Z. Wahrscheinlichkeitstheorie verw. Gebiete
70, 335–340 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Lifetime of Conditioned Brownian Motion in Lipschitz Domains

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Summary. If $D \subseteq \mathbb{R}^d$, $d \geqq 3$, is bounded and has Lipschitz boundary then the expected lifetime of any Brownian h-path process in D is finite.

Let (P_x, X_t) be Brownian motion killed on exiting D. If $p(t, x, y)$ is the transition density of Brownian motion killed on exiting D and $h > 0$ is harmonic in D, set

$$p^h(t, x, y) = h(x)^{-1} \, p(t, x, y) \, h(y)$$

and let P_x^h denote the measure on continuous paths induced by p^h. These are the h-paths of Doob [5].

In this paper we show that Brownian h-paths, $h > 0$, harmonic in $D \subseteq \mathbb{R}^d$, $d \geqq 3$, have finite expected lifetime provided the bounded region D has a Lipschitz boundary. We use the boundary Harnack principle and an estimate of Dahlberg on harmonic measure.

Define for $B \subseteq \mathbb{R}^d$, $\tau_B = \inf\{t > 0 \colon X(t) \notin B\}$.

Theorem 1. *Let D be a bounded domain in $\mathbb{R}^d$, $d \geqq 3$, and suppose ∂D is Lipschitz. Then there is a constant $c(D)$ such that if h is positive and harmonic in D,*

$$E_x^h \tau_D \leqq c(D), \qquad x \in D.$$

This result is companion to Cranston-McConnell [3] where the following theorem was proved.

Theorem 2. *If D is a domain in $\mathbb{R}^2$, $h > 0$ is harmonic in D, then*

$$E_x^h \tau_D \leqq c m(D)$$

where c is an absolute constant and m is Lebesgue measure.

Note there is no assumption on the smoothness of the boundary in Theorem 2.

In addition, an example was given in [3] of a bounded domain D in $\mathbb{R}^3$ together with an h where the lifetime of the h-path process was infinite almost surely. One of the properties of Lipschitz domains is (2) below, that the number of balls in a chain connecting two points in D can be bounded in terms of the distance of those points from the boundary of D. This property is violated in the above mentioned example.

Any constants that appear will depend only on D and d. Their value may change from line to line.

We collect results of Jerison-Kenig [7] and Dahlberg [4] and add a few that will be useful. The domain D will be assumed Lipschitz and bounded. First, Hunt and Wheeden [6] showed the minimal Martin boundary, Δ_1, is equal to the Euclidean boundary for domains D with Lipschitz boundary and there are no nonminimal Martin boundary points. Thus if $x_0 \in D$ is fixed and $w^x(\cdot)=P^x(X(\tau_D)\in\cdot)$ is harmonic measure and $\Delta(Q,r)=B(Q,r)\cap D, B(Q,r)=\{y: |y-Q|<r\}$ then in [7] it is shown that all minimal harmonic functions arise as

$$K(x,Q)=\lim_{r\to 0}\frac{w^x(\Delta(Q,r))}{w^{x_0}(\Delta(Q,r))}, \quad Q\in\partial D.$$

We state two properties of Lipschitz domains needed in the proofs of the lemmas.

There exist positive numbers M and r_0 such that

(1) If $r<r_0$ and $Q\in\partial D$, there exists $A=A_r(Q)\in D$ with

$$M^{-1}r<|A-Q|<r.$$
$$M^{-1}r<\operatorname{dist}(A,\partial D).$$

(2) If $x_1, x_2\in D$ and $\operatorname{dist}(x_j,\partial D)>\varepsilon$, $j=1,2$, and $|x_1-x_2|<2^k\varepsilon$ then there is a chain of Mk balls $B_1,\ldots,B_{Mk}$ connecting x_1 and x_2 where x_1 is the center of B_1, x_2 is the center of B_{Mk}, $B_j\cap B_{j+1}\neq\phi$ for $j=1,\ldots,Mk-1$, $B_j\subset D$, $M^{-1}\operatorname{diam} B_j<\operatorname{dist}(B_j,\partial D)<M\operatorname{diam} B_j$.

The proof of (1) and (2) can be found in [7]. We will assume $\operatorname{dist}(x_0,\partial D)>Mr_0$.

