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A Note on Invariant Measures for Markov Maps of an Interval

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Summary. We present two examples of Markov maps which satisfy the expanding condition, Rényi's condition and do not admit any absolutely continuous invariant measures. These examples are counterexamples to the theorem formulated in [1, p. 1].

1. Introduction

We start with the definition of Markov map.

Definition 1.1. A many-to-one transformation φ from an interval I (bounded or not) into itself is called a *Markov map* with respect to a finite or countable family $\{I_k\colon k \in K\}$ of disjoint, open intervals I_k iff it satisfies the following conditions:

(1.M1) φ is defined on $\tilde{\tilde{I}} = \bigcup_{k \in K} I_k$ and $cl(\tilde{\tilde{I}}) = I$;

(1.M2) for each $k \in K$, the function $\varphi_k = \varphi | I_k$ is strictly monotonic, differentiable, and its derivative φ'_k is a locally Lipschitzean function which can be extended to $cl(I_k)$;

(1.M3) for each $j, k \in K$, if $\varphi(I_j) \cap I_k \neq \emptyset$, then $I_k \subseteqq \varphi(I_j)$;

(1.M4) for each $(j, k) \in K^2$, there exists an integer $n > 0$ such that $I_k \subseteqq \varphi^n(I_j)$.

The following conditions:

(1.H1) $\inf\{|(\varphi^n)'(x)|\colon x \in \tilde{\tilde{I}}\} > 1$ for some n,

(1.H2) $\sup\{|\varphi''(z)|/(\varphi'(y))^2\colon y, z \in \tilde{\tilde{I}}\} < \infty$ are called *expanding condition*, and *Rényi's condition*, respectively.

In this note we present two examples of Markov maps which satisfy the conditions (1.H1), (1.H2) and do not admit any absolutely continuous invariant measures. These examples are counterexamples to the theorem formulated in [1, p. 1].

It seems worth mentioning that the theorem in question holds true under the following two additional conditions:

(1.H3) $\inf\{|\varphi(I_k)|\colon k\in K\}>0$;

(1.H4) if φ is defined on unbounded interval I, then $\lim_{y\to\infty} R(y)=0$ where $R(y) = \sup_{k\in K}(|I_k|^{-1}\int_{A_y}\sigma_k(x)\,dx)$, $|I_k|$ denotes the length of I_k, $\sigma_k=|(\varphi_k^{-1})'|$ and $A_y = \{x\in I\colon |x|>y\}$. The proof of this fact can be obtained by a suitable modification of the proofs of Theorems 1 [3] and 3.1 [2].

2. Counterexamples

We disprove here the theorem on the existence of absolutely continuous invariant measures for Markov maps in [1, p. 1] by providing two counterexamples.

Counterexample 1. First, let us consider the stochastic matrix

$$R=[r_{ij}]_{i,j=1}^{\infty} \qquad (\text{i.e., } r_{ij}\geqq 0,\ \sum_j r_{ij}=1)$$

which consists of the following elements:

$$r_{ij}=\begin{cases} c\,2^{-(2^i-j+1)} & \text{if } j=1,2,\ldots,2^i;\\ c\,2^{-2^i} & \text{if } j=2^i+1;\\ (1-2c) & \text{if } j=2^i+2;\\ c\,2^{-(j-2^i-2)} & \text{if } j=2^i+3,\ 2^i+4,\ldots; \end{cases} \tag{2.1}$$

for $i=1,2,\ldots$; where $0<c<1/2$.

We associate with the matrix R a piecewise linear transformation $\psi\colon I\to I$ ($I = [0,\infty)$) in the following way. For $i=1,2,\ldots$, we put $I_{ij}=[a_{ij},a_{ij+1})$ where

$$a_{ij}=(i-1)+\sum_{k=1}^{j-1} r_{ik} \quad \text{if } j=2,3,\ldots, \text{ and } a_{i1}=i-1.$$

Then we determine, for each pair (i,j), a linear mapping ψ_{ij} (increasing or decreasing) from I_{ij} onto whole interval $[j-1,j)$. Finally, we define the desired transformation by $\psi(x)=\psi_{ij}(x)$ iff $x\in I_{ij}$.

