\eta/2\nu}(\sum a_r)\int_{e^{ms}}^{\infty} t\,G(dt) \\ \leqq K(\sum a_r)\int_{e^{ms}}^{\infty} t(\log t)^{\eta/2\nu}\,G\,dt) &= K\,W^{(s)}(\phi)\int_{e^{ms}}^{\infty} t(\log t)^{\eta/2\nu}\,G(dt) \\ \to 0 \quad \text{as} \quad j \to \infty \end{aligned}$$

if η/v is less than three for then by the results of the preceding Sect. (2.6) implies that

$$\int_0^\infty t(\log t)^{\frac{3}{2}+\varepsilon} G(dt) < \infty.$$

Therefore, using (5.1),

$$\sqrt{n} \sum_r a_r \mathscr{E}[\tilde{Y}_i] \to 0 \quad \text{a.s.}$$

as $n \to \infty$.

2) It follows from (3.11) and the definition of $\tilde{Y}_r$ that

$$\mathscr{P}[\sum a_r(Y_r - \tilde{Y}_r) \neq 0 \mid \mathfrak{F}^{(s)}] \leqq \sum \int_{a_r^{-1}}^\infty G(dt)$$

and so, using the fact that $a_r = m(\phi)^{-s} \exp(-\phi z_r^{(s)})$ and (3.2),

$$\begin{aligned}\mathscr{P}[\sum a_r(Y_r - \tilde{Y}_r) \neq 0] &\leqq \mathscr{E}\left[\sum \int_{a_r^{-1}}^\infty G(dt)\right] \\ &= \int_{-\infty}^\infty e^{\phi y} m(\phi)^s \left(\int_{e^{\phi y} m(\phi)^s}^\infty G(dt)\right) \bar{F}^{s*}(dy).\end{aligned} \tag{5.2}$$

It is convenient to introduce a simple transformation of $\bar{F}$ defined by

$$\tilde{F}(\phi\, d y + \log m(\phi)) = \bar{F}(d y)$$

($\tilde{F}$ is degenerate at $\log m(0)$ when $\phi = 0$); $\tilde{F}$ has mean $\log m(\phi)$ and variance $\phi^2 \sigma^2$. Now (5.2) becomes

$$\begin{aligned}\mathscr{P}[\sum a_r(Y_r - \tilde{Y}_r) \neq 0] &\leqq \int_{-\infty}^\infty e^y \int_{e^y}^\infty G(dt) \bar{F}^{s*}(d y) \\ &= \int_1^\infty \left(\int_{-\infty}^{\log t} e^y \tilde{F}^{s*}(dy)\right) G(dt) \leqq \int_1^\infty t \left(\int_{-\infty}^{\log t} \tilde{F}^{s*}(d y)\right) G(dt).\end{aligned}$$

Therefore, using (3.9),

$$\sum_n \mathscr{P}[\sum a_r(Y_r - \tilde{Y}_r) \neq 0] \leqq \int_1^\infty t \left(\sum_{j=1}^\infty K j^{\eta-1} \int_{-\infty}^{\log t} \tilde{F}^{s*}(dy)\right) G(dt),$$

where $s = j^v$. Let $0 < m < \log m(\phi)$ then if $w = |\phi|^{-1}(\log m(\phi) - m)$

$$\tilde{F}^{s*}(sm) \leqq \bar{F}^{s*}(I_s).$$

Let $J = ((m^{-1} \log t)^{1/v})_\wedge$ then, using Lemma 4(ii)

$$\sum_{j=1}^{\infty} j^{\eta-1}\tilde{F}^{s*}(\log t) \leqq \sum_{j=1}^{\infty} j^{\eta-1}\tilde{F}^{s*}(sm) + \sum_{j=1}^{J} j^{\eta-1}\tilde{F}^{s*}(\log t)$$
$$\leqq \sum_{j=1}^{\infty} j^{\eta-1} c^{j^{\nu}} + \sum_{j=1}^{J} j^{\eta-1}$$
$$\leqq K(1+J^{\eta}) \leqq K(1+(\log t)^{\eta/\nu}).$$

Therefore, if η/ν is less than 3/2

$$\sum_n \mathscr{P}[\sum a_r(Y_r - \tilde{Y}_r) \neq 0] \leqq K \int_1^{\infty} t(\log t)^{\eta/\nu} G(dt) < \infty,$$

when (2.6) holds, and so

$$\sqrt{n}\sum a_r(Y_r - \tilde{Y}_r) = 0$$

for all but finitely many n.