The chain of balls in (2) is called a Harnack chain. Note that each ball in the chain may have its radius increased by a constant factor, namely $(1+M^{-1})$, and each dilated ball will also be in D. Thus Harnack's inequality implies that if h is positive and harmonic in D,

$$h(x)\leqq c_1 h(y), \quad x,y\in B_j, \quad j=1,\ldots,Mk,$$

where $c_1>1$ depends only on M. Therefore with x_1 and x_2 as in (2),

$$c_1^{-Mk}h(x_2)\leqq h(x_1)\leqq c_1^{Mk}h(x_2). \tag{3}$$

We will use (3) to obtain upper and lower bounds for $K(x,Q)$. Neil Falkner has pointed out that one can apply (3) directly to general harmonic h instead of $K(\cdot,Q)$ with $x_2=x_0$, $x_1=x$ and obtain Lemma 5 with the inclusion (9) replaced by $A_n\subseteq\{x: \operatorname{dist}(x,\partial D)\leqq M2^{-n\gamma}\}$ for some $\gamma>0$. Our method gives additional information about the kernel function $K(\cdot,Q)$.

The next lemma is a version of the boundary Harnack principle.

Lemma 1 (Jerison-Kenig [7]). *There exists a positive constant c such that if $A_r(Q)$ is as in* (1), *for all $x \in D \setminus B(Q, Mr)$, $Q \in \partial D$,*

$$c^{-1} K(A_r(Q), Q)\, w^x(\Delta(Q,r)) \leqq K(x,Q) \leqq c\, K(A_r(Q), Q)\, w^x(\Delta(Q,r)). \tag{4}$$

Another result we need is the following due to Dahlberg [4] though we will only use the upper bound.

Lemma 2. *There exist positive constants $\alpha > \frac{1}{2}$, c and β such that*

$$c^{-1} r^{\beta(d-1)} \leqq w^{x_0}(\Delta(Q,r)) \leqq c\, r^{\alpha(d-1)}. \tag{5}$$

Finally, the following lemma is an easy consequence, using the Harnack chain condition, of a lemma in [7].

Lemma 3. *There is a positive constant c such that if $r < r_0$ and $Q \in \partial D$ then*

$$w^{A_r(Q)}(\Delta(Q,r)) \geqq c. \tag{6}$$

In view of Lemma 1, upper and lower bounds for $K(x,Q)$ will involve controlling the growth of $K(A_r(Q), Q)$.

Lemma 4. *There exist positive constants c, a, b such that for $r < r_0$, $Q \in \partial D$,*

$$c^{-1} r^{-a} \leqq K(A_r(Q), Q) \leqq c\, r^{-b}. \tag{7}$$

Proof. For the upper bound observe that there is a Harnack chain connecting $A_r(Q)$ to x_0 of length Mk where

$$k \cong \log_2 \left[\frac{|x_0 - A_r(Q)|}{M^{-1} r}\right] \leqq \log_2 \left[\frac{M \operatorname{diam} D}{r}\right].$$

Thus by (3)

$$\begin{aligned} K(A_r(Q), Q) &\leqq c_1^{Mk} K(x_0, Q) \\ &= c_1^{Mk} \\ &= c\, r^{-b} \end{aligned}$$

with $b = M \log_2 c_1 > 0$, since $c_1 > 1$.

To get the lower bound observe that by (4)

$$c\, K(A_r(Q), Q)\, w^{x_0}(\Delta(Q,r)) \geqq K(x_0, Q)$$

and since $K(x_0, Q) = 1$, (5) implies

$$K(A_r(Q), Q) \geqq c\, r^{-a}$$

with $a = \alpha(d-1)$. $\square$

In the proof of Theorem 1 we need to control the Lebesgue measure, m, of the sets

$$A_n = \{x : 2^n < K(x,Q)\}$$

and

$$B_n = \{x : K(x,Q) < 2^{-n}\}$$

for n large. For A_n we need an upper bound for $K(x,Q)$ and for B_n we need a lower bound for $K(x,Q)$. From Lemmas 1 and 4

$$c^{-1} r^{-a} w^x(\Delta(Q,r)) \leqq K(x,Q) \leqq c\, r^{-b}, \quad x \in D \setminus B(Q, Mr). \tag{8}$$

Notice the term $w^x(\Delta(Q,r))$ has disappeared from the right hand side of (8) as it is bounded by one.

