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The Central Limit Theorem for Non-Separable Valued Functions

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1. Introduction

The main purpose of this paper is to formulate and investigate the central limit theorem for functions which are not assumed to be separable-valued nor measurable. The inspiration is a part of a paper by Dudley and Philipp [2, Theorem 1.1] but the aim is to use the setting and some results in Hoffmann-Jørgensen [5, Chap. 7].

Let f be a function from a probability space $(S, \mathscr{S}, \mu)$ into a Banach space $(B, \|\cdot\|)$. We say that f satisfies the central limit theorem if there exists a Radon probability measure γ on $(B, \|\cdot\|)$ so that

$$\lim_n \int^* g\left(n^{-\frac{1}{2}} \sum_{i=1}^n f(s_i)\right) \mu^{\mathbb{N}}(ds) = \lim_n \int_* g\left(n^{-\frac{1}{2}} \sum_{i=1}^n f(s_i)\right) \mu^{\mathbb{N}}(ds) = \int g\, d\gamma$$

for all bounded, real-valued continuous functions g on $(B, \|\cdot\|)$, where $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}}, \mu^{\mathbb{N}})$ is the countable product of $(S, \mathscr{S}, \mu)$. It turns out that f satisfies the central limit theorem if and only if the normalized sums are eventually tight, i.e. if for all $\varepsilon > 0$ there exists $K \subseteq B$, compact, so that

$$\limsup_n (\mu^{\mathbb{N}})^* \left(n^{-1/2} \sum_{i=1}^n f(s_i) \notin G\right) < \varepsilon \qquad \forall G \supseteq K \text{ open},$$

and if and only if the normalized sums can, in limit be approximated by finite dimensionally, measurable functions. Furthermore, it turns out that the statements above imply that f is weakly integrable and that the limit measure γ is a gaussian measure, whose covariance function is determinated by f.

In the following section, I shall describe the notation and the basic definitions and results. In Sect. 3, I shall state and prove the main results.

2. Notation and Basic Definitions and Results

Let $(S, \mathscr{S}, \mu)$ be a probability space, (M, d) a linear metric space equipped with the Baire σ-algebra $\mathscr{M}$, M' the dual space of M, $C(M)$ the set of bounded, real-valued continuous functions on M, and $\mathscr{K}(M)$ the set of compact subsets of M. We say that $f: S\to M$ is μ-measurable or a random variable on $(S,\mathscr{S},\mu)$ if f is $(\mathscr{S}(\mu),\mathscr{M})$-measurable, where $\mathscr{S}(\mu)$ is the μ-measurable sets. We say that f is weakly μ-measurable (μ-integrable) if $x'(f)$ is μ-measurable (μ-integrable) for all $x'\in M'$ and that f is Bochner μ-measurable if f is μ-measurable and $f(S\setminus N)$ is separable for some μ-nullset $N\in\mathscr{S}$. We let

$$L_w^2(M,\mu)=\{f: S\to M \mid x'(f)^2 \text{ is } \mu\text{-integrable } \forall x'\in M'\}.$$

The outer resp. inner μ-measure is denoted μ^* resp. μ_* and if f is a $\bar{\mathbb{R}}$-valued function on S, where $\bar{\mathbb{R}}=[-\infty,\infty]$, then the upper resp. lower μ-integral of f is denoted $\int_S^* f\,d\mu$ resp. $\int_{*S} f\,d\mu$ and the upper resp. lower μ-envelope of f by f^* resp. f_*. Furthermore we denote the μ-hull resp. μ-kernel of a subset A of S by A^* resp. A_*.

We say that an S-valued random variable φ, defined on some probability space $(\Omega,\mathscr{F},P)$ and with distribution μ, is P-perfect if

$$P^*(\varphi\in A)=\mu^*(A) \qquad \forall A\subseteq S.$$

Non-measurable sets and functions, envelopes and perfect random variable are investigated closely in [1].

The following definitions are due to Hoffmann-Jørgensen (see Chap. 7 in [5]).

Definition 2.1. Let $\{f_n\}$ be a sequence of M-valued functions on a probability space $(\Omega,\mathscr{F},P)$. We say that $\{f_n\}$ converges weakly ($\tilde{\to}$) to γ, a Baire probability measure on M, if

$$\int g\,d\gamma=\lim_n \int^* g\circ f_n\,dP=\lim_n \int_* g\circ f_n\,dP \qquad \forall g\in C(M) \tag{2.1.1}$$

and we say that $\{f_n\}$ is eventually tight if

$$\forall \varepsilon>0 \quad \exists K\in\mathscr{K}(M)\colon \limsup_n P^*(f_n\notin G)<\varepsilon \qquad \forall G\supseteq K, \text{ open.} \quad \square \tag{2.1.2}$$

Let us end this section with some results from [1]:

Proposition 2.2. *Let f be a $\bar{\mathbb{R}}$-valued function on S and let φ be an S-valued random variable defined on some probability space $(\Omega,\mathscr{F},P)$ and with distribution μ. We have*

$$\mu^*(f>t)=\mu(f^*>t) \qquad \forall t\in\mathbb{R} \tag{2.2.1}$$

$$\int^* f\,d\mu=\int f^*\,d\mu \qquad (\textit{if they exist}) \tag{2.2.2}$$

$$f_*\circ\varphi\leqq(f\circ\varphi)_*\leqq(f\circ\varphi)^*\leqq f^*\circ\varphi \qquad P\text{-a.s.} \tag{2.2.3}$$

$$f_*\circ\varphi=(f\circ\varphi)_*, \quad (f\circ\varphi)^*=f^*\circ\varphi \qquad P\text{-a.s. } \textit{if } \varphi \textit{ is } P\textit{-perfect} \tag{2.2.4}$$

where f^* and f_* are the μ-envelopes of f and $(f\circ\varphi)^*$ and $(f\circ\varphi)_*$ the P-envelopes of $f\circ\varphi$.

Proof. Look at I.(2.6.6), I.(2.3.1), II.(2.1.1) and Theorem II.2.2, all in [1]. □

Proposition 2.3. *Let f be a $\bar{\mathbb{R}}$-valued function on S and let $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}}, \mu^{\mathbb{N}})$ be the countable product of $(S, \mathscr{S}, \mu)$. We then have*

$$\mu^*(\|f\|>t)=(\mu^{\mathbb{N}})^*(\{s=(s_j)\in S^{\mathbb{N}} \mid \|f(s_i)\|>t\}) \qquad \forall t\in\mathbb{R},\ \forall i\in\mathbb{N} \tag{2.3.1}$$

$$(\mu^{\mathbb{N}})^*\left(\max_{k\leq n}\left\|\sum_{j=1}^{k} f(s_j)\right\|>3t\right)\leqq 2(1+C_n(t))(\mu^{\mathbb{N}})^*\left(\left\|\sum_{j=1}^{n} f(s_j)\right\|>t\right)$$
$$\forall t\in\mathbb{R},\ \forall n\in\mathbb{N} \tag{2.3.2}$$

where $C_n(t)=\min_{i\leqq n}(\mu^{\mathbb{N}})^*\left(\left\|\sum_{j=1}^{i} f(s_j)\right\|\leqq t\right)^{-1}$.

Proof. Since the projection from $S^{\mathbb{N}}$ into S^M is $\mu^{\mathbb{N}}$-perfect for all $M\subseteq\mathbb{N}$ (see Proposition II.3.1 in [1]) (2.3.1) follows from Proposition 2.2 and since furthermore $b\to\|\sum b_i\|$ for $b\in B^n$ is even and subadditive (2.3.2) follows from Theorem III.1.2 in [1]. □

3. The Central Limit Theorem

In all of this section we let $(B, \|\cdot\|)$ be a Banach space, $(S, \mathscr{S}, \mu)$ a probability space and $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}}, \mu^{\mathbb{N}})$ the countable product of $(S, \mathscr{S}, \mu)$.

Furthermore, we let f be a B-valued function on $(S, \mathscr{S}, \mu)$ and we define the sequence of B-valued functions $\{U_n(f)\}_{n\in\mathbb{N}}$ on $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}}, \mu^{\mathbb{N}})$ by

$$U_n(f,s)=n^{-\frac{1}{2}}\sum_{j=1}^{n} f(s_j) \qquad \forall s=(s_j)_{j\in\mathbb{N}}\in S^{\mathbb{N}}\ \forall n\in\mathbb{N}.$$

Definition 3.1. We say that f satisfies the central limit theorem or $f\in CLT(B,\mu)$, if there exists a Radon measure γ_f on $(B, \|\cdot\|)$ such that $U_n(f)\xrightarrow{\sim}\gamma_f$. □

Proposition 3.2. *Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\pi=(\pi_n)_{n\in\mathbb{N}}$ a sequence of independent, identically distributed S-valued random variables on $(\Omega, \mathscr{F}, P)$, with common distribution μ, and let for all $n\in\mathbb{N}$ the B-valued function W_n on $(\Omega, \mathscr{F}, P)$ be defined in the following way*

$$W_n(\omega)=U_n(f)\circ\pi(\omega)=U_n(f,\pi(\omega))=n^{-\frac{1}{2}}\sum_{j=1}^{n} f(\pi_j(\omega)) \qquad \forall\omega\in\Omega$$

then the following two statements hold:

$$f\in CLT(B,\mu)\Rightarrow W_n\xrightarrow{\sim}\gamma_f \tag{3.2.1}$$

$$\left.\begin{array}{l} W_n\xrightarrow{\sim}\gamma_f, \quad \text{for some Radon measure} \\ \gamma_f \text{ on } (B, \|\cdot\|), \text{ and } \pi \text{ is } P\text{-perfect}\end{array}\right\}\Rightarrow f\in CLT(B,\mu). \tag{3.2.2}$$