3) To complete the first part of the proof we will need the following inequality which is based on the ideas in [13]; it holds for $1<\alpha<2$,

$$\mathscr{P}[|\sum a_r(\tilde{Y}_r - \mathscr{E}[\tilde{Y}_r])| > v \mid \mathfrak{F}^{(s)}] \leqq \frac{K}{v^{\alpha}}\left\{\sum a_r^{\alpha}\int_0^{a_r^{-1}} t^{\alpha} G(dt) + \sum \int_{a_r^{-1}}^{\infty} G(dt)\right\}. \tag{5.3}$$

To prove this let $\psi_r(u)$ be the characteristic function of $\tilde{Y}_r - \mathscr{E}[\tilde{Y}_r]$ then, using Proposition 8.2.9 in [8],

$$\mathscr{P}[|\sum a_r(\tilde{Y}_r - \mathscr{E}[\tilde{Y}_r])| > v \mid \mathfrak{F}^{(s)}] \leqq K v \int_0^{v^{-1}} |1 - \prod_r \psi_r(a_r u)|\, du$$
$$\leqq K\sum_r v \int_0^{v^{-1}} a_r^{\alpha} u^{\alpha}\left|\frac{1-\psi_r(a_r u)}{a_r^{\alpha} u^{\alpha}}\right| du \leqq K v \sum_r \int_0^{v^{-1}} a_r^{\alpha} u^{\alpha}\mathscr{E}[|\tilde{Y}_r - \mathscr{E}[\tilde{Y}_r]|^{\alpha}]\, du$$
$$\leqq K v^{-\alpha}\sum_r a_r^{\alpha}\mathscr{E}[|\tilde{Y}_r - \mathscr{E}[\tilde{Y}_r]|^{\alpha}] \leqq K v^{-\alpha}\sum_r a_r^{\alpha}\mathscr{E}[|\tilde{Y}_r|^{\alpha}],$$

and, from the definition of $\tilde{Y}_r$ and (3.11) it can be seen that

$$\mathscr{E}[|\tilde{Y}_r|^{\alpha}] \leqq \int_1^{a_r^{-1}} t^{\alpha} G(dt) + a_r^{-\alpha}\int_{a_r^{-1}}^{\infty} G(dt),$$

proving (5.3.).

By taking the expected value of (5.3) we deduce that

$$\mathscr{P}[|\sqrt{n}\sum a_r(\tilde{Y}_r - \mathscr{E}[\tilde{Y}_r])| > v]$$
$$\leqq \frac{K n^{\alpha/2}}{v^{\alpha}}\int_{-\infty}^{\infty}\left\{e^{-y(\alpha-1)}\int_1^{e^y} t^{\alpha} G(dt) + e^y \int_{e^y}^{\infty} G(dt)\right\}\tilde{F}^{s*}(dy)$$
$$= \frac{K}{v^{\alpha}}\left\{\int_1^{\infty}\left(t^{\alpha}\int_{\log t}^{\infty} n^{\alpha/2} e^{-y(\alpha-1)}\tilde{F}^{s*}(dy) + \int_{-\infty}^{\log t} n^{\alpha/2} e^y \tilde{F}^{s*}(dy)\right) G(dt)\right\}.$$

We will need to sum this inequality over n.