Lemma 5. *If $n \geqq N$ and $Q \in \partial D$, then*

$$A_n \subseteq B(Q, M r_n), \quad \text{with} \quad r_n = c\, 2^{-n/b}, \tag{9}$$

and

$$B_n \subseteq \{x: \operatorname{dist}(x, \partial D) \leqq s_n\}, \quad \text{with} \quad s_n = c\, 2^{-n\gamma} \tag{10}$$

for some $\gamma > 0$. Here N is taken to satisfy $r_N \leqq r_0$ and $s_N \leqq r_0$.

Proof. The inclusion (9) is immediate from the upper bound in (8).

To establish (10) we need a lower bound for $w^x(\Delta(Q,r))$ for $\operatorname{dist}(x,\partial D) > r$. For such x there is a Harnack chain connecting x and $A_r(Q)$ of length at most Mk, $k \cong \log_2 \left(\frac{M \operatorname{diam} D}{r}\right)$. Thus by (3) and (6), for $\operatorname{dist}(x,\partial D) > r$,

$$\begin{aligned} w^x(\Delta(Q,r)) &\geqq c_1^{-Mk} w^{A_r(Q)}(\Delta(Q,r)) \\ &\geqq c\, r^b, \quad \text{for } r < r_0, \end{aligned} \tag{11}$$

where $b = M \log_2 c_1 > 0$. Combining (8) and (11) leads to

$$K(x,Q) \geqq c\, r^{b-a} \quad \text{for } \operatorname{dist}(x,\partial D) > r.$$

Then taking $r = s_n = 2^{-n\gamma}$, and $\operatorname{dist}(x,\partial D) > s_n$ we have $K(x,Q) > c\, s_n^{b-a} = c\, 2^{-n\gamma(b-a)} \geqq 2^{-n}$, $n \geqq N$, provided $\gamma > 0$ is chosen sufficiently small. This gives the inclusion (10). □

We need the following probabilistic lemmas which do not assume ∂D is Lipschitz.

Lemma 6. *If $D \subseteq \mathbb{R}^d$ and $h(x) = \int_{\Delta_1} K(x,Q)\, \mu(dQ)$ for μ a positive measure on Δ_1 then*

$$E_x^h \tau_D = h(x)^{-1} \int_{\Delta_1} K(x,Q)\, E_x^{K(\cdot,Q)} \tau_D\, \mu(dQ).$$

Proof. See P.-A. Meyer [9] p. 96. □

Lemma 7. *Let $D \subseteq \mathbb{R}^d$ be open. There exists a constant c_d, independent of D, such that*

$$E_x \tau_D \leqq c_d\, m(D)^{2/d}, \quad x \in D,$$

where m is Lebesgue measure.

Proof. See Chung [1] or Cranston-McConnell [3] for the two dimensional result which extends readily to higher dimensions. □

Proof (of Theorem 1). We will show

$$E_x^h \tau_D \leqq c(D) \tag{12}$$

for minimal h. The result for general h follows using Lemma 6 since if

$$h(x) = \int_{\Delta_1} K(x,Q)\,\mu(dQ),$$

then

$$\begin{aligned} E_x^h \tau_D &= h(x)^{-1} \int_{\Delta_1} K(x,Q)\, E_x^{K(\cdot,Q)} \tau_D\, \mu(dQ) \\ &\leqq c(D)\, h(x)^{-1} \int_{\Delta_1} K(x,Q)\,\mu(dQ) \quad \text{by (2.14)} \\ &= c(D). \end{aligned}$$

Now for (12) take $h(\cdot) = K(\cdot, Q)$ and define

$$C_n = \{x : K(x,Q) = 2^n\},$$
$$D_n = \{x : 2^{n-1} < K(x,Q) < 2^{n+1}\}.$$

Then set

$$U_n = \#\{\text{upcrossings of } [2^{-n-1}, 2^{-n}] \text{ by } h(X_t)^{-1}\}$$
$$V_n = \#\{\text{downcrossings of } [2^{-n}, 2^{-n+1}] \text{ by } h(X_t)^{-1}\}.$$

