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Stability of P. Lévy's Characterization Theorem

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§ 1. Introduction and the Main Theorems

Stable distributions play an evergrowing role as a natural generalization of the normal law. For the description of them let $X, X_1, X_2, \ldots$ be i.i.d. random variables, $S_n = X_1 + X_2 + \ldots + X_n$ and let F_X be the distribution function of random variable X.

Definition (see [1]). The distribution function F_X is called strictly stable if it is not concentrated in zero and for each n there exists constant $d_n > 0$ such that

$$F_{S_n}(x) = F_{d_n X}(x). \tag{1}$$

It follows (see e.g. Feller [1]) that the constants $d_n = n^{1/\alpha}$ are the only possible normalizing constants and $0 < \alpha \leqq 2$. Thus, formula (1) is rewritten as follows:

$$F_{S_n}(x) = F_{n^{1/\alpha} X}(x). \tag{2}$$

P. Lévy's Theorem. *F_X is strictly stable if* (2) *is realized at $n=2$ and $n=3$.*

It is interesting that, generally speaking, the realization of (2) only for $n=2$ is yet insufficient for the stability of F_X.

P. Lévy's Example (Feller [1], Chap. 17). *The characteristic function $f(t) = \exp\{\psi(t)\}$,*

$$\psi(t) = 2 \sum_{k=-\infty}^{\infty} 2^{-k}(\cos 2^k t - 1)$$

has the property $f(t) = f^2(t/2)$, i.e. $F_{S_2}(x) = F_{2X}(x)$, but it is not strictly stable.

Our aim is to investigate the stability of P. Levy theorem, i.e. having assumed that the assumptions of the theorem

$$f(t) = f^k(t/k^{1/\alpha}), \qquad \alpha \in (0, 2], \; k = 2, 3, \tag{3}$$

$f(t) = \mathbf{E}\exp(itX)$, are valid only approximately (with some error ε) we shall prove that its conclusions are also approximately realized and shall measure this approximation.

Depending on the fact in which domain (finite or infinite) the perturbed version of the relation (3) is considered, two theorems are formulated.

Let us make the following evident remark before the formulation of the theorems. For any characteristic function $f(t)$ and any number $p\in(0,1)$ there exists a positive numbers s (depending, generally speaking, on f and p, finite or infinite) such that

$$s=\inf\{|t|:|f(t)|=p\}. \tag{4}$$

When $|f(t)|>p$, $t\in(-\infty,\infty)$ it is evidently assumed that $s=\infty$.

Theorem 1. *Let for some $\alpha\in(0,2]$ and $T\geqq 1$ the following two conditions be valid ($k=2,3$):*

$$|f(t)-f^k(t/k^{1/\alpha})|\leqq\varepsilon \quad \textit{for } |t|\leqq T. \tag{5}$$

Then there exists constant C depending only on α such that for $|t|\leqq\min(T,\varepsilon^{-\eta/(1+\alpha)})$:

$$|f(t)-\exp\{-|A|\exp(iD\operatorname{sign}t)s_*^{-\alpha}|t|^\alpha\}|\leqq C\tau_p\varepsilon^{\eta/(1+\alpha)} \tag{6}$$

where $A=2\ln f(2^{-1/\alpha}s_*)$, $D=\operatorname{arctg}(\operatorname{Im}A/\operatorname{Re}A)$, $s_*=s$ when $s<T$ and $s_*=1$ when $s\geqq T$,

$$\eta=1/(b+\max(1,\alpha)),\qquad \tau_p=\{p^3\ln(2/(p+1))\}^{-1}$$

and, finally, b is some absolute constant[1], *$b>1$.*

The result of Theorem 1 may be improved in the case $s<T$ in the following way.

Corollary. *Let for some $\alpha\in(0,2]$ and $1<T\leqq\infty$ the conditions (5) be valid for $k=2,3$. If $s<T$ then there exists a constant C depending only on α such that for $|t|\leqq T$*

$$|f(t)-\exp\{-|A_s|\exp(iD_s\operatorname{sign}t)s^{-\alpha}|t|^\alpha\}|\leqq C\tau_p\varepsilon^\eta, \tag{7}$$

where $A_s=2\ln f(2^{-1/\alpha}s)$, $D_s=\operatorname{arctg}(\operatorname{Im}A_s/\operatorname{Re}A_s)$.

If the condition on the perturbed version of Eq. (3) is satisfied on the whole axis, then estimation (8), stronger than (6), is valid. The order of ε in estimation (8) is the same as in estimation (7).

Theorem 2. *If (5) is valid for $k=2,3$ and $T=\infty$ then either*

$$|f(t)-\exp\{-\mathbf{A}|t|^\alpha\}|\leqq C\varepsilon^\eta,\qquad |t|\leqq\infty, \tag{8}$$

where $\mathbf{A}$ is a constant depending only on f, α, $\operatorname{sign}t$, and C depending only on α, or $f(t)$ is almost degenerate, namely for $\varepsilon\leqq 1/4$

$$|f(t)|\geqq(1+\sqrt{1-4\varepsilon})/2\geqq 1-2\varepsilon. \tag{9}$$

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[1] Constant b is considered in detail in [2]

§ 2. The Main Lemmas

In the Diophantine approximation theory it is known (see Fel'dman [2]) that there exist absolute constants b and b' such that for any natural r and k

$$|r\ln 2 - k\ln 3| > b' r^{-b}.$$

These absolute constants b and b' are considered in the same paper [2].