It is evident that ψ is a Markov map (in the sense of the Def. 1.1) which satisfies both the conditions (1.H1) and (1.H2). It is also evident that ψ does not satisfy (1.H4). We shall see later (cf. Th. 2.1) that ψ is without any absolutely continuous invariant measure.

Counterexample 2. We obtain the second counterexample by a simple modification of the previous one.

To this end, we first define an auxiliary piecewise linear transformation λ from $I=[0,\infty)$ onto whole unit interval $\tilde{I}=[0,1]$ by the following formula:

$$\lambda(x)=\lambda_i(x) \qquad \text{iff } x\in[i-1,i)$$

where λ_i: $[i-1, i) \to \tilde{I}_i$ is a linear mapping (e.g., increasing) from $[i-1, i)$ onto the whole interval $\tilde{I}_i = [1-2^{-(i-1)}, 1-2^{-i})$, $i=1, 2, \ldots$. Note that $\lambda_i(I_{ij}) = \tilde{I}_{ij} = [\tilde{a}_{ij}, a_{ij+1})$ where

$$\tilde{a}_{ij} = (1-2^{-(i-1)}) + 2^{-i} \sum_{k=1}^{j-1} r_{ik} \quad \text{if } j \geqq 2, \quad \text{and} \quad \tilde{a}_{i1} = 1-2^{-(i-1)}.$$

Then we define $\tilde{\psi} = \lambda \circ \psi \circ \lambda^{-1}$.

The transformation $\tilde{\psi}$ is a Markov map which satisfies the conditions (1.H1) and (1.H2). Also it is evident that $\tilde{\psi}$ does not satisfy (1.H3).

Now we prove that both ψ and $\tilde{\psi}$ do not admit any absolutely continuous invariant measures.

Theorem 2.1. *Let $P, \tilde{P}$ be the Frobenius-Perron operators corresponding to the Markov maps $\psi, \tilde{\psi}$, respectively. Then the equations $Pg=g$ and $\tilde{P}g=g$ have no non-negative, integrable, non-trivial solutions.*

Proof. We first give the proof in the case when the Frobenius-Perron operator is associated with ψ. For an arbitrary (but fixed) $i \geqq 1$, let us put $\psi_i(x) = \psi_{ij}(x)$ iff $x \in I_{ij}$ (ψ_i is a one-to-one piecewise linear mapping from $[i-1, i)$ onto $[0, \infty)$).

From the definitions of the Frobenius-Perron operator (see Def. 2.2 in [2]), and ψ_i it follows that ($g \geqq 0$ and $\|g\|=1$)

$$\|1_{[j-1,j)} P^n g\| = \sum_{i=1}^{s} \|1_{I_{ij}} P^{n-1} g\| + \sum_{i>s} \|1_{I_{ij}} P^{n-1} g\|$$

where

$$I_{ij} = \psi_i^{-1}([j-1, j)) = \psi_{ij}^{-1}([j-1, j)).$$

(Here and in what follows we denote by $\|g\|$ the integral over I of an integrable function g.)

Using this equality successively, we find that

$$\begin{aligned}
\|1_{[j-1,j)} P^n g\| = &\sum_{j_n=1}^{s} \cdots \sum_{j_1=1}^{s} C_{j_1 \ldots j_n, j}(g) \\
&+ \sum_{j_n > s} \sum_{j_{n-1}=1}^{s} \cdots \sum_{j_1=1}^{s} C_{j_1 \ldots j_n, j}(g) \\
&+ \sum_{j_n=1}^{\infty} \sum_{j_{n-1}>s} \sum_{j_{n-2}=1}^{s} \cdots \sum_{j_1=1}^{s} C_{j_1 \ldots j_n, j}(g) \\
&+ \ldots + \sum_{j_n=1}^{\infty} \cdots \sum_{j_2=1}^{\infty} \sum_{j_1>s} C_{j_1 \ldots j_n, j}(g)
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
C_{j_1 \ldots j_n, j}(g) &= \|1_{I_{j_1 \ldots j_n, j}} g\|, \\
I_{j_1 \ldots j_n, j} &= \psi_{j_n}^{-1} \circ \ldots \circ \psi_{j_1}^{-1}([j-1, j)).
\end{aligned} \tag{2.3}$$