Proof. Let $g \in C(B)$ then by (2.2.3)

$$(g \circ U_n(f))_* \circ \pi \leqq (g \circ W_n)_* \leqq (g \circ W_n)^* \leqq (g \circ U_n(f))^* \circ \pi \qquad P\text{-a.s.}$$

and if π is P-perfect we have by (2.2.4)

$$(g \circ U_n(f))_* \circ \pi = (g \circ W_n)_*, \qquad (g \circ W_n)^* = (g \circ U_n(f))^* \circ \pi, \qquad P\text{-a.s.}$$

so using (2.1.1) and (2.2.1) the proof is completed. □

Before investigating $CLT(B, \mu)$ we need the following nice results:

Proposition 3.3. *Let g be a real-valued function on $(S, \mathcal{S}, \mu)$. If*

$$\lim_n (\mu^{\mathbb{N}})^* \left(\{ \in S^{\mathbb{N}} \left| n^{-1} \sum_{i=1}^{n} g(s_i) - a \right| > \varepsilon \right) = 0 \qquad \forall \varepsilon > 0$$

then g is μ-measurable.

Proof. See Example II.3.2 in [1]. □

Lemma 3.4. *Let g be a real-valued random variable on $(S, \mathcal{S}, \mu)$ satisfying*

$$\forall \xi > 0 \qquad \exists T: (\mu^{\mathbb{N}})(|U_n(g)| > T) < \xi \qquad \forall n \in \mathbb{N} \tag{3.4.1}$$

then $\int_S g^2 d\mu < \infty$.

Proof. By a standard symmetrization argument we can assume that g is symmetric. Let φ resp. φ_n be the characteristic function of g resp. $U_n(g)$, then φ is a real-valued positive function and $\varphi_n(t) = \varphi(t \cdot n^{-\frac{1}{2}})^n$ for all $t \in \mathbb{R}$.

Let $\xi = 1/12$, choose T in (3.4.1) and choose δ so that

$$|(1 - e^{it})| < \xi \qquad \forall t: |t| \leqq \delta$$

then since

$$1 - \varphi_n(t) \leqq \int_{S^{\mathbb{N}}} |1 - e^{itU_n(g)}| d\mu^{\mathbb{N}} \leqq 2\xi + \int_{\{|U_n(g)| \leqq T\}} |1 - e^{itU_n(g)}| d\mu^{\mathbb{N}}$$

$$0 \leqq 1 - \varphi(tn^{-\frac{1}{2}})^n \leqq \tfrac{1}{4} \qquad \forall |t| \leqq \delta/T, \ \forall n \in \mathbb{N}. \tag{3.4.2}$$

For all $z \in [0, 1]$ satisfying $(1 - z^n) \leqq \frac{1}{4}$ we have that

$$(1 - z^n) = (1 - z)(1 + z + \ldots + z^{n-1}) \geqq n 3/4 (1 - z). \tag{3.4.3}$$

Now let $t_0 \in [-\delta/T, \delta/T]$ be fixed then using (3.4.3) and (3.4.2) we get

$$\begin{aligned} \liminf_{t \to 0} (1 - \varphi(t)) \cdot t^{-2} &\leqq \liminf_{n \to \infty} n(1 - \varphi(t_0 n^{-\frac{1}{2}}) t_0^{-2} \\ &\leqq \liminf_{n \to \infty} \tfrac{4}{3}(1 - \varphi(t_0 n^{-\frac{1}{2}})^n) t_0^{-2} \\ &\leqq (3t_0^2)^{-1} \end{aligned}$$

so by Theorem 2.3.1 in [6] the proof is completed. □

Lemma 3.5. *If* $\{U_n(f)\}$ *is eventually tight then* $f \in L_w^2(B, \mu)$.

Proof. By (2.3.1)

$$\mu^*(\|f\| > t) \leqq (\mu^{\mathbb{N}})^*(\|U_n(f)\| > t/n) + (\mu^{\mathbb{N}})^*(\|U_{n-1}(f)\| > t/n-1) \quad (3.5.1)$$

for $\forall n \geqq 1$ and if $\{U_n(f)\}$ is eventually tight

$$\forall \varepsilon > 0 \quad \exists n_0 \geqq 1 \quad \exists T: (\mu^{\mathbb{N}})^*(\|U_n(f)\| > T) < \varepsilon \quad \forall n \geqq n_0 \quad (3.5.2)$$

and if we use (3.5.1) and (3.5.2) we get

$$\forall \varepsilon > 0 \quad \exists T: (\mu^{\mathbb{N}})^*(\|U_n(f)\| > T) < \varepsilon \quad \forall n \geqq 1 \quad (3.5.3)$$

$$\forall x' \in B' \quad \forall \xi > 0 \quad \exists T_{x'} \in \mathbb{R}: (\mu^{\mathbb{N}})^*(|U_n(x'(f))| > T_{x'}) < \xi \quad \forall n \geqq 1 \quad (3.5.4)$$

$$\lim_n (\mu^{\mathbb{N}})^*(\{s \in S^{\mathbb{N}} \,|\, |n^{-\frac{1}{2}} U_n(x'(f))| > \varepsilon\}) = 0 \quad \forall \varepsilon > 0. \quad (3.5.5)$$

By (3.5.5) and Proposition 3.3 we have that $x'(f)$ is μ-measurable, and by (3.5.4) and Lemma 3.4 that $\int_S x'(f)^2 \, d\mu$ is finite i.e. $f \in L_w^2(B, \mu)$. □

We can now prove the following important result:

Theorem 3.6. *The following two statements are equivalent:*

$$f \in CLT(B, \mu) \quad (3.6.1)$$

$$\{U_n(f)\} \quad \text{is eventually tight.} \quad (3.6.2)$$

Proof.