Lemma 1 (O. Yanushkevichiene [3]). *Let m be an arbitrary integer and $\kappa > 0$ and let ε be an arbitrary small positive number satisfying the condition*

$$\varepsilon^{-\kappa b} M - (b' M/\ln 3)^{1/b} \varepsilon^{-\kappa} \geqq 2^{b(1-b)} (b'/\ln 3)^{1/b-1}, \tag{10}$$
$$M = 2\cdot 3^b (\ln 3)^{1+b} / ((2^{1/b} - 1) b' \alpha^b).$$

Then there exist an integer m' and an integer n' corresponding to m' such that

$$|m'\beta_2 - n'\beta_1| < \varepsilon^{\kappa}, \tag{11}$$

$$0 \leqq m - m' < M\varepsilon^{\kappa b}, \tag{12}$$

where $\beta_1 = -\alpha^{-1}\ln 2$, $\beta_2 = -\alpha^{-1}\ln 3$.

Let us note that there is a positive ε satisfying (10) since, according to [2], $b > 1$. Lemma 1 is trivial in the case $m = 0$ - then $m' = n' = 0$ is chosen.

The next lemma, probably of independent interest, is very useful in the following:

Lemma 2 [2]. *Let $\chi(t)$ be a complex-valued function and let $\mathfrak{n}(\delta) = \sup_{|t|\leqq\delta} |\chi(t)|$. If we have*

$$|\chi(t)| \geqq 2|\chi(t/2^{1/\alpha})| - |r(t)|, \qquad \alpha > 0$$

and $|r(t)| \leqq \varepsilon$ for $-T \leqq t \leqq T$, $T \geqq 1$, then for any δ in the interval $(0, 1]$:

$$\mathfrak{n}(\delta) \leqq 2\mathfrak{n}(1)\delta^{\alpha} + \varepsilon.$$

Corollary. *Let a characteristic function $f(t)$ satisfy*

$$\ln f(t) = 2\ln f(t/2^{1/\alpha}) + r(t), \tag{13}$$

$f(t) \neq 0$ and $|r(t)| \leqq \varepsilon$ for $-T \leqq t \leqq T$, $T \geqq 1$. If $|z| \leqq T$, $|t_0| \leqq T - |t_0 - z|$, then

$$|\ln f(t_0) - \ln f(z)| = H|t_0 - z|^{\alpha} + 2\varepsilon, \tag{14}$$

where $H = 2 \sup_{z:|t_0 - z| \leqq 1,\, z\in[-T,T]} |\ln f(t_0) - \ln f(z)|$.

Proof of Lemma 2. According to the condition of the Lemma

[2] Lemma 2 is proved together with L. Klebanov

$$\sup_{|t|\leqq\delta}|\chi(t)|\geqq 2\sup_{|t|\leqq\delta}|\chi(t/2^{1/\alpha})|-\sup_{|t|\leqq\delta}|r(t)|,$$
$$\eta(\delta)\geqq 2\eta(\delta/2^{1/\alpha})-\varepsilon, \tag{15}$$
$$\eta(\delta/2^{1/\alpha})\leqq\eta(\delta)/2+\varepsilon/2.$$

Having supposed in (15) that $\delta=1$ we obtain:

$$\eta(2^{-1/\alpha})\leqq\eta(1)/2+\varepsilon/2.$$

Continuing in a similar way we get that

$$\eta(2^{-2/\alpha})\leqq\eta(2^{-1/\alpha})/2+\varepsilon/2\leqq\eta(1)/2^2+\varepsilon/2^2+\varepsilon/2,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\eta(2^{-k/\alpha})\leqq\eta(1)/2^k+\varepsilon/2^k+\dots+\varepsilon/2^2+\varepsilon/2.$$

Let us note that for any $\delta\in(0,1)$ there exists a non-negative integer $k=k(\delta)$ such that

$$2^{-k/\alpha}\geqq\delta\geqq 2^{-(k+1)/\alpha}.$$

Consequently by the monotonicity of the function $\eta(\delta)$

$$\eta(\delta)\leqq\eta(2^{-k/\alpha})\leqq\eta(1)/2^k+\varepsilon$$
$$\leqq 2\eta(1)\delta^\alpha+\varepsilon$$

which was to be proved.

For the proof of the corollary let χ be the complex valued function defined on the interval $[-T,T]$ by

$$\chi(u)=\ln f(u+t_0)-\ln f(t_0),$$

and put $\eta(\delta)=\sup_{|u|\leqq\delta}|\chi(u)|$.

Making use of Eq. (13) and Lemma 2, we conclude that

$$\eta(\delta)\leqq 2\delta^\alpha\sup\{|\ln f(u)-\ln f(t_0)|: u\in[-T,T]\cap[t_0-1,t_0+1]\}+2\varepsilon.$$

Now it is enough to let $\delta=|t_0-z|$.

The corollary is proved.

Lemmas 1 and 2 are used essentially in the proof of Lemma 3 on which, in turn, the proofs of the theorems are based.

Lemma 3. *Let for some* $\alpha\in(0,2]$ *the following two conditions be satisfied* ($k=2,3$):

$$|f(t)-f^k(t/k^{1/\alpha})|\leqq\varepsilon,\qquad |t|\leqq 1. \tag{16}$$

Furthermore let there be a constant $p\in(0,1)$ *such that*

$$|f(t)|\geqq p\qquad \textit{for}\ |t|\leqq 1. \tag{17}$$

Then there exists a constant C_1 *depending only on* α *such that for* $|t|\leqq 1$

$$|f(t)-\exp\{-|A_1|\exp(iD_1\operatorname{sign}t)|t|^\alpha\}|\leqq C_1p^{-3}(\varepsilon+\varepsilon^\eta|t|^\alpha),$$

where

$$A_1 = 2\ln f(2^{-1/\alpha}), \qquad \eta = 1/(b+\max(1,\alpha)),$$
$$D_1 = \operatorname{arc\,tg}(\operatorname{Im} A_1/\operatorname{Re} A_1),$$

$b > 1$ *is some absolute constant.*

Concluding the paragraph we note that it is sufficient to prove Lemma 3 as well as the theorems only for $\varepsilon \leqq \varepsilon_0(\alpha)$ where $\varepsilon_0(\alpha)$ is a small positive number depending only on α. Indeed in the case $\varepsilon > \varepsilon_0(\alpha)$ it suffices to increase correspondingly the constant $C = C(\alpha)$ or the constant $C_1 = C_1(\alpha)$ which are present in the formulations of the theorems and Lemma 3.