From (2.3) it follows that

$$C_{j_1 \ldots j_n, j}(g_1) = r_{1 j_n} r_{j_n j_{n-1}} \ldots r_{j_2 j_1} r_{j_1 j},$$

for $g_1 = \sum_{j=1}^{\infty} r_{1j} 1_{[j-1, j)}$.

From this last equation, (2.1) and (2.2) it follows that

$$\|1_{[j-1, j)} P^n g_1\| \leqq \tilde{c}(q_s^n + (1 + q_s + \ldots + q_s^{n-2}) Q_s) + r_{s+1 j}, \tag{2.4}$$

provided that $s \geqq 5$ where $\tilde{c} = \max\{(1-2c), c/2\}$,

$$q_s = \sum_{j=1}^{s} r_{1j} = (1 - c\, 2^{-(s-4)}), \qquad Q_s = \sum_{j=1}^{s} r_{s+1 j} \leqq c\, 2^{-(2^{s+1} - s)}.$$

Now, if for each $s \geqq 5$ we take n_s such that $q_s^{n_s} < 1/s$, then, by the inequality (2.4), we obtain

$$0 \leqq \|1_{[j-1, j)} P^{n_s} g_1\| \leqq \tilde{c}(1/s + 2^{-(2^{s+1} - 2s + 4)}) + r_{s+1 j}.$$

Hence, for an arbitrary interval $[j-1, j)$ we have

$$\liminf_{n \to \infty} \|1_{[j-1, j)} P^n g_1\| = 0. \tag{2.5}$$

We now want to show that (2.5) holds true for an arbitrary non-negative, integrable function f. To this end, we remark that the inequality $f \leqq s g_1 + (f - s g_1)^+$ implies

$$0 \leqq \|1_{[j-1, j)} P^n f\| \leqq s \|1_{[j-1, j)} P^n g_1\| + \|(f - s g_1)^+\|$$

because P^n is a contraction. From this inequality and (2.5) we get the desired result because $\|(f - s g_1)^+\| \downarrow 0$, as $s \to \infty$.

If $Pf = f$ for some non-negative, integrable f, then

$$0 \leqq \|1_{[j-1, j)} f\| = \liminf_{n \to \infty} \|1_{[j-1, j)} P^n f\| = 0$$

for each $j = 1, 2, \ldots$. So, we get $f = 0$ (a.e.). This finishes the proof concerning P.

Regarding $\tilde{P}$, since $\tilde{\psi} = \lambda \circ \psi \circ \lambda^{-1}$, the following equality is valid:

$$\|1_{\tilde{I}_j} \tilde{P}^n f\| = \|1_{[j-1, j)} P^n (P_{\lambda^{-1}} f)\| \qquad \text{for any non-negative,}$$

integrable function f defined on $[0, 1]$, and for each $j = 1, 2, \ldots$.

We already know (from the previous case) that

$$\liminf_{n \to \infty} \|1_{[j-1, j)} P^n (P_{\lambda^{-1}} f)\| = 0, \qquad \text{for any non-negative,}$$

integrable function f defined on $[0, 1]$, and $j = 1, 2, \ldots$. Thus, for any such a function f we have

$$\liminf_{n \to \infty} \|1_{\tilde{I}_j} \tilde{P}^n f\| = 0.$$

From this it follows that if $\tilde{P}f = f$ for some $f \geqq 0$, then $f = 0$ (a.e.). This finishes the proof in the second case, and completes the proof of the theorem.

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