(3.6.1) ⇒ (3.6.2): Proposition 7.17 in [5].

(3.6.2) ⇒ (3.6.1): Let $\psi = \{e^{ix'} \,|\, x' \in B'\}$, then ψ is a selfadjoint semigroup of bounded, continuous and complex-valued functions on B (see e.g. Definition 1.10 in [5]). By Lemma 3.5 $f \in L_w^2(B, \mu)$ so $x'(f)$ satisfies the real central limit theorem i.e.

$$\lim_{n \to \infty} \int_{S^{\mathbb{N}}} e^{ix'(U_n(f))} d\mu^{\mathbb{N}} \quad \text{exists } \forall x' \in B'.$$

Now let $b_1, b_2 \in B$ so that $b_1 \neq b_2$, then by Hahn-Banach theorem (see e.g. [3] Sect. II.3) we can find $x' \in B'$ such that $x'(b_1) \neq x'(b_2)$. Furthermore it is possible to find t so that $e^{itx'(b_1)} \neq e^{itx'(b_2)}$ (take $t = 1$ or $\sqrt{2}$). Since $tx' \in B'$ we find that ψ separates points in B, so by Theorem 7.11 (case 3, Remark (1)) in [5] $f \in CLT(B, \mu)$. □

We will use the equivalence shown in Theorem 3.6 to find necessary and/or sufficient conditions for $CLT(B, \mu)$.

Proposition 3.7. *If* $f \in CLT(B, \mu)$ *then*

$$\forall \xi > 0 \quad \exists T \in \mathbb{R}: (\mu^{\mathbb{N}})^*(\|U_n(f)\| > T) < \xi \quad \forall n \geqq 1 \quad (3.7.1)$$

$$\gamma_f \text{ is a centered gaussian measure} \quad (3.7.2)$$

$$\int_S x'(f) d\mu = 0, \quad \int_S x'(f) y'(f) d\mu = \int_B x' y' d\gamma_f < \infty \quad \forall x', y' \in B' \quad (3.7.3)$$

$$\exists\, k\in\mathbb{R}: (\mu^{\mathbb{N}})^*(\|U_n(f)\|>t)\leqq k/t^2 \qquad \forall t\geqq 0\ \forall n\geqq 1 \tag{3.7.4}$$

$$\int_S^* \|f\|^p d\mu<\infty, \qquad \sup_n \int_{S^{\mathbb{N}}}^* \|U_n(f)\|^p d\mu^{\mathbb{N}}<\infty \qquad \forall p<2 \tag{3.7.5}$$

$$(\mu^{\mathbb{N}})_*(n^{-\frac{1}{2}}\|U_n(f)\| \xrightarrow[n\to\infty]{} 0)=1. \tag{3.7.6}$$

Proof. From Lemma 3.5 and Theorem 3.6 follow (3.7.1)–(3.7.3) and (3.7.6) is a consequence of a theorem by M. Talagrand in [7]. Furthermore, (3.7.5) follows from (3.7.4). Now choose T in (3.7.1) with $\varepsilon=1/9$ and let

$$C_n(t)^{-1}=\inf_{j\leqq n} \mu^{\mathbb{N}}(\|U_j(f)\|^*\leqq t/\sqrt{j}),$$

then by (2.2.1) and (2.3.2)

$$\begin{aligned}
\mu(\|f\|^*\leqq t)^n &= 1-\mu^{\mathbb{N}}(\max_{k\leqq n} \|f(s_k)\|^*>t)\\
&\geqq 1-2\mu^{\mathbb{N}}\left(\max_{k\leqq n}\left\|\sum_{j=1}^k f(s_j)\right\|^*>t/2\right)\\
&\geqq 1-2\mu^{\mathbb{N}}(\|U_n(f)\|^*>t/6\sqrt{n})(1+C_n(t/6))\\
&\geqq 1-2\cdot 1/9\cdot(1+8/9)\\
&\geqq \tfrac{1}{2} \qquad \forall t\in[6T\sqrt{n}, 6T\sqrt{n+1}) \qquad \forall n\geqq 1
\end{aligned}$$

i.e. by (2.2.1)

$$\mu(\|f\|^*>t)\leqq 1-2^{-1/n} \qquad 6\sqrt{n}T\leqq t<6\sqrt{n+1}T \qquad \forall n\geqq 0 \tag{3.7.7}$$

so we can copy the proof of Lemma 4.9 in [4] (p. 94) where we use (3.7.7) instead of symmetrization. □