If J and m are defined as in part (2) of this section then

$$\sum_{n=1}^{\infty}\int_{\log t}^{\infty} n^{\alpha/2}\, e^{-y(\alpha-1)}\tilde{F}^{s*}(dy) \leqq K\sum_{j=1}^{\infty} j^{\eta\left(1+\frac{\alpha}{2}\right)-1}\int_{\log t}^{\infty} e^{-y(\alpha-1)}\tilde{F}^{s*}(dy)$$

$$\leqq K\left\{\sum_{j=1}^{J} j^{\eta\left(1+\frac{\alpha}{2}\right)-1}\, t^{-\alpha+1}(1-\tilde{F}^{s*}(\log t))\right.$$

$$\left. + \sum_{j=J+1}^{\infty} j^{\eta\left(1+\frac{\alpha}{2}\right)-1}(t^{-\alpha+1}\tilde{F}^{s*}(sm)+e^{-sm(\alpha-1)})\right\}$$

$$\leqq K\left\{t^{-\alpha+1} J^{\eta\left(1+\frac{\alpha}{2}\right)} + t^{-\alpha+1}\sum_{j=1}^{\infty} j^{\eta\left(1+\frac{\alpha}{2}\right)-1} c^{j^{\nu}} + \sum_{j=J}^{\infty} j^{\eta\left(1+\frac{\alpha}{2}\right)-1} e^{-j^{\nu} m(\alpha-1)}\right\}$$

$$\leqq K\{t^{-\alpha+1} J^{\eta\left(1+\frac{\alpha}{2}\right)} + t^{-\alpha+1} + J^{\eta\left(1+\frac{\alpha}{2}\right)-1} e^{-J^{\nu} m(\alpha-1)}\}$$

$$\leqq K\{t^{-\alpha+1}(\log t)^{\eta\left(1+\frac{\alpha}{2}\right)/\nu} + t^{-\alpha+1}\}$$

and

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\log t} n^{\alpha/2}\, e^{y}\tilde{F}^{s*}(dy) \leqq K t(1+(\log t)^{\eta\left(1+\frac{\alpha}{2}\right)/\nu}).$$

Therefore

$$\sum_{n=1}^{\infty}\mathscr{P}[|\sqrt{n}\sum a_r(\tilde{Y}_r-\mathscr{E}[\tilde{Y}_r])|>v] \leqq \frac{K}{v^{\alpha}}\int_{1}^{\infty} t(\log t)^{\eta\left(1+\frac{\alpha}{2}\right)/\nu} G(dt)$$

which is finite when (2.6) holds, provided that α and η/ν are both sufficiently near one. Hence

$$\sqrt{n}\sum a_r(\tilde{Y}_r-\mathscr{E}[\tilde{Y}_r]) \to 0 \quad \text{a.s.}$$

as $n\to\infty$.

This completes the proof that (3.5) tends to zero almost surely as n tends to infinity.

We must now consider the behaviour of

$$\sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}\sqrt{n}\int f(x+z_r^{(s)}-c_n)\,\bar{F}^{(n-s)*}(dx)$$

as n tends to infinity. All of the points of increase of F^{s*} lie in c_s+L_d and so $z_r^{(s)}\in c_s+L_d$ a.s., therefore

$$z_r^{(s)}+c_{n-s}\in c_{n-s}+c_s+L_d=c_n+L_d.$$

Hence $z_r^{(s)}-c_n+c_{n-s}\in L_d$ and so by (2.4)

$$\left|\sigma\sqrt{n-s}\int f(x+(z_r^{(s)}-c_n+c_{n-s})-c_{n-s})\,\bar{F}^{(n-s)*}(dx)\right.$$
$$\left. -\int f(x)\,\xi\left(\frac{x-z_r^{(s)}+c_n}{\sigma\sqrt{n-s}}\right)\ell_d(dx)\right| \leqq A_{n-s},$$

where $A_{n-s}\to 0$ as $n\to\infty$.