§ 3. Proof of Lemma 3

Let at first t be positive, i.e. $0 < t \leqq 1$. Having noted $u = \ln t$ and adopting $\psi(\ln t) = \ln f(t)$ from (16) and (17) we obtain for $\varepsilon \leqq p^k/2$

$$\psi(u) = k\psi(u + \beta_{k-1}) + R_{k-1}(\exp u), \tag{18}$$
$$\beta_{k-1} = -(1/\alpha)\ln k, \qquad k = 2, 3;$$
$$|R_{k-1}(t)| \leqq 2p^{-k}\varepsilon, \qquad 0 < t \leqq 1. \tag{19}$$

The statement of Lemma 3 is trivial for $\varepsilon > p^k/2$ when $C_1 = 2$.

In further considerations the following notation is useful:

$$G(u) = \psi(u)\exp(-\alpha u).$$

Substituting this expression in (18) we obtain

$$G(u) = G(u + \beta_j) + R_j(\exp u)\exp(-\alpha u), \qquad j = 1, 2. \tag{20}$$

Further investigation can be divided into five parts.

1°. Let us consider the following sets:

$$\mathfrak{U}_1 = \{u\colon u = n\beta_1 + m\beta_2;\ n, m \text{ - } \textit{nonnegative integers}\},$$
$$\mathfrak{U}_2 = \{u\colon \beta_1 < u = n\beta_1 + m\beta_2 \leqq 0;\ n, m \text{ - } \textit{integers}\},$$
$$\mathfrak{U} = \{u\colon u = n\beta_1 + m\beta_2 \leqq 0,\ n, m \text{ - } \textit{integers}\}.$$

Let us note that the set $\mathfrak{U}_2$ is dense in the interval $(\beta_1, 0)$ and $\mathfrak{U}$ is dense in the interval $(-\infty, 0)$.

It is easy to see that for arbitrary $u \in \mathfrak{U}$ the representation

$$u = u_1 + u_2, \qquad u_1 \in \mathfrak{U}_1,\ u_2 \in \mathfrak{U}_2 \tag{21}$$

is true. In fact, let $u \in \mathfrak{U}$ and $u \leqq \beta_2$. Let us determine $m_0 = \min\{m\colon m$ - *natural*, $m\beta_2 \geqq u\}$. If $|u - m_0\beta_2| \geqq \beta_1$, then obviously

$$\beta_1 < u - m_0\beta_2 - \beta_1 = u_0 \leqq 0. \tag{22}$$

Since $u_0 \in \mathfrak{U}$ according to the structure of u_0, we conclude from (22) that $u_0 \in \mathfrak{U}_2$. If $|u-m_0\beta_2|<|\beta_1|$, then instead of (22) we should consider

$$\beta_1 < u - m_0\beta_2 \leqq 0.$$

Thus the validity of (21) is proved.

2°. We are going to estimate $|G(u)-G(\beta_1)|$. First of all let us assume that $u \in \mathfrak{U}_1$. From the formula (20) we obtain:

$$\begin{aligned} G(u) &= G(n\beta_1+m\beta_2) = G((n-1)\beta_1+m\beta_2) \\ &\quad - R_1(\exp\{(n-1)\beta_1+m\beta_2\})\exp\{-\alpha(n-1)\beta_1+m\beta_2)\} \\ &= \ldots = G(\beta_1+m\beta_2) - \sum_{j=1}^{n-1} R_1(\exp(j\beta_1+m\beta_2)) \\ &\quad \times \exp\{-\alpha(j\beta_1+m\beta_2)\} = G(\beta_1) - \sum_{j=0}^{m-1} R_2(\exp(\beta_1+j\beta_2))\exp\{-\alpha(\beta_1+j\beta_2)\} \\ &\quad - \sum_{j=1}^{n-1} R_1(\exp(j\beta_1+m\beta_2))\exp\{-\alpha(j\beta_1+m\beta_2)\}. \end{aligned} \tag{23}$$

Let us note that according to (19)

$$\begin{aligned} &\left|\sum_{j=1}^{n-1} R_1(\exp(j\beta_1+m\beta_2))\exp\{-\alpha(j\beta_1+m\beta_2)\}\right| \\ &\qquad \leqq 2p^{-2}\varepsilon(\exp\{-\alpha(n\beta_1+m\beta_2)\} - \exp\{-\alpha(\beta_1+m\beta_2)\}) \\ &\qquad \leqq 2p^{-2}\varepsilon\exp(-\alpha u). \end{aligned}$$

Similarly,

$$\begin{aligned} &\sum_{j=0}^{m-1} R_2(\exp(\beta_1+j\beta_2))\exp(-\alpha(\beta_1+j\beta_2)) \\ &\qquad \leqq p^{-3}\varepsilon\exp(-\alpha(\beta_1+m\beta_2)). \end{aligned}$$

Consequently taking into account (23) we have

$$|G(u)-G(\beta_1)| \leqq C_2\varepsilon\exp(-\alpha u), \tag{24}$$
$$C_2 = 3p^{-3}.$$

3°. Now let $u \in \mathfrak{U}_2$. According to the definition of the set $\mathfrak{U}_2$ either $u=n\beta_1-m\beta_2$ or $u=m\beta_2-n\beta_1$ where n and m are non-negative integers. Both cases are similar, therefore, for definiteness let us assume that

$$\mathfrak{U}_2 \ni u = n\beta_1 - m\beta_2, \qquad n \geqq 0,\ m \geqq 0.$$

Let us determine

$$\begin{aligned} &m_i = m_{i-1}-1 = m_{i-2}-2 = \ldots = m-i; \qquad n_0 = n, \\ &n_i = \min\{n^*\colon n^*\beta_1 - m_i\beta_2 \leqq 0,\ n^* \textit{ is natural}\}. \end{aligned} \tag{25}$$