Theorem 3.8. *If $f\in CLT(B,\mu)$, ψ is a $\mathbb{R}_+$-valued function on $\mathbb{R}_+$ and φ is a $\mathbb{R}$-valued function on B such that*

$$\psi \text{ is increasing, continuously differentiable} \tag{3.8.1}$$

$$\psi(0)=0, \qquad \int_0^\infty \psi'(t)/t^2\,dt<\infty \tag{3.8.2}$$

$$\varphi \text{ is continuous } \gamma_f\text{-a.s.} \tag{3.8.3}$$

$$\exists\, k\in\mathbb{R} \qquad \forall x\in B: |\varphi(x)|\leqq k+\psi(\|x\|) \tag{3.8.4}$$

then

$$\begin{aligned}
\lim_n \int_{S^{\mathbb{N}}}^* \varphi(U_n(f))\,d\mu^{\mathbb{N}} &= \lim_n \int_{*S^{\mathbb{N}}} \varphi(U_n(f))\,d\mu^{\mathbb{N}}\\
&= \int_B \varphi\, d\gamma_f<\infty.
\end{aligned} \tag{3.8.5}$$

Proof. By (3.8.1) there exists a $\mathbb{R}_+$-valued, continuous and increasing function $\tilde{\psi}$ on $\mathbb{R}_+$ so that

$$\tilde{\psi}(0)=1, \qquad \tilde{\psi}(t)\xrightarrow[t\to\infty]{}\infty, \qquad \int_0^\infty \psi'(t)\tilde{\psi}(t)/t^2\,dt<\infty.$$

Let ψ_0 be defined by $\psi_0'(t)=\psi'(t)\tilde{\psi}(t)+\sqrt{t}$ for all $t\in\mathbb{R}_+$, with the condition $\psi_0(0)=1$, then

$$\psi_0(t)\xrightarrow[t\to\infty]{}\infty, \qquad \lim_{t\to\infty}\frac{\psi(t)}{\psi_0(t)}=\lim\frac{\psi'(t)}{\psi_0'(t)}=0$$

i.e.

$$\forall\xi>0 \quad \exists t_0\colon \psi(t)\leqq\psi(t_0)+\xi\psi_0(t) \quad \forall t\in\mathbb{R}_+$$

i.e.

$$\psi_0 \text{ satisfies (3.8.1) and (3.8.2)} \tag{3.8.6}$$

$$\forall\xi>0 \quad \exists K_1 \quad \forall x\in B\colon |\varphi(x)|\leqq K_1+\xi\psi_0(\|x\|). \tag{3.8.7}$$

By Example II.1.5 and Theorem II.1.2 both in [1] we have that $\psi_0(\|U_n(f)\|)^* =\psi_0(\|U_n(f)\|^*)$ $\mu^{\mathbb{N}}$-a.s. so if we use (2.2.2), (2.2.1) and (3.7.4) we get

$$\begin{aligned}\exists\tilde{k}\in\mathbb{R}\colon \int_{S^{\mathbb{N}}}^{*}\psi_0(\|U_n(f)\|)d\mu^{\mathbb{N}}&=\int_0^\infty\psi_0'(t)\mu^{\mathbb{N}}(\|U_n(f)\|^*>t)dt\\ &\leqq\tilde{k} \quad \forall n\in\mathbb{N}.\end{aligned} \tag{3.8.8}$$

Since γ_f is gaussian there exists $c\in\mathbb{R}$, so that $\gamma_f\{x\in B|\,\|x\|>t\}\leqq c/t^2$ for all $t\in\mathbb{R}_+$, so using (3.8.7), (3.8.6) and (3.8.3) we get

$$\int_B\psi_0(\|x\|)\gamma_f dx<\infty, \qquad \int_B\varphi d\gamma_f<\infty \tag{3.8.9}$$

i.e. (3.8.7)–(3.8.9) prove (3.8.5) by (7.8.12) in [5]. □

Remark. Theorem 3.8 tells us that the convergence (2.1.1) can be extended to some unbounded functions. Remark that φ given by $\varphi(x)=\psi(\|x\|)$ for $x\in B$ satisfies (3.8.3) and (3.8.4) and that $\psi(t)=t^p$ satisfies (3.8.1) and (3.8.2) for $p<2$, i.e. (3.8.5) holds for $\|U_n(f)\|^p$ for $p<2$.

Proposition 3.9. *$CLT(B,\mu)$ is a linear space.*

Proof. Let $f_1, f_2\in CLT(B,\mu)$, $a\in\mathbb{R}$ and $\xi>0$, then by Theorem 3.6 and (2.1.2)

$$\begin{aligned}&\forall i=1,2 \quad \exists K_i\in\mathscr{K}(B)\colon \limsup_n(\mu^{\mathbb{N}})^*(U_n(f_i)\notin G)<\xi\\ &\forall G\supseteq K_i, \quad \text{open.}\end{aligned} \tag{3.9.1}$$

Since K_1/a is compact and $G\supseteq K_1$ iff $G/a\supseteq K_1/a$, we have that $\{U_n(af_1)\}$ is eventually tight, i.e. by Theorem 3.6, that $af_1\in CLT(B,\mu)$.

