Z. Wahrscheinlichkeitstheorie verw. Gebiete
70, 341–344 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

An Iterated Logarithm Law for Families of Brownian Paths

Raoul LePage and Bertram M. Schreiber*

Department of Statistics and Probability, Michigan State University, Wells Hall, East Lansing, MI 48824-1027, USA

Summary. If n is large, a plot of $\log n$ independent Brownian paths over $[0, n]$ is nearly certain to give the appearance of a shaded region having square root boundaries.

Introduction

Consider a sequence b_i, $i=1, 2, \dots$ of mutually independent Brownian motions in one dimension ([3], Sect. 1.2). Write $Lx=\log_e x$ and $a(t)=(2tLLt)^{1/2}$ for $x>0$, $t \geqq e$. For an arbitrary $c>0$, define plots $\mathscr{P}_n$, the region $\mathscr{R}$, and the "distance" $|\mathscr{P}_n-\mathscr{R}|$ by

$$\begin{aligned} \mathscr{P}_n &= \{(t, \gamma_i(n,t)) \colon 0 \leqq t \leqq 1,\ 1 \leqq i \leqq cLn\}, \qquad n=3,4,\dots \\ \mathscr{R} &= \{(t,y) \colon 0 \leqq t \leqq 1,\ |y| \leqq \sqrt{t}\} \\ |\mathscr{P}_n - \mathscr{R}| &= U_n + V_n, \end{aligned} \tag{1}$$

where

$$\gamma_i(n,t) = b_i(nt)/a(n)$$

$$U_n = \max_{(t,y)\in\mathscr{R}} \ \min_{1 \leqq i \leqq cLn} |\gamma_i(n,t) - y|$$

$$V_n = \max_{1 \leqq i \leqq cLn} \ \max_{0 \leqq t \leqq 1} \ \min_{|y| \leqq \sqrt{t}} |\gamma_i(n,t) - y|.$$

We prove that $|\mathscr{P}_n-\mathscr{R}|$ *converges to zero in probability as* $n \to \infty$.

To visualize this result, suppose cLn paths are simultaneously plotted over $[0, n]$, for a large n. Along with this plot we wish to show the iterated logarithm boundaries $\pm a(t)$, $e \leqq t \leqq n$. In order to properly view the latter, it is necessary to select scales for vertical *vs* horizontal plotting which are nearly in the proportion $a(n)\colon n$. In these scales, the iterated logarithm boundaries appear

* Research of this author was partially supported by the National Science Foundation under grant MCS 82-00786

to the viewer like square root boundaries, since $a(nt)/a(n)=\sqrt{t}+o(1)$, $e/n\leqq t\leqq 1$. Remarkably, it is nearly certain that the normalized paths $\{(t,\gamma_i(n,t))\colon 0\leqq t\leqq 1\}$, $1\leqq i\leqq cLn$ will appear (at finite resolution) to have completely filled the interior region between the boundaries, without crossing outside them.

As might be expected, finer resolution requires a larger n. At coarse resolution, the phenomenon is visible in plots for moderate n, as the following simulations show.

It is tempting to think of this result as providing a model for the plume of particles left by a rocket in air.

Outer law. We have need of the following result, which is Proposition 1 of [2] stated in self-contained form (see also Lemma 2.2 of [1]).

Proposition. *Let μ be any symmetric Gaussian measure on the Borel sigma-algebra of a real separable Banach space B, and denote by K_ε the ε-neighborhood in B of the unit ball K of the reproducing kernel Hilbert space in B determined by μ. Then for every $\varepsilon>0$ there exist r_0, $n_0>1$ such that for all $r<r_0$ and $n>n_0$,* $\mu(\sqrt{2LLn}\,K_\varepsilon)\geqq 1-(Ln)^{-r^2}$. □

In the present context $B=C[0,1]$, μ is Wiener measure,

$$K=\left\{h\in B\colon \exists\, g\in L^2[0,1],\int_0^1 g^2(s)\,ds\leqq 1,\ h(t)=\int_0^t g(s)\,ds,\ 0\leqq t\leqq 1\right\},$$

and $f\in K_\varepsilon$ implies

$$|f(t)|<\sqrt{t}+\varepsilon,\qquad \forall\, t\in[0,1].$$

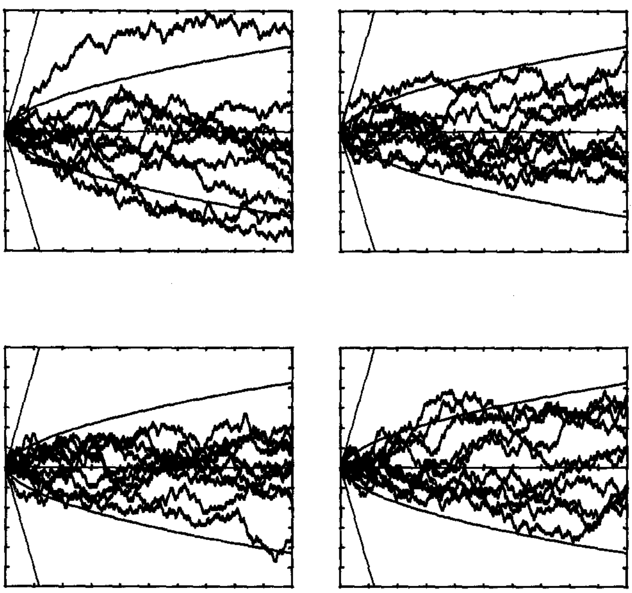

Fig. 1. Four independent plots, each of ten independent paths for $n=500$

Now $b(nt)/\sqrt{n}$, $0 \leqq t \leqq 1$ is a Brownian path over $[0, 1]$. It follows from the proposition above that for every $\varepsilon > 0$ there exist $r_0, n_0 > 1$ such that for $1 < r < r_0$ and $n > n_0$,

$$P(\exists\, i, t \text{ with } 1 \leqq i \leqq cLn, 0 \leqq t \leqq 1, |\gamma_i(n, t)| > \varepsilon + \sqrt{t})$$
$$\leqq c(Ln)(Ln)^{-r^2} \to 0, \quad \text{as} \quad n \to \infty.$$

This implies the outer law: For every $\varepsilon > 0$, $P(V_n > \varepsilon) \to 0$ *as* $n \to \infty$.