It is not difficult to show that n_i can be changed within the following bounds[3]:

$$[m_i\beta_2/\beta_1] \leqq n_i \leqq [m_i\beta_2/\beta_1]+1. \tag{26}$$

Using (20) for $u \in \mathfrak{U}_2$ we obtain:

$$\begin{aligned} G(u) &= G(n\beta_1 - m\beta_2) = G(n\beta_1 - (m-1)\beta_2) \\ &\quad + R_2(\exp(n\beta_1 - m\beta_2))\exp(-\alpha(n\beta_1 - m\beta_2)) \\ &= G(n_1\beta_1 - m_1\beta_2) + R_2(\exp(n\beta_1 - m\beta_2)) \\ &\quad \times \exp(-\alpha(n\beta_1 - m\beta_2)) - \sum_{j=n_1}^{n-1} R_1(\exp(j\beta_1 - m_1\beta_2))\exp(-\alpha(j\beta_1 - m_1\beta_2) \\ &= \ldots = G(n_i\beta_1 - m_i\beta_2) + \sum_{l=0}^{i-1} R_2(\exp(n_l\beta_1 - m_l\beta_2))\exp(-\alpha(n_l\beta_1 - m_l\beta_2)) \\ &\quad - \sum_{l=1}^{i} \sum_{j=n_l}^{n_{l-1}-1} R_1(\exp(j\beta_1 - m_l\beta_2))\exp(-\alpha(j\beta_1 - m_l\beta_2)). \end{aligned} \tag{27}$$

Now let us apply Lemma 1 for our m. Since in this Lemma m' is natural, $m' \leqq m$ and according to our definition $m_j = m-1$ then obviously there exists i such that $m' = m_i$. Let us show that for n' determined by Lemma 1 one of the following two conditions is true: either $n' = n_i$ or $n' = n_i - 1$. In fact, according to (25) for fixed m_i the natural number n_i is chosen so that the difference $n_i\beta_1 - m_i\beta_2$ were as close to zero on the left as possible. In Lemma 1 n' is chosen by the condition of immediate proximity to zero of the number $n'\beta_1 - m'\beta_2$ without regard to the direction of this proximity, i.e. without regard to the sign. Consequently, there are only two possibilities: if $|n'\beta_1 - m_i\beta_2| = n'\beta_1 - m_i\beta_2$ then $n_i = n' - 1$; if $|n'\beta_1 - m_i\beta_2| = -(n'\beta_1 - m_i\beta_2)$ then $n_i = n'$.

Now it is not difficult to estimate the sums in (27). First of all let us compare the case $u \in \mathfrak{U}_2$ with the case $u \in \mathfrak{U}_1$ or more exactly with the relation (23). In the latter case, i.e. in (23), as we have seen, the role of exponents is essentially taken into account in all estimations. In (27) an exponential multiplier is not to be trusted since both n_l and m_l can be very large but their difference $n_l\beta_1 - m_l\beta_2$ can be very small and at the same time $\exp\{-\alpha(n_l\beta_1 - m_l\beta_2)\}$ is close to 1. Since m_l, n_l together with l can be very large, the estimation of sums in (27) is impossible without lemmas of the Lemma 1 type.

According to the relation (12) and the above remark there exists i such that

$$i < M\varepsilon^{-\kappa b} \tag{28}$$

and in addition

$$|m_i\beta_2 - n_i\beta_1 - \beta_1| < \varepsilon^{\kappa} \tag{29}$$

(we assume $n_i = n' - 1$ at first).

Remembering (26) we easily find that the exponential multipliers in (27) are bounded: for $j = n_l, \ldots, n_{l-1} - 1$,

$$\begin{aligned} \exp(-\alpha(j\beta_1 - m_l\beta_2)) &\leqq \exp(-\alpha((n_{l-1}-1)\beta_1 - m_l\beta_2)) \\ &\leqq \exp(-\alpha(\beta_1 + \beta_2)) = 6. \end{aligned} \tag{30}$$

[3] $[a]$ is the integer part of a

By (26), in the last expression of the equalities in (27) the number of summands other than $G(n_i\beta_1-m_i\beta_2)$ does not exceed the number $i+i([\beta_2/\beta_1]+2)=4i$. It follows from (18), (28) and (30) that

$$|G(n\beta_1-m\beta_2)-G(n_i\beta_1-m_i\beta_2)|<C_3\varepsilon^{1-\kappa b}, \tag{31}$$

where

$$C_3=48M(p+1)p^{-3}.$$

Now, taking into account (29) let us estimate the modulus of the difference $G(n_i\beta_1-m_i\beta_2)-G(\beta_1)$. Obviously, for $0<t_j\leqq 1$ and $u_j=\ln t_j$, $j=1,2$

$$\begin{aligned}|G(u_1)-G(u_2)|&=|t_1^{-\alpha}\ln f(t_1)-t_2^{-\alpha}\ln f(t_2)|\\&\leqq t_1^{-\alpha}|\ln f(t_1)-\ln f(t_2)|+|\ln f(t_2)|\,|t_1^{-\alpha}-t_2^{-\alpha}|.\end{aligned} \tag{32}$$

According to (29) and the corollary of Lemma 2, we have, for $t_1=\exp\beta_1$ and $t_2=\exp(n_i\beta_1-m_i\beta_2)$, the inequality

$$|\ln f(t_1)-\ln f(t_2)|\leqq 4C_4|t_1-t_2|^\alpha+2\varepsilon$$

where

$$C_4=(\ln^2(1/p)+4\pi^2)^{1/2}.$$

Then from (17), (29) and (32) we conclude that for $u_1=\beta_1$, $u_2=n_i\beta_1-m_i\beta_2$

$$\begin{aligned}|G(u_1)-G(u_2)|&\leqq 8C_4t_2^\alpha|e^{u_1-u_2}-1|^\alpha+4\varepsilon+2C_4|e^{\alpha(u_1-u_2)}-1|\\&\leqq 16C_4\varepsilon^{\alpha\kappa}+4\varepsilon+4C_4\alpha\varepsilon^\kappa\end{aligned}$$

if $\varepsilon^\kappa\max(1,\alpha)\leqq 1/2$.