Let $G\supseteq K_1+K_2$, then using that K_1, K_2 and K_1+K_2 are compact we have for all $a\in K_1$ and all $b\in K_2$

$$\exists r(a,b)>0\colon x+y\in G \quad \forall x\in B(a,r(a,b)), \quad \forall y\in B(b,r(a,b))$$

where $B(x,r)=\{y\in B|\,\|x-y\|<r\}$, $r\in\mathbb{R}_+$. Using the compactness we get

$$\forall b\in K_2 \quad \exists n_1(b)\in\mathbb{N} \quad \exists\{a_j(b)\}_{j=1}^{n_1(b)}\subseteq K_1\colon K_1\subseteq\bigcup_{j=1}^{n_1(b)}B(a_j(b),r(a_j(b),b)).$$

Put $r(b)=\inf\{r(a_j(b),b)\mid 1\leqq j\leqq n_1(b)\}$, then $r(b)>0$ and by compactness

$$\exists\, n_2\in\mathbb{N},\quad \{b_i\}_{i=1}^{n_2}\subseteq K_2\colon K_2\subseteq\bigcup_{i=1}^{n_2} B(b_i, r(b_i)).$$

Let $G_1=\bigcap_{i=1}^{n_2}\bigcup_{j=1}^{n_1(b_i)} B(a_j(b_i), r(a_j(b_i), b_i))$ and $G_2=\bigcup_{i=1}^{n_2} B(b_i, r(b_i))$ then $G_1\supseteq K_1$ and $G_2\supseteq K_2$, and if $x\in G_1$, $y\in G_2$ then

$$\exists\, b\in\{b_i\}_{i=1}^{n_2},\quad \exists\, a\in\{a_j(b)\}_{j=1}^{n_1(b)}\colon y\in B(b, r(b)),\ x\in B(a, r(a,b)),$$

and since $r(b)\leqq r(a,b)$ we have that $x+y\in G$. Thus $G_1\cup G_2\subseteq G$ so by 3.9.1 and subadditivity of the outer measure we get

$$\begin{aligned}\limsup_n (\mu^{\mathbb{N}})^*(U_n(f+g)\notin G)&\leqq\limsup_n (\mu^{\mathbb{N}})^*(U_n(f)\notin G_1 \text{ or } U_n(f)\notin G_2)\\ &\leqq 2\xi\end{aligned}$$

i.e. $\{U_n(f+g)\}$ is eventually tight so by Theorem 3.6 the proof is completed. □

The next two lemmas and two theorems show that f belongs to $CLT(B,\mu)$ if and only if $U_n(f)$ can, in limit, be approximated by finite dimensional, measurable functions, which can be taken as continuous projections.

Lemma 3.10. *If $f\in CLT(B,\mu)$ and h is a B-valued, linear, continuous function on B so that* $\dim h(B)<\infty$, *then*

$$h\circ f\in CLT(B,\mu),\qquad g=(f-h\circ f)\in CLT(B,\mu) \tag{3.10.1}$$

$$\gamma_g=\mathscr{L}(q) \tag{3.10.2}$$

where $q\colon B\to B$ is given by $q(x)=x-h(x)$ for $x\in B$ and $\mathscr{L}(q)$ is the distribution of q under γ_f.

Proof. For all $x'\in B'$ we have that $x'\circ h\in B'$ and since $\dim(h\circ f(S))<\infty$ we have that $h\circ f\in CLT(B,\mu)$ and then by Proposition 3.9 $g\in CLT(B,\mu)$. Furthermore $x'\circ q\in B'$ so q is a gaussian B-valued random variable and by (3.7.3) we have that

$$\begin{aligned}\sigma_q(x_1', x_2')&=\int_B (x_1'\circ q)(x_2'\circ q)\,d\gamma_f=\int_S (x_1'\circ q(f))(x_2'\circ q(f))\,d\mu\\ &=\int_S (x_1'(f-h\circ f))(x_2'(f-h\circ f))\,d\mu\\ &=\int_B x_1' x_2'\,d\gamma_g,\qquad \forall\, x_1', x_2'\in B'\end{aligned}$$

i.e. $\mathscr{L}(q)$ and γ_g have the same covariance function and therefore equal. □

Lemma 3.11. *Let γ be a gaussian Radon measure on a Banach space $(B,\|\cdot\|)$ with mean zero and covariance-function σ, then there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ of finite dimensional continuous projections on B such that*

$$h_n\colon B\to B \quad\textit{and}\quad h_{n+1}\circ h_n=h_n \qquad \forall\, n\in\mathbb{N} \tag{3.11.1}$$

$$h_n(x)\xrightarrow[n\to\infty]{} x \qquad \gamma\text{-a.a.}\quad x\in B. \tag{3.11.2}$$

