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Products of Markovian Semi-Groups of Operators

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Summary. Let A and B denote the generators of two contraction semi-groups of operators (P^t) and (Q^t) acting on some Banach space. If the operator $A+B$ has a closure generating a third semi-group (R^t), then it is known (Trotter) that $R^t = \lim_{h\to 0}(P^h Q^h)^{[t/h]}$. The existence and identification of this limit is of interest even when the closure of $A+B$ is not a generator. A probabilistic version of this problem is given here in the case of Markovian transition semi-groups when the corresponding processes have identical hitting distributions. Sufficient conditions for the existence of (R^t) are given, and in special cases its generator is identified.

Introduction

If (P^t) and (Q^t) are two continuous contraction semi-groups of operators on a Banach space V, the product $(P\circ Q)^t$ may be defined as

$$(P\circ Q)^t = \lim_{h\to 0}(P^h Q^h)^{[t/h]} \tag{0.1}$$

if this limit exists in some appropriate topology, ($[t/h]$ is the greatest integer in t/h). Trotter considered the existence of such products in [6], and obtained the following theorem.

Theorem (Trotter). Suppose that $\mathbf{A}$ and $\mathbf{B}$ are the strong generators of (P^t) and (Q^t) and that the intersection $\mathscr{D}_\mathbf{A}\cap\mathscr{D}_\mathbf{B}$ of their domains is dense in V. Then $\mathbf{A}+\mathbf{B}$ has a closure which generates a continuous semi-group

$$R^t = \text{strong}\lim_{h\to 0}(P^h Q^h)^{[t/h]}$$

if and only if, the range of $\lambda\mathbf{I}-(\mathbf{A}+\mathbf{B})$ is dense in V for some $\lambda>0$.

In the study of Markov processes products that are similar to those considered by Trotter arise. The simplest example of this is Lévy's decomposition for the characteristic functions of processes with independent increments, which may be interpreted as a theorem about sums of Markovian generators.

The motivation for this work arises from the fact that in many problems of this type, the conditions of Trotter's Theorem are not satisfied, and yet, one can check explicitly that the limit in (0.1) exists. The example of uniform translations in opposite directions, which is given in Section 4, shows that $\mathscr{D}_\mathbf{A}\cap\mathscr{D}_\mathbf{B}$ may contain only one element while the limit in (0.1) exists and equals the identity operator $\mathbf{I}$.

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In this paper we give a probabilistic analysis of the convergence of $(P^h Q^h)^{[t/h]}$ for a class of Markovian semi-groups. In contrast to Trotter's results involving analytic assumptions about the generators, we will impose probabilistic conditions on the corresponding Markov processes. In particular, we assume that (P^t) and (Q^t) are Hunt semi-groups on the same space E, and that the (Q^t) process (Y_t) is obtainable from the (P^t) process (X_t) by means of a random time change.

With this assumption we will find stopping times $\{\gamma_h(t)\}$ for the process (X_t) such that the kernels $(P^h Q^h)^{[t/h]}$ are represented explicitly as

$$(P^h Q^h)^{[t/h]} f(x) = E_x f(X_{\gamma_h(t)}).$$

Conditions are found which guarantee the convergence of $\gamma_h(t)$ as $h \to 0$, and this gives corresponding theorems about the kernels $(P^h Q^h)^{[t/h]}$.

When restricted to non-singular diffusions on $\mathbb{R}^1$ the results assume the following simplified form (Section 3).

Let (X_t) and (Y_t) be two diffusions on $\mathbb{R}^1$ with transition semi-groups (P^t) and (Q^t) and with generators given by two of Feller's generalized differential operators; namely

$$\mathbf{A} = D_m D_x \quad \text{and} \quad \mathbf{B} = D_n D_x.$$

If the measures m and n are not orthogonal, then

$$(P \circ Q)^t f(x) = \lim_{h \to 0} (P^h Q^h)^{[t/h]} f(x)$$

exists for each bounded continuous function f. The semi-group $(P \circ Q)^t$ corresponds to a process whose generator is formally given by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = D_\mu D_x$$

where $d\mu = \frac{dm}{dv} \cdot \frac{dn}{dv} dv$ and $dv = dm + dn$.