Consequently, by (31) we have for $u\in\mathfrak{U}_2$ and $n_i=n'-1$

$$|G(u)-G(\beta_1)|\leqq C_5\varepsilon^\eta, \tag{33}$$

where

$$C_5=C_3+4C_4(4+\alpha)+4,$$
$$\eta=\begin{cases}1/(b+1) & \text{for } 0<\alpha\leqq 1,\\ 1/(b+\alpha) & \text{for } 1\leqq\alpha\leqq 2.\end{cases}$$

It remains to consider the case $u\in\mathfrak{U}_2$ and $n_i=n'$. Obviously in this case in (27) one should consider not the term $G(n_i\beta_1-m_i\beta_2)$ but the term $G((n_i+1)\beta_1-m_i\beta_2)$, i.e. one should make one step less in (27). Thus, having noted the remainder term in (27) by R_3 we obtain:

$$\begin{aligned}G(n\beta_1-m\beta_2)=G((n_i+1)\beta_1-m_i\beta_2)+R_3\\+R_1(\exp(n_i\beta_1-m_i\beta_2))\exp(-\alpha(n_i\beta_1-m_i\beta_2)).\end{aligned}$$

Since, in this relation as well as in (27), Lemma 1 is applied only to the term m_i which in both cases is the same, we see that it is sufficient to repeat the arguments of (28)–(33).

Consequently, the estimation (33) holds true for any $u\in\mathfrak{U}_2$.

4°. Let finally $u \in \mathfrak{U}$. Expressing this number in the form (21) let us repeat (23):

$$\begin{aligned} G(u) &= G(u_1+u_2) = G(n\beta_1+m\beta_2+u_2) = G(u_2) \\ &- \sum_{j=0}^{m-1} R_2(\exp(\beta_1+j\beta_2+u_2))\exp(-\alpha(\beta_1+j\beta_2+u_2)) \\ &- \sum_{j=0}^{n-1} R_1(\exp(j\beta_1+m\beta_2+u_2))\exp(-\alpha(j\beta_1+m\beta_2+u_2)). \end{aligned}$$

Consequently, as in (24)

$$|G(u)-G(u_2)| \leqq C_2\varepsilon\exp(-\alpha u).$$

Hence, as in (33) we conclude

$$|G(u)-G(\beta_1)| \leqq C_2\varepsilon\exp(-\alpha u)+C_5\varepsilon^\eta, \qquad u \in \mathfrak{U}. \tag{34}$$

At the end of the proof in the case $0 \leqq t \leqq 1$ it remains to note that for any $u \in (-\infty, 0]$

$$|G(u)-G(\beta_1)| = \lim_{i\uparrow\infty} |G(u_i)-G(\beta_1)|,$$

because $G(u)$ is continuous on $(-\infty,\infty)$ and relation $u=\lim_{i\uparrow\infty} u_i$, $u_i \in \mathfrak{U}$ is valid for any $u \in (-\infty, 0]$, i.e. (33) is valid for an arbitrary $u \in (-\infty, 0]$. Since

$$\psi(\ln t) = \ln f(t), \qquad \psi(u) = \exp(\alpha u)G(u),$$

(34) means that

$$|\psi(u)-G(\beta_1)\exp(\alpha u)| \leqq C_2\varepsilon + C_5\varepsilon^\eta\exp(\alpha u), \qquad u \in (-\infty, 0]$$

i.e. that

$$|\ln f(t) - 2t^\alpha \ln f(2^{-1/\alpha})| \leqq C_2\varepsilon + C_5\varepsilon^\eta t^\alpha. \tag{35}$$

5°. Now let $-1 \leqq t < 0$. Letting $t=-z$, $0<z\leqq 1$ we rewrite (35) in the following form:

$$\ln f(z) = Az^\alpha + r(z), \qquad 0<z\leqq 1,$$

where $A = 2\ln f(2^{-1/\alpha})$,

$$|r(z)| \leqq C_2\varepsilon + C_5\varepsilon^\eta z^\alpha. \tag{36}$$

Note that

$$f(t) = f(-z) = \overline{f(z)} = \exp(\bar{A}z^\alpha)\exp(\overline{r(z)}) \tag{37}$$

and put $A = -|A|\exp(iD)$ where

$$\operatorname{tg} D = \operatorname{Im} A / \operatorname{Re} A$$

(while selecting the value D from the last equation one should take into consideration the signs of $\operatorname{Re} A$ and $\operatorname{Im} A$).

Hence and from relations (36), (37) we obtain for $|t| \leqq 1$

$$f(t) = \exp\{-|A|\exp(iD\operatorname{sign} t)|t|^\alpha\}\exp(r_0(t)),$$

where for $|r_0(t)|$ the estimation (36) holds true as it does for $|r(t)|$. This means that for $|t| \leqq 1$

$$|f(t) - \exp(-|A| \exp(iD \operatorname{sign} t)|t|^\alpha)| \leqq 2C_2\varepsilon + 2C_5\varepsilon^\eta |t|^\alpha. \tag{38}$$

§ 4. Proof of the Theorems and the Corollary

It is obvious, that

$$s = s(f, p) = \inf\{|t| : |f(t)| = p\} > 0. \tag{39}$$

As already mentioned, two cases are possible: $s < T$ and $s \geqq T$.