Proof. Since γ is Radon we have that $L^2(\gamma)$ is a separable Hilbert space and B' is a linear subspace of $L^2(\gamma)$. Let $\mathscr{H}' = cl_{L^2_{(\cdot,\cdot)}}(B')$, then $\mathscr{H}'$ is separable and therefore there exist, an orthonormal base $\{l'_n\}$ for $\mathscr{H}'$ so that $\{l'_n\} \subseteq B'$. Let $l_n = \int_B l'_n(x)\, x\, d\gamma$ for all $n \in \mathbb{N}$ then since $l'_n(x)\, x$ is Bochner-integrable we have $\{l_n\}$, and

$$l'_j(l_i) = \int_B l'_i(x)\, l'_j(x)\, d\gamma = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases} \tag{3.11.3}$$

Now for all $n \in \mathbb{N}$ define h_n by

$$h_n(x) = \sum_{j=1}^{n} l'_j(x)\, l_j \qquad \forall x \in B$$

then h_n is a linear, continuous function, $\dim h_n(B) < \infty$ and

$$\begin{aligned} h_n(h_n(x)) &= \sum_{j=1}^{n} l'_j \left(\sum_{i=1}^{n} l'_i(x)\, l_i \right) l_j \\ &= \sum_{j=1}^{n} l'_j(x)\, l_j \\ &= h_n(x) \end{aligned} \tag{3.11.4}$$

$$\begin{aligned} h_{n+1}(h_n(x)) &= \sum_{j=1}^{n+1} l'_j \left(\sum_{i=1}^{n} l'_i(x)\, l_i \right) l_j \\ &= \sum_{j=1}^{n} l'_j(x)\, l_j \\ &= h_n(x) \end{aligned} \tag{3.11.5}$$

i.e. (3.11.4) shows that h_n is a projection and (3.11.5) shows (3.11.1). Now if $x' = \sum_{j \in \mathbb{N}} a_j l'_j \in B'$ then

$$x' \circ h_n = \sum_{j \in \mathbb{N}} a_j l'_j \left(\sum_{i=1}^{n} l'_i(x)\, l_i \right) = \sum_{j=1}^{n} a_j l'_j \qquad \forall n \in \mathbb{N}$$

i.e.

$$x' \circ h_n \to x' \text{ in } L^2(\gamma) \qquad \forall x' \in B' \tag{3.11.6}$$

which by Theorem 5.3 in [4] (p. 99) (3.11.6) implies that

$$x' \circ h_n \xrightarrow[n \to \infty]{} x' \qquad \gamma\text{-a.s.} \qquad \forall x' \in B'. \tag{3.11.7}$$

Using that $\{l'_n\}$ are independent $N(0,1)$ random variables we have that x' and h_n are gaussian so by (3.11.7) and Theorem 5.3 case (1) in [4] (p. 99)

$$\exists h: B \to B: h_n \xrightarrow[n \to \infty]{} h \qquad \gamma\text{-a.s.} \tag{3.11.8}$$

Now there exists a separable, linear subspace B_0 of B so that $h \in B_0$ γ-a.s. and $\gamma(B_0) = 1$. Let $\{x_n\}_{n \in \mathbb{N}}$ be dense in B_0, then by Corollary 2.III.14 in [3], there exists $\{x_n'\}_{n\in\mathbb{N}}\subseteq B_0'$ so that for all $n\in\mathbb{N}$ we have $x_n'(x_n)=\|x_n\|$ and $\|x_n'\|=1$, i.e.

$$\forall x\in B_0\setminus\{0\} \quad \exists n\in\mathbb{N}: \|x-x_n\|<\tfrac{1}{4}\|x\|$$

and

$$\begin{aligned}\forall x\in B_0\setminus\{0\} \quad \exists n\in\mathbb{N}: x_n'(x)&=x_n'(x_n)+x_n'(x-x_n)\\ &\geqq \|x_n\|-\|x-x_n\|\\ &\geqq \tfrac{1}{2}\|x\|\end{aligned}$$

i.e. we have

$$\exists\{x_n'\}\subseteq B' \quad \forall x\in B_0\setminus\{0\} \quad \exists n\in\mathbb{N}: x_n'(x)>0 \tag{3.11.9}$$

and by (3.11.7) and (3.11.8)

$$x_n'(h(x)-x)=0 \quad \forall n\in\mathbb{N} \quad \mu\text{-a.a.} \quad x\in B. \tag{3.11.10}$$

Now using the properties of B_0, (3.11.9) and (3.11.10) we find that $h(x)=x$ for μ-almost all $x\in B$. □