Inner law. We prove that the plot $\mathscr{P}_n$ is, with high probability, close to all points of $\mathscr{R}$. For an arbitrary $0 < \varepsilon < 1$, consider m_n random vertical closed line segments of length $\varepsilon/2$ whose centers lie in $\mathscr{R}$. The centers of these segments are taken to be i.i.d. samples from the uniform distribution on $\mathscr{R} - \{(t, y): 0 \leqq t < \varepsilon/2\}$, and independent of the Brownian paths. Clearly, if $m_n \to \infty$ then

$$P(\max_{(t, y) \in \mathscr{R}} \min_{1 \leqq i \leqq m_n} \|(t, y) - \pi_i\| > \varepsilon) \to 0, \quad \text{as} \quad n \to \infty \tag{2}$$

where π_i, $i = 1, 2, \ldots$ denote the segment centers.

Lemma. *For each* $\varepsilon > 0$ *there is a choice of* $m_n \to \infty$ *for which* $P(A_n) \to 1$ *where* $A_n = A_n(\varepsilon)$ *is defined to be the event that every one of the* m_n *segments is intersected by at least one of the paths* $\{(t, \gamma_i(n, t)), 0 \leqq t \leqq n\}$, $1 \leqq i \leqq cLn$. □

Proof of the Lemma. Let $\mathscr{F}$ be the σ-algebra generated by the cLn paths, and A be the event that the first random segment is intersected by at least one of the cLn paths. The center π_1 of the first segment will be written (T, Y). Since the paths are conditionally i.i.d. given (T, Y),

$$\begin{aligned} P(A^c) &= EP(\text{all } cLn \text{ paths miss the first segment} \mid T, Y) \\ &= E(1 - P(b(nT) \in a(n)[Y - \varepsilon/4, Y + \varepsilon/4] \mid T, Y))^{cLn} \\ &= E(1 - P(b(1) \in \sqrt{2LLn/T}\,[Y - \varepsilon/4, Y + \varepsilon/4] \mid T, Y))^{cLn}. \end{aligned}$$

By the symmetry of the standard normal density and its monotonicity over $[0, \infty)$, and since $(\sqrt{2LLn/T})(\sqrt{T} - \varepsilon/4) > 0$ a.s., the last line above is

$$\begin{aligned} &\leqq E(1 - P(b(1) \in \sqrt{2LLn/T}\,[\sqrt{T} - \varepsilon/4, \sqrt{T} + \varepsilon/4] \mid T, Y))^{cLn} \\ &= E(1 - P(b(1) \in \sqrt{2LLn}\,[1 - \varepsilon/4\sqrt{T}, 1 + \varepsilon/4\sqrt{T}] \mid T, Y))^{cLn}. \end{aligned}$$

Since $P(b(1) \in [u - \alpha v, u + \alpha v])$ is increasing in α if $\alpha > 0$ and $\alpha v < u$, the above is

$$\leqq (1 - P(b(1) \in \sqrt{2LLn}\,[1 - \varepsilon/4, 1 + \varepsilon/4]))^{cLn}.$$

Provided $\sqrt{2LLn}(1 - \varepsilon/4) > 1$, the line above is

$$\begin{aligned} &\leqq (1 - (\varepsilon/2)\sqrt{2LLn}(1/\sqrt{2\pi})(Ln)^{-1})^{cLn} \\ &\leqq \exp -(c\varepsilon\sqrt{LLn}/2\sqrt{\pi}) \to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Using the fact that the m_n segments are conditionally i.i.d. given $\mathscr{F}$, and writing p_n for the bound obtained on the last line above,

$$\begin{aligned} P(A_n) &= EP(A_n \mid \mathscr{F}) \\ &= E(P(A \mid \mathscr{F}))^{m_n} \\ &\geqq (P(A))^{m_n} \\ &\leqq (1-p_n)^{m_n} \to 1, \end{aligned}$$

provided $m_n \to \infty$ slowly enough. □

From (2) *and the lemma, follow the inner law: For every* $\varepsilon > 0$, $P(U_n > \varepsilon) \to 0$ *as* $n \to \infty$.

Main Result. Combining the inner and outer laws gives the following result.

Theorem. *For every* $\varepsilon > 0$, $P(|\mathscr{P}_n - \mathscr{R}| > \varepsilon) \to 0$ *as* $n \to \infty$. □

This work was undertaken while B. Schreiber, of Wayne State University, was a Visiting Professor of Mathematics at Michigan State University, whose hospitality he wishes to acknowledge with gratitude.

References

1. deAcosta, A.: A new proof of the Hartman-Winter law of the iterated logarithm. Ann. Probab. **11**, 270–276 (1983)
2. LePage, R.: Log log law for Gaussian processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **25**, 103–108 (1973)
3. McKean, H.P.: Stochastic integrals. New York-London: Academic Press 1969

Received May 20, 1983