Let us consider, first of all, the case $s < T$. Then instead of the characteristic function $f(t)$ we shall introduce the characteristic function $f_s(t) = f(ts)$, for which

$$s(f_s, p) = \inf\{|t| : |f(st)| = p\} = \frac{1}{s} \inf\{s|t| : |f(st)| = p\} = 1. \tag{40}$$

It is not difficult to verify that if $f(t)$ satisfies (5) for $|t| \leqq T$, then $f_s(t)$ satisfies (5) for $|t| \leqq T/s$. By virtue of (40) relation (17) is true. Therefore for f_s we can apply Lemma 3. Then for $|t| \leqq 1$ we have:

$$|f_s(t) - \exp\{-|A_s| \exp(iD_s \operatorname{sign} t)|t|^\alpha\} \leqq C_1 \varepsilon^\eta, \tag{41}$$

where $A_s = 2 \ln f_s(2^{-1/\alpha})$, $D_s = \operatorname{arc\,tg}(\operatorname{Im} A_s / \operatorname{Re} A_s)$.

Thus it remains to consider the domain $1 < |t| \leqq T/s$. For this purpose we shall denote

$$r(t) = f_s(t) - f_s^2(t/2^{1/\alpha}). \tag{42}$$

According to the conditions of the theorem, $|r(t)| \leqq \varepsilon$ for $|t| \leqq T/s$. The second inequality in the conditions, namely,

$$|f(t) - f^3(t/3^{1/\alpha})| \leqq \varepsilon, \qquad |t| \leqq T$$

is not used in the consideration of the case $|t| > 1$. Put

$$h(t) = f_s(t) - \exp(B_t |t|^\alpha),$$

where

$$B_t = B_t(s) = -|A_s| \exp(iD_s \operatorname{sign} t) = 2 \ln f_s(2^{-1/\alpha} \operatorname{sign} t).$$

According to (42),

$$h(t) = h^2(t/2^{1/\alpha}) + 2h(t/2^{1/\alpha}) \exp(B_t |t|^\alpha / 2) + r(t). \tag{43}$$

Having assumed that

$$|h(t)|^2 \leqq \Delta = \sup_{|t| < 1} |f_s(t) - \exp(B_t |t|^\alpha)| \tag{44}$$

for $|t| < t_0$, where t_0 is an arbitrary number from the interval $[1, T/s)$, we shall prove that this inequality holds true for $|t| \leqq \min(t_0 2^{1/\alpha}, T/s)$. Since $\alpha > 0$ and (44) holds for $t_0 = 1$ on account of (41), this will then mean that (44) is valid for any real $t \in [-T/s, T/s]$. Put

$$d = 1 + [\alpha \log_2 t_0].$$

Note that for $|t| \leqq 2^{1/\alpha} t_0$

$$2^{-(d+1)/\alpha} |t| < 1. \tag{45}$$

Since according to the assumption, (44) holds true for $|t| < t_0$, for any natural m

$$|h(t/2^{m/\alpha})|^2 \leqq \Delta \tag{46}$$

for $|t| < t_0 2^{1/\alpha}$.

According to (43), (44) and (46) in the interval $|t| < t_0 2^{1/\alpha}$ we obtain:

$$\begin{aligned}
|h(t/2^{(m-1)/\alpha})| &\leqq 2|h(t/2^{m/\alpha})| \exp(\operatorname{Re} B_t |t|^\alpha / 2^m) + \Delta + \varepsilon, \\
|h(t)| &\leqq 2|h(t/2^{1/\alpha})| \exp(\operatorname{Re} B_t |t|^\alpha/2) + \Delta + \varepsilon \\
&\leqq 2^2 |h(t/2^{2/\alpha})| \exp(\operatorname{Re} B_t |t|^\alpha (2^{-1} + 2^{-2})) \\
&\quad + 2(\Delta + \varepsilon) \exp(\operatorname{Re} B_t |t|^\alpha/2) + \Delta + \varepsilon \leqq \dots \\
&\leqq 2^{d+1} |h(t/2^{(d+1)/\alpha})| \exp(\operatorname{Re} B_t |t|^\alpha/2) \\
&\quad + 2^{d+1}(\Delta + \varepsilon) \exp(\operatorname{Re} B_t |t|^\alpha/2) + \Delta + \varepsilon.
\end{aligned} \tag{47}$$

As we have seen $B_t = 2 \ln f_s(2^{-1/\alpha} \operatorname{sign} t)$, we obtain

$$\operatorname{Re} B_t = 2 \ln |f_s(2^{-1/\alpha} \operatorname{sign} t)| = 2 \ln |f_s(2^{-1/\alpha})| = B.$$

Having noted

$$G^*(t) = 4|t|^\alpha \exp(B|t|^\alpha/2)$$

we see that for $|t| \geqq t_0$

$$2^{d+1} \exp(B|t|^\alpha/2) \leqq G^*(t_0).$$

Since $G^*(t_0)$ is even, it is easy to verify that the maximum $G^*(t_0)$ is reached at the points t_* and $(-t_*)$ where

$$t_* = (2/(-B))^{1/\alpha}.$$

Therefore,

$$|h(t)| \leqq G^*(t_*)(|h(t/2^{(d+1)/\alpha})| + \Delta + \varepsilon) + \Delta + \varepsilon \tag{48}$$

for $t_0 \leqq |t| < \min(2^{1/\alpha} t_0, T/s)$. Further by (40) $f_s(1) = p$. Therefore from (5) we have

$$|f_s(2^{-1/\alpha})|^2 \leqq p + \varepsilon.$$

Consequently for $\varepsilon < (1-p)/2$

$$|B|^{-1} \leqq |\ln(p+\varepsilon)|^{-1} \leqq |\ln((p+1)/2)|^{-1}.$$

Since $G^*(t_*) = 8/(e|B|)$, (48) denotes that for $t_0 \leqq |t| < \min(2^{1/\alpha} t_0, T/s)$

$$|h(t)| \leqq \left| \ln \frac{p+1}{2} \right|^{-1} (|h(t/2^{(d+1)/\alpha})| + \Delta + \varepsilon) + \Delta + \varepsilon. \tag{49}$$