If they are finite, U_n and V_{n+1} differ by at most one. Since $h(X_t)^{-1}$ is a P_x^h-supermartingale, the upcrossing lemma[1] gives

$$E_x^h U_n \leqq \frac{2^{-n-1}}{2^{-n} - 2^{-n-1}} = 1$$

and thus

$$E_x^h V_n \leqq 2.$$

Suppose $x \in D_k$, as it must be for some k. Then

$$\begin{aligned} E_x^h \tau_D &= E_x^h \tau_{D_k} + E_x^h [E_{X_{\tau_{D_k}}} \tau_D ; X_{\tau_{D_k}} = 2^{k-1}] + E_x^h [E_{X_{\tau_{D_k}}} \tau_D ; X_{\tau_{D_k}} = 2^{k+1}] \\ &\leqq E_x^h \tau_{D_k} + \sup_{z \in C_{k-1}} E_z^h \tau_D + \sup_{z \in C_{k+1}} E_z^h \tau_D. \end{aligned}$$

Using the strong Markov property at the successive hitting times to the sets C_n, we have for $z \in C_j$, any j,

$$\begin{aligned} E_z^h \tau_D &\leqq \sum_{n=-\infty}^{\infty} E_z^h (U_n + V_n) \sup_{y \in C_n} E_y^h \tau_{D_n} \\ &\leqq 3 \sum_{n=-\infty}^{\infty} \sup_{y \in C_n} E_y^h \tau_{D_n}. \end{aligned}$$

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[1] Kai Lai Chung pointed out this simplification of the arguments in [2]; see [2], where the E^x in (3) and (4) should be replaced by E_h^x

Now for $y \in C_n$,

$$E_y^h \tau_{D_n} = \int_0^\infty P_y^h(\tau_{D_n} > \lambda)\, d\lambda$$
$$= \int_0^\infty E_y \left(\frac{h(X_\lambda)}{h(y)}; \tau_{D_n} > \lambda \right) d\lambda$$
$$\leqq 2 \int_0^\infty P_y(\tau_{D_n} > \lambda)\, d\lambda$$
$$= 2 E_y \tau_{D_n}$$
$$\leqq c\, m(D_n)^{2/d}, \quad \text{by Lemma 7.}$$

Similarly, $E_x^h \tau_{D_k} \leqq c\, m(D_k)^{2/d}$. If $n \geqq N+1$ then $D_n \subseteq A_{n-1} \subseteq B(Q, M r_{n-1})$ with $r_n = c\, 2^{-n/b}$ by Lemma 5. If $n \leqq -N-1$ then $D_n \subseteq B_{|n|-1} \subseteq \{x\colon \operatorname{dist}(x, \partial D) \leqq s_{|n|-1}\}$ with $s_n = c\, 2^{-n\gamma}$, $\gamma > 0$, again by Lemma 5. Thus,

$$E_x^h \tau_D \leqq c \{ \sum_{n \geqq N+1} m(B(Q, M r_{n-1}))^{2/d} + \sum_{|n| \leqq N} m(D_n)^{2/d}$$
$$+ \sum_{n \leqq -N-1} m(\{x\colon \operatorname{dist}(x, \partial D) \leqq s_{|n|-1}\})^{2/d} \} + c\, m(D_k)^{2/d}$$
$$\leqq c(D)$$

by the choice of r_n and s_n. □

References

1. Chung, K.L.: Lectures from Markov processes to Brownian motion. Berlin-Heidelberg-New York: Springer 1982
2. Chung, K.L.: The lifetime of conditional Brownian motion in the plane. Ann. Inst. Henri Poincaré **20**, 349–351 (1984)
3. Cranston, M., McConnell, T.R.: The lifetime of conditioned Brownian motion. Z. Wahrscheinlichkeitstheor. Verw. Geb. **65**, 1–11 (1983)
4. Dahlberg, B.E.J.: Estimates of harmonic measure. Arch. Rat. Mech. **65**, 275–288 (1977)
5. Doob, J.L.: Conditional Brownian motion and the boundary limits of harmonic functions. Bull. Soc. Math. France **85**, 431–458 (1957)
6. Hunt, R.A., Wheeden, R.L.: Positive harmonic functions on Lipschitz domains. Trans. Am. Math. Soc. **132**, 307–322 (1968)
7. Jerison, D.S., Kenig, C.E.: Boundary behavior of harmonic functions in non-tangentially accessible domains. Adv. Math. **146**, 80–147 (1982)
8. Martin, R.S.: Minimal positive harmonic functions. Trans. Am. Math. Soc. **49**, 137–172 (1941)
9. Meyer, P.-A.: Processus de Markov: la frontière de Martin. Lect. Notes Math. **77**. Berlin-Heidelberg-New York: Springer 1968

Received February 15, 1985