If m and n are smooth, in which case Trotter's theorem applies, the operators $\mathbf{A}$ and $\mathbf{B}$ are of the form

$$\mathbf{A} f(x) = a(x) f''(x), \qquad \mathbf{B} f(x) = b(x) f''(x)$$

and $\mathbf{C}$ is simply given by

$$\mathbf{C} f(x) = \big(a(x) + b(x)\big) f''(x).$$

Intuitively, this may be interpreted by saying that the product process is obtained by adding the "speeds" of (X_t) and (Y_t).

In general, our conditions are independent of Trotter's but when both are applicable they yield analogous results. Unfortunately, the results are tied to a special class of semi-groups and the methods do not seem to be easily generalized. The general semi-group practitioner will perhaps find our conditions of interest as a guide to, and an upper bound on, what is true in general.

Section 1 is a collection of standard definitions and notations. The main results are stated in Section 2 and proved in Section 5. These are applied to non-singular diffusions on $\mathbb{R}^1$ in Section 3. Examples of atypical behavior which occurs with translation semi-groups are given in Section 4.

§ 1. Definitions and Notations [1]

Let E denote a locally compact separable metric space. We adjoin an ideal point Δ to E so that $\bar{E}=E\cup\{\Delta\}$ is the one point compactification of E when E is not compact and Δ is isolated in $\bar{E}$ if E is compact. The Borel σ-algebra of E (resp. $\bar{E}$) is denoted as $\mathscr{B}$ (resp. $\bar{\mathscr{B}}$).

The notation (P^t) is used to denote a Borel measurable sub-Markovian semi-group of transition kernels on E defined for $t\geqq 0$ with P^0 equal the identity kernel **I.** We also write (P^t) for the extension of (P^t) to $\bar{E}$ determined by

$$P^t(\Delta,\{\Delta\})=1, \qquad P^t(x,\{\Delta\})=1-P^t(x,E) \qquad \text{for } x\in E.$$

For convenience we take our basic sample space to be the space Ω of right continuous functions ω: $[0,\infty)\to\bar{E}$ which satisfy: if $\omega(s)=\Delta$ then $\omega(t)=\Delta$ for all $t\geqq s$. To denote the process defined on Ω we write $X_t(\omega)=X(t,\omega)=\omega(t)$. The terminal time of (X_t) is defined as $\xi=\inf\{t\colon X_t=\Delta\}$ if such t exist and $+\infty$ otherwise. The basic σ-algebras $\mathscr{M}$ and $(\mathscr{M}_t)_{t\geqq 0}$ on Ω are those generated by the sets $\{X_s\in B\}$ for $B\in\mathscr{B}$ and $s\geqq 0$, (resp. $t\geqq s\geqq 0$). The shift operator Θ_t is given on Ω by $X_s(\Theta_t\omega)=X_{s+t}(\omega)$. This induces an operator acting on functions defined on Ω by composition, $f\circ\Theta_t$.

We say (P^t) admits (X_t) as a realization if for each initial distribution μ on $\bar{E}$, there exists a probability measure P_μ on $(\Omega,\mathscr{M})$ satisfying the following three properties.

(1.1) $$P_\mu\{X_0\in A\}=\mu(A) \qquad \text{for each } A\in\bar{\mathscr{B}}.$$

(1.2) (Markov property): For each $\bar{\mathscr{B}}$-measurable $f\geqq 0$ and each pair s,t with $0\leqq s\leqq t$,

$$E_\mu[f(X_t)|\mathscr{M}_s]=P^{t-s}f(X_s) \quad \text{a.s.}$$

Here $E_\mu[\cdot|\mathscr{M}_s]$ is the conditional expectation given $\mathscr{M}_s$.

(1.3) $P_x(A)$ is a $\bar{\mathscr{B}}$-measurable function of x for each A in $\mathscr{M}$, and satisfies

$$P_\mu(A)=\int_{\bar{E}} P_x(A)\,\mu(dx),$$

where P_x is the measure corresponding to the initial point distribution at x.

If (P^t) admits the realization (X_t) we may complete the σ-algebra $\mathscr{M}$ with respect to the measure P_μ. Taking the intersection over all μ, we obtain a new σ-algebra $\mathscr{N}\supseteq\mathscr{M}$. Also, we write $\mathscr{N}_t$ to denote the intersection over μ of the σ-algebras $\mathscr{M}_t^\mu$; where $\mathscr{M}_t^\mu$ consists of the elements of $\mathscr{N}$ which differ from an element of $\mathscr{M}_t$ only by a P^μ null set. The process (X_t) also has the Markov property (1.2) with respect to the σ-algebras $(\mathscr{N}_t)$.

1. A standard reference for this material is R.M. Blumenthal and R.K. Getoor, Markov processes and potential theory. New York 1968.

A stopping time of (X_t) is a $\mathscr{N}$-measurable function $T\colon \Omega\to[0,\infty]$ such that $\{T<t\}\in\mathscr{N}_t$ for each $t\geqq 0$. The shift operator Θ_T is also defined for stopping times T by the relation $X_t(\Theta_T\,\omega)=X(t+T(\omega),\omega)$. With each stopping time T we associate the σ-algebra

$$\mathscr{N}_T=\{A\in\mathscr{M}\colon\ A\cap\{T<t\}\in\mathscr{M}_s \ \text{ for all } \ s>t\geqq 0\}.$$

The process (X_t) is called a Hunt process and (P^t) a Hunt semi-group provided that (X_t) is a strong Markov process and is quasi-left continuous. That is,

(1.4) For each $\bar{\mathscr{B}}$-measurable $f\geqq 0$ and stopping time T, we have

$$E_x[f(X_s)\cdot 1_{\{T\leqq s\}}|\mathscr{M}_T]=E_{X_T}[f(X_{s-T})]\cdot 1_{\{T\leqq s\}}$$

a.s. for each fixed s;

and

(1.5) for each increasing sequence $T_n\uparrow T$ of stopping times we have $X_{T_n}\to X_T$ a.s. on $\{T<\infty\}$, as $n\to\infty$.

A *continuous additive functional*, abbreviated *caf*, A of the Hunt process (X_t) is a a.s. continuous non-decreasing real valued process $\{A_t;\, t\geqq 0\}$ defined on Ω and satisfying the following relations.

(i) $A_0=0$ and $A_t=\lim_{s\downarrow\xi} A_s$ on $\{t\geqq\xi\}$ a.s.

(ii) A_t is $\mathscr{M}_t$-measurable.

(iii) $A_{t+s}=A_t+A_s\circ\Theta_t$ a.s. for $t,s>0$.

The sum of two continuous additive functionals is a caf. Also, if f is a non-negative measurable function on $\bar{E}$ and A is a caf one may operate on A with f to obtain the caf $f\cdot A$ given by $(f\cdot A)_t=\int_0^t f(X_s)\,dA_s$. The functional A satisfies the relation

$$A(T(\omega)+R(\omega),\omega)=A(T(\omega),\omega)+A(R(\omega),\Theta_T(\omega)) \quad \text{a.s.} \tag{1.6}$$

for each stopping time T and random variable $R\geqq 0$.

Associated with A is its functional inverse $\alpha_t=\alpha(t)$ defined by, $\alpha(t,\omega)=\inf\{s\colon A_s(\omega)>t\}$ if such s exists or $+\infty$ otherwise. For each t, $\alpha(t)$ is a stopping time and as a function of t it is right continuous. Eq. (1.6) easily implies

$$\alpha(t+s)=\alpha(t)+\alpha(s)\circ\Theta_{\alpha(t)} \quad \text{a.s.} \tag{1.7}$$

We use the "time change" $\alpha(t)$ to define a new process $Y_t=X_{\alpha(t)}$. Because of (1.7), (Y_t) is seen to be a strong Markov process.

§ 2. The Main Theorems

In this section we will discuss the limit behavior of the $(P^h Q^h)^{[t/h]}$, where (P^t) and (Q^t) are Markovian semi-groups on a locally compact separable space E. It is also necessary for us to make the assumptions:

(2.1) (P^t) and (Q^t) admit representations (X_t) and (Y_t) that are Hunt processes.

(2.2) There exists a caf A of (X_t) with inverse α_t such that $Y_t=X_{\alpha(t)}$.

Thus by (2.2) for each $\bar{\mathscr{B}}$-measurable $f \geqq 0$ we have

(2.3)
$$P^t f(x) = E_x f(X_t)$$
$$Q^t f(x) = E_x f(X_{\alpha(t)}),$$

and for $h \geqq 0$

$$P^h Q^h f(x) = E_x[Q^h f(X_h)]$$
$$= E_x[E_{X_h} f(X_{\alpha(h)})].$$

Setting

$$\gamma_{h,1} = h + \alpha(h) \circ \Theta_h$$

the last identity takes on the form

$$P^h Q^h f(x) = E_x f(X_{\gamma_{h,1}}).$$

Introduce stopping times $\{\gamma_{h,k}\}$ by

$$\gamma_{h,k+1} = \gamma_{h,k} + \gamma_{h,1} \circ \Theta_{\gamma_{h,k}}. \tag{2.4}$$

A repeated application of the strong Markov property yields

$$(P^h Q^h)^k f(x) = E_x f(X_{\gamma_{h,k}}). \tag{2.5}$$

Eq. (2.5) enables us to make arguments involving the stopping times $\{\gamma_{h,k}\}$ rather than the kernels (P^t) and (Q^t). For technical reasons however we will work with a related family $C_h(t)$ of *non-additive* functionals of (X_t). These are given by

$$C_h(t) \begin{array}{ll} = (2k)\,h + (I_t - I_{\gamma_{h,k}}) & \text{if } \gamma_{h,k} \leqq t < \gamma_{h,k} + h \\ = (2k+1)\,h + (A_t - A_{\gamma_{h,k}+h}) & \text{if } \gamma_{h,k} + h \leqq t < \gamma_{h,k+1}, \end{array} \tag{2.6}$$

where I is defined as $I_t = t \wedge \xi$.

The functionals C_h are related to the $\{\gamma_{h,k}\}$ by the obvious identity

$$\gamma_{h,k} = \inf\{s \colon C_h(s) > 2kh\}. \tag{2.7}$$

Since for almost all ω the functions $I_t(\omega)$ and $A_t(\omega)$ are continuous and increasing we may define the Hellinger integral

$$\langle I, A \rangle_t = \lim_{h \to 0} \sum_{k=1}^{[t/h]} \frac{[I_{kh} - I_{(k-1)h}][A_{kh} - A_{(k-1)h}]}{[I_{kh} - I_{(k-1)h}] + [A_{kh} - A_{(k-1)h}]}. \tag{2.8}$$

This limit is easily seen to exist (see [3]) and defines a caf of (X_t).

The following theorem is basic.

Theorem 1. *There exists a set $\Omega' \subset \Omega$ with $P_x(\Omega') = 1$ for each $x \in E$ and such that on Ω'*

$$\lim_{h \to 0} C_h(t) = 2 \langle I, A \rangle_t \qquad \textit{for each } t \geqq 0.$$

This theorem is proven in Section 5. In the remainder of this section we discuss the implications when applied to $\{(P^h Q^h)^{[t/h]}\}$.

We first note that under mild conditions Theorem 1 implies that the stopping times $\gamma_{h,[t/h]}$ will converge a.s.

Corollary 1. *Let γ be the functional inverse of $\langle I, A\rangle$. Then for each $x\in E$ and $t\geqq 0$,*

$$\lim_{h\to 0}\gamma_{h,[t/h]}=\gamma(t) \qquad a.s.\ (P_x)$$

on the set of ω for which γ is continuous at t.

Proof. The theorem implies that $C_h(\gamma_t)\to 2\langle I, A\rangle_{\gamma(t)}=2t$, while the relation (2.7) gives $C_h(\gamma_{h,[t/h]})=2t$. It then follows directly that $\gamma_{h,[t/h]}\to\gamma(t)$ if γ is continuous at t.

Using the fact that (X_t) is a Hunt process we obtain

Proposition 1. *If γ is a.s. (P_x) continuous at t, then*

$$X_{\gamma_{h,[t/h]}}\to X_{\gamma(t)} \qquad a.s.\ (P_x)\ \text{on}\ \{\gamma(t)<\infty\}.$$

Proof. It follows easily from the definition that the Hunt process (X_t) has left hand limits a.s. on $\{\xi<\infty\}$. From this and the fact that (X_t) is right continuous it suffices to show that $X_{\gamma_n'}\to X_{\gamma(t)}$ a.s., where the sequence γ_n' is given by $\gamma_n'=\gamma_{1/n,[t\cdot n]}\wedge\gamma_t$. This, in turn, follows at once from the definition of a Hunt process.

The time change $\gamma(t)$ is said to have *no fixed discontinuities* if for each fixed $t>0$ and $x\in E$, γ is continuous at t, a.s. (P_x). (In example (a) of Section 4 γ has a fixed discontinuity.) If γ has no fixed discontinuities then $X_{\gamma_{h,[t/h]}}\to X_{\gamma(t)}$ a.s. on $\{\gamma(t)<\infty\}$. We summarize with two propositions which are immediate consequences of this remark and Proposition 1.

Proposition 2. *Suppose γ has no fixed discontinuities and that $\langle I, A\rangle_\infty=\infty$ a.s. on $\{\xi=\infty\}$. Then $\gamma(t)<\infty$ a.s. on $\{\xi=\infty\}$ and*

$$\lim_{h\to 0}X_{\gamma_{h,[t/h]}}=X_{\gamma(t)} \qquad a.s.\ \text{for each}\ t\geqq 0.$$

Moreover,

$$(P^h Q^h)^{[t/h]}f(x)\to E_x f(X_{\gamma(t)}) \qquad \text{as}\ h\to 0$$

for each $x\in E$, $t>0$, and bounded continuous f on $\bar{E}$.

If the process (X_t) is transient, ($X_t\to\Delta$ a.s. as $t\to\infty$), then $X_{\gamma_{h,[t/h]}}\to\Delta$ on $\{\gamma(t)=\infty\}$. In this case we have the following modification of Proposition 2.

Proposition 3. *If (X_t) is transient and γ has no fixed discontinuities then*

$$\lim_{h\to 0}(P^h Q^h)^{[t/h]}f(x)\to E_x f(X_{\gamma(t)})$$

for each $x\in E$, $t>0$, and continuous f vanishing at Δ.

The process $(X_{\gamma(t)})$ may be formally interpreted as being obtained from (X_t) by adding the "speeds" of (X_t) and (Y_t). To see this we set $B_t=I_t+A_t$ and note that for each sample path ω, the Radon-Nikodym theorem guarantees the existence of non-negative functions $i(s)$ and $a(s)$ with $1 \equiv i(s)+a(s)$ and such that

$$I_t = \int_0^t i(s)\,dB_s, \qquad A_t = \int_0^t a(s)\,dB_s.$$

The functional $\langle I, A\rangle$ is then seen to satisfy

$$\langle I, A\rangle_t = \int_0^t i(s)\,a(s)\,dB_s. \tag{2.9}$$

Now the representation $Y_t = X_{\gamma(t)}$ gives the formal interpretation of $\frac{d\alpha}{dt} = i/a$ as the "speed" of (Y_t) relative to (X_t). The "speed" of (X_t) should then be identically 1 and $1/a$ is the "speed" of $X_{\gamma(t)}$. This however equals the sum $1+i/a$ of the "speeds" of (X_t) and (Y_t).

It is worthwhile noting that the natural state space of the limit process $(X_{\gamma(t)})$ is not in general E but rather a proper subset of E. We give a brief description, due to Getoor [2], of this situation. For a set $A \subset \bar{E}$, $T_A \equiv \inf\{t>0\colon X_t \in A\}$ or $+\infty$ if this set is empty, is called the *hitting time* of A. If A is analytic it is known that T_A is a stopping time and that $P_x\{T_A=0\}$ is either zero or 1. If A is analytic and $P_x\{T_A=0\}=1$, then x is said to be *regular* for A. For a general set A we say x is *regular* for A if A contains an analytic set B with x regular for B. The set A is said to be *finely closed* if A contains all points that are regular for A. A is called nearly Borel (relative to X_t), if for each initial distribution μ, there are Borel sets B_1 and B_2 with $B_1 \subset A \subset B_2$ and such that $P_\mu\{X_t \in B_2 - B_1$ for some $t \geqq 0\} = 0$.

The *fine support* F of the caf $\langle I