According to (41) and (45) we correspondingly obtain:

$$\Delta=\Delta(\varepsilon)\downarrow 0 \quad \text{for } \varepsilon\downarrow 0,$$
$$|h(t/2^{(d+1)/\alpha})|\leqq\Delta.$$

Therefore, there exists a positive constant $\varepsilon_*=\varepsilon_*(\alpha,p)$ such that for $0\leqq\varepsilon\leqq\varepsilon_*$

$$\sqrt{\Delta}\leqq\left(3\left|\ln\frac{p+1}{2}\right|^{-1}+2\right)^{-1}. \tag{50}$$

According to Lemma 3 and notation (44) we have $\Delta\geqq\varepsilon$. Then (49) and (50) mean that (44) is valid for $|t|\leqq\min(2^{1/\alpha}t_0, T/s)$ and hence (44) is valid for any real t in the interval $[-T/s, T/s]$. Therefore according to (49) we have in this interval

$$|h(t)|\leqq\left(2\left|\ln\frac{p+1}{2}\right|^{-1}+1\right)(\varepsilon^\eta+\varepsilon). \tag{51}$$

For the completion of the case $s<T$ (and also of the Corollary of Theorem 1) it is sufficient to realize a conversion from $f_s(t)$ to $f(t)$ and make use of (51).

Now suppose that $s\geqq T$. That means

$$|f(t)|\geqq p \quad \text{for } |t|\leqq T. \tag{52}$$

Let us slightly change the proof concerning the case $s<T$. Namely, considering only $|t|\leqq\min(T,\varepsilon^{-\eta/(1+\alpha)})$, i.e. assuming a priori $t_0\leqq\varepsilon^{-\eta/(1+\alpha)}$, we shall use the following estimation instead of (47):

$$|h(t)|\leqq 2^{d+1}|h(t/2^{(d+1)/\alpha})|+(2^{d+1}+1)(\Delta+\varepsilon) \tag{53}$$

(remember that $\operatorname{Re}B_t=2\ln|f(2^{-1/\alpha})|<0$ and that, for $s\geqq T$, it is not necessary to introduce function $f_s(t)$). Having noted now that

$$2^{d+1}\leqq 4t_0^\alpha\leqq 4\varepsilon^{-\eta\alpha/(1+\alpha)}$$

we obtain from (53) that

$$|h(t)|\leqq 4\varepsilon^{-\eta\alpha/(1+\alpha)}(|h(t/2^{(d+1)/\alpha})|+\Delta+\varepsilon)+\Delta+\varepsilon.$$

According to (45) and Lemma 3 (which is applicable because of (52)) we get:

$$|h(t)|\leqq 5C_1\varepsilon^{\eta/(1+\alpha)}+(4\varepsilon^{-\eta\alpha/(1+\alpha)}+1)(\Delta+\varepsilon).$$

Now it remains only to change assumption (44) into

$$|h(t)|^2\leqq\Delta=\varepsilon^{\eta/(1+\alpha)}$$

for any $|t|<t_0$ where t_0 is an arbitrary number from the interval $[1,\min(T,\varepsilon^{-\eta/(1+\alpha)})]$.

Theorem 1 and its Corollary are completely proved.

Let us pass to the proof of Theorem 2. According to (5) for $k=2$ we have:

$$f(t)=f^2(t/2^{1/\alpha})+r(t), \qquad |r(t)|\leqq\varepsilon, \qquad |t|\leqq\infty. \tag{54}$$

Let us denote

$$I=\inf\{|f(t)|: |t|\leqq\infty\}.$$

From (54)

$$I\leqq I^2+\sup\{|r(t)|: |t|\leqq\infty\}\leqq I^2+\varepsilon,$$
$$I^2-I+\varepsilon\geqq 0.$$

Thus for $\varepsilon\leqq 1/4$ either

$$I\geqq(1+\sqrt{1-4\varepsilon})/2 \tag{55}$$

or

$$I\leqq(1-\sqrt{1-4\varepsilon})/2. \tag{56}$$

Inequality (55) means that $f(t)$ is almost degenerate and relation (9) is valid.

If inequality (56) is valid, then there exists a point $0<z_0<\infty$ such that for $\varepsilon\leqq 1/4$

$$|f(z_0)|=1/2. \tag{57}$$

Let us denote

$$|\mathfrak{f}(t)|=|f(t z_0)|.$$

According to (57)

$$\mathfrak{s}=s(\mathfrak{f}, 1/2)=\min\{|t|: |\mathfrak{f}(t)|=1/2\}\leqq 1.$$

Now it remains to apply the Corollary of Theorem 1 to the characteristic function $\mathfrak{f}(t)$ and to carry out the reverse conversion from $\mathfrak{f}$ to f which is trivial because of the relation $T=\infty$.

Theorem 2 is proved.

§ 5. Supplements

1°. It is well known that the class of stable distributions forms a four parameter set of functions $F(x;\alpha,\beta,\gamma,\lambda)$ with parameters $0<\alpha\leqq 2$, $-1\leqq\beta\leqq 1$, $-\infty<\gamma<\infty$, $\lambda>0$. Besides, the characteristic function of the stable law can be written as follows:

$$\ln y(t)=it\gamma-\lambda|t|^{\alpha}\omega(t,\alpha,\beta), \tag{58}$$

where

$$\omega(t,\alpha,\beta)=\begin{cases}\exp\left(i\frac{\pi}{2}\beta K(\alpha)\operatorname{sign} t\right) & \text{for } \alpha\neq 1,\\ 1+i\beta\frac{2}{\pi}\ln|t|\operatorname{sign} t & \text{for } \alpha=1,\end{cases}$$
$$K(\alpha)=1-|1-\alpha|=\min(\alpha, 2-\alpha).$$

Substituting (58) for (3) we see that $\gamma=0$ for $\alpha\neq 1$ and $\beta=0$ for $\alpha=1$. This means that P. Levy's theorem characterizes in reality the subclass $\mathfrak{M}$ of the class of stable laws, investigated by V.M. Zolotarev [5] in detail. The characteristic functions of the class $\mathfrak{M}$ can be written in the following way:

$$\ln y(t) = -\exp\{v^{-1/2}\{\ln|t| + \tau - \tfrac{1}{2} i\pi\theta \operatorname{sign} t + \mathbb{C}(1 - v^{1/2})\}\},$$
$$v \geqq 1/4, \quad |\theta| \leqq \min(1, 2\sqrt{v} - 1), \quad -\infty < \tau < \infty \tag{59}$$

and $\mathbb{C} = -\Gamma'(1) = 0.577\ldots$ is Euler's constant.

A natural question arises: under which conditions a function of the form $\exp(-|A|\exp(iD \operatorname{sign} t) s^{-\alpha}|t|^{\alpha})$ that appears in (6)-(8), is an element of class $\mathfrak{M}$? It is obvious that it is always an element of class $\mathfrak{M}$ if we know in addition that $\operatorname{Im} A = 0$. This case is considered in greater detail below in 2°. Having noticed that

$$\theta = -2Dv^{1/2}/\pi$$

we shall consider the general case. Thus, from the conditions (59) it remains to check that

$$|D| \leqq \frac{\pi}{2} \min(v^{-1/2}, 2 - v^{-1/2}). \tag{60}$$

The right-hand side of inequality (60) will be denoted by ξ. (60) is equivalent to the following inequality

$$|\operatorname{Im} A / \operatorname{Re} A| \leqq \operatorname{tg} \xi. \tag{61}$$

Let us recall that $A = \ln f(x_0)$, where x_0 is defined in the course of the proof of the theorems. Let us note that if in (3) $\alpha \neq 2$ (i.e. the case of characterization of the normal law is excluded), then, for any characteristic function f satisfying the conditions of Theorems 1 or 2, there exists a number $z_0 > 0$ such that

$$|\operatorname{Im} \ln f(z_0) / \operatorname{Re} \ln f(z_0)| \leqq \operatorname{tg} \xi. \tag{62}$$

In fact, we can obtain the smallness of $\operatorname{Im}^2 \ln f(z_0) + \operatorname{Re}^2 \ln f(z_0)$ from the corollary of Lemma 2, more exactly from (14) assuming there that z_0 is sufficiently close to zero. However, one should not conclude from here about the smallness of the ratio in (62). The above smallness can be obtained if one notices that $\operatorname{Re} \ln f(z_0) = \ln|f(z_0)|$. Now it remains to apply Theorem 1 or Theorem 2 to characteristic function $|f(t)|^2$ and to obtain closeness with the class of symmetric stable laws.

Let us suppose now that for $\alpha \neq 2$ $(v \neq 1/4)$ we found a number z_0 such that (62) is valid. Can we take x_0 for z_0? To be more exact, can we prove the theorems in such a way that x_0 is chosen not in the course of proof, but it is assumed a priori that $x_0 = z_0$? The answer is positive. It can be done by substituting the point $x_0 = 2^{-1/\alpha}$ for the required point z_0 in Lemma 3 (such a change was made in the paper [3] by O. Yanushkevichiene). One succeeds to do so rather often by means of replacing $f(t)$ by $f_{y_0}(t) = f(ty_0)$ where $y_0 = 2^{1/\alpha} z_0$.

Thus the case $\alpha = 2$ (or which is the same $v = 1/4$) remained unconsidered because in this case the right-hand side of (60) equals zero. Under the conditions in (3) this is a well known theorem of G. Polya and besides it is sufficient to require the realization of (3) only for $k=2$. This case was widely investigated (see the paper [4] of R. Yanushkevichius and O. Yanushkevichiene and references in that paper) and for this reason we will not consider it in greater detail. Let us only denote that there are some limitations on moments in all the papers on the estimations of stability of G. Polya's theorem. As we have seen already they are absent in our Theorems 1 and 2.

2°. Let us recall the definition of the λ-metric

$$\lambda(X, Y)=\min_{T>0} \max\left\{\frac{1}{2} \max_{|t|\leqq T} |\mathbf{E}\exp(itX)-\mathbf{E}\exp(itY)|, \frac{1}{T}\right\},$$

which is equivalent to the Levy metric L in the sense that L-convergence of the sequence $\{X_n\}$ implying λ-convergence of this sequence and vice versa. This metric is especially convenient in those cases where the proofs of statements are carried out in terms of characteristic functions.

Proposition. *Let X_1, X_2, X_3 be i.i.d. symmetrical random variables. If*

$$\lambda\left(X_1, k^{-1/\alpha} \sum_{j=1}^{k} X_j\right) \leqq \varepsilon$$

for $k=2, 3$ and some $\alpha\in(0, 2]$, then there exist a symmetrical stable random variable Y and a constant C, depending only on α, such that

$$\lambda(X_j, Y) \leqq C\varepsilon^{\eta/(1+\alpha)}.$$

For the proof of this proposition it is sufficient to note that $\operatorname{Im} A=0$, to assume $p=1/2$, and to make use of Theorem 1.

3°. The function $\exp(-|A|\exp(iD\operatorname{sign} t)s_*^{-\alpha}|t|^\alpha)$ in (6) and analogously in (7) and (8) can be written in another form: $\exp(2\ln f(2^{-1/\alpha}s_* \operatorname{sign} t)s_*^{-\alpha}|t|^\alpha)$. The latter representation has been already used in the proof of Theorem 1.

Finally, let us note that for $T<\infty$, instead of the characteristic s defined in (4) it makes sense to use the characteristic

$$s_T=\min\{|t|\leqq T: |f(t)|=p\},$$

since for its finding less information than for s is required. This, however, was not our task because we wanted to simplify the proof. The modification of the proof is obvious.

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