Theorem 3.12. *If $f\in CLT(B,\mu)$ then there exists a sequence $\{h_k\}$ of finite dimensional, continuous projections on B so that*

$$h_k(x)\to x \quad \gamma_f\text{-a.s.}, \quad h_{k+1}(h_k)=h_k \quad \forall k\in\mathbb{N} \tag{3.12.1}$$

$$\limsup_n \int_{S^{\mathbb{N}}}^{*} \varphi(\|U_n(f-h_k\circ f)\|)\,d\mu^{\mathbb{N}} \xrightarrow[k\to\infty]{} 0 \tag{3.12.2}$$

whenever φ is a $\mathbb{R}_+$-valued, increasing, continuously differentiable function on $\mathbb{R}_+$ satisfying

$$\varphi(0)=0, \quad \int_0^\infty \varphi'(t)/t^2\,dt<\infty. \tag{3.12.3}$$

Proof. Using Lemma 3.11 on γ_f, there exists a sequence $\{h_k\}_{k\in\mathbb{N}}$ of finite dimensional, continuous projections on B so that (3.12.1) is satisfied, and since φ is continuous and $\varphi(0)=0$ we have by (3.11.2)

$$\varphi(\|x-h_k(x)\|)\to 0 \quad \gamma_f\text{-a.a.} \quad x\in B. \tag{3.12.4}$$

From Theorem 3.8 and Lemma 3.10 we get

$$\forall k\in\mathbb{N}: \lim_n \int_{S^{\mathbb{N}}}^{*} \varphi(\|U_n(f-h_k\circ f)\|)\,d\mu^{\mathbb{N}}=\int_B \varphi(\|x-h_k(x)\|)\,d\gamma_f. \tag{3.12.5}$$

Now let $q(x)=\sup_n\|x-h_k(x)\|$, then q is a seminorm and by (3.11.2) and Theorem 3.4 in ([4], p. 79) we have that $\int_B q^2\,d\gamma_f$ is finite. Since φ is increasing and satisfies (3.12.3) we get that there exist $K\in\mathbb{R}$

$$\varphi(t)\leqq\varphi(1)+t^2\int_0^\infty \frac{\varphi'(s)}{s^2}\,ds\leqq K(1+t^2) \quad \forall t\in\mathbb{R}_+$$

i.e.

$$\varphi(\|x-h_k(x)\|)\leqq K(1+\|x-h_k(x)\|^2)\leqq K(1+q^2(x))$$

so (3.12.2) follows from (3.12.4), (3.12.5) and the Lebesgue dominated convergence Theorem. □

Theorem 3.13. *If for all* $\varepsilon>0$ *there exists* $g\in CLT(B,\mu)$ *so that*

$$\limsup_n \left| \int_{S^{\mathbb{N}}}^{*} \varphi(U_n(f))\,d\mu^{\mathbb{N}} - \int_{S^{\mathbb{N}}}^{*} \varphi(U_n(g))\,d\mu^{\mathbb{N}} \right| < \varepsilon$$

whenever φ *is a real valued function on* B *satisfying*

$$|\varphi(b)|\leqq 1, \qquad |\varphi(a)-\varphi(b)|\leqq \|a-b\| \qquad \forall\, a,b\in B$$

then $f\in CLT(B,\mu)$.

Proof. Follows from Corollary 8.11 in [5].

Example 3.14. Assume that f satisfies the conditions in Theorem 1.1 in [2], i.e. for all $m\in\mathbb{N}$ exists a B-valued function Λ_m on B so that

$$\Lambda_m\circ f \quad \text{is measurable}, \quad \dim \Lambda_m(B)<\infty \tag{3.14.1}$$

$$\int_S \Lambda_m\circ f\,d\mu=0, \quad \int_S \|\Lambda_m\circ f\|^2\,d\mu<\infty \tag{3.14.2}$$

$$\exists\, n_0(m) \quad \forall\, n\geqq n_0:\ (\mu^{\mathbb{N}})^*(\|U_n(f-\Lambda_m\circ f)\|>1/m)\leqq 1/m. \tag{3.14.3}$$

Now (3.14.1) and (3.14.2) imply that $\Lambda_m\circ f\in CLT(B,\mu)$ and if φ is a function satisfying the conditions in Theorem 3.13 then

$$\begin{aligned}
&\left| \int_{S^{\mathbb{N}}}^{*} \varphi(U_n(f))\,d\mu^{\mathbb{N}} - \int_{S^{\mathbb{N}}} \varphi(U_n(\Lambda_m\circ f))\,d\mu^{\mathbb{N}} \right| \\
&\quad\leqq \left| \int_{S^{\mathbb{N}}}^{*} \varphi(U_n(f-\Lambda_m\circ f))\,d\mu^{\mathbb{N}} \right| \\
&\quad\leqq \int_{S^{\mathbb{N}}}^{*} 1\wedge \|U_n(f-\Lambda_m\circ f)\|\,d\mu^{\mathbb{N}} \\
&\quad\leqq 1/m+(\mu^{\mathbb{N}})^*(\|U_n(f-\Lambda_m\circ f)\|>1/m)
\end{aligned}$$

i.e. $f\in CLT(B,\mu)$ by Theorem 3.13. Furthermore Theorem 3.12 tells us that Λ_m can be taken as a continuous projection, i.e. Theorem 3.12 and Theorem 3.13 gives us necessary and sufficient conditions for $f\in CLT(B,\mu)$. □

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