Furthermore

$$\sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s} \int f(x)\left\{\xi\left(\frac{x-z_r^{(s)}+c_n}{\sigma\sqrt{n-s}}\right)-\xi\left(\frac{x}{\sigma\sqrt{n-s}}\right)\right\}\ell_d(dx)$$

$$\leqq K\sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}\frac{\{|z_r^{(s)}|+|c_n|\}}{\sigma\sqrt{n-s}}$$

$$\leqq \frac{K}{\sqrt{n}}\left\{\sum_r |z_r^{(s)}|\frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}+W^{(s)}(\phi)\right\} \quad \text{for } n>1$$

$$\leqq \frac{K}{j^{\eta/2}}\left\{\sum_r |z_r^{(s)}|\frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}+W^{(s)}(\phi)\right\}, \tag{5.4}$$

but

$$\sum_{j=1}^{\infty}\frac{1}{j^{\eta/2}}\mathscr{E}\left[\sum_r |z_r^{(s)}|\frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}\right]=\sum_{j=1}^{\infty}\frac{1}{j^{\eta/2}}\int |x|\,\bar{F}^{s*}(dx)$$

$$\leqq \sigma\sum_{j=1}^{\infty}\frac{s^{\frac{1}{2}}}{j^{\eta/2}}=\sigma\sum_{j=1}^{\infty}\frac{j^{\nu/2}}{j^{\eta/2}}<\infty$$

if $\eta-\nu>2$, which we will assume, therefore (5.4) tends to zero as n tends to infinity. Consequently

$$\left|\sum_r \frac{\exp(-\phi z_r^{(s)})}{m(\phi)^s}\sqrt{n}\int f(x+z_r^{(s)}-c_n)\,\bar{F}^{(n-s)*}(dx)\right.$$
$$\left.-W^{(s)}(\phi)\int f(x)\,\xi\left(\frac{x}{\sigma\sqrt{n-s}}\right)\ell_d(dx)\right|\to 0 \quad \text{a.s.}$$

as $n\to\infty$. Combining this with the fact, proved in the first part of this section, that (3.5) tends to zero, we deduce that

$$|\sigma\sqrt{n}\int f(x-c_n)\,\bar{Z}^{(n)}(dx)-W(\phi)(2\pi)^{-\frac{1}{2}}\int f(x)\,\ell_d(dx)|\to 0 \quad \text{a.s.}$$

as $n\to\infty$, and as was noted at (3.3) this is equivalent to the assertion of Theorem B.

References

1. Asmussen, S.: Some martingale methods in the limit theory of supercritical branching processes. In Advances in Probability and Related Topics (ed. P. Ney). New York: Marcel Dekker 1978
2. Asmussen, S., Kaplan, N.: Branching random walks II. Stochastic Processes Appl. **4**, 15–31 (1976)
3. Bahadur, R.R., Ranga Rao, R.: On deviation of the sample mean. Ann. Math. Statist. **31**, 1015–1027 (1960)
4. Biggins, J.D.: Martingale convergences in the branching random walk. J. Appl. Probability **14**, 25–37 (1977)
5. Biggins, J.D.: Chernoff's theorem in the branching random walk. J. Appl. Probability **14**, 630–636 (1977)

6. Bingham, N.H., Doney, R.A.: Asymptotic properties of supercritical branching processes. I: the Galton-Watson process. Advances in Appl. Probability **6**, 711–731 (1974)
7. Bingham, N.H., Doney, R.A.: Asymptotic properties of supercritical branching processes. II: Crump-Mode and Jirina processes. Advances in Appl. Probability **7**, 66–82 (1975)
8. Breiman, L.: Probability. London: Addison-Wesley 1968
9. Feller, W.: An Introduction to Probability Theory and its Applications. Volume II. New York: Wiley 1971
10. Gnedenko, B.V., Kolmogorov, A.N.: Limit distributions for sums of independent random variables London: Addison-Wesley 1954
11. Kesten, H., Stigum, B.P.: Limit theorems for decomposable multi-dimensional Galton-Watson processes. J. Math. Anal. Appl. **17**, 309–338 (1967)
12. Kingman, J.F.C.: The first birth problem for an age-dependent branching process. Ann. Probability **3**, 790–801 (1975)
13. Kurtz, T.G.: Inequalities for the law of large numbers. Ann. Math. Statist. **43**, 1874–1883 (1972)
14. Matthes, K., Kerstan, J., Mecke, J.: Infinitely divisible point processes. Translated from the German edition. New York: Wiley 1978
15. Stone, C.J.: A local limit theorem for multidimensional distribution functions. Ann. Math. Statist. **36**, 546–551 (1965)

Received June 2 1978; in revised form November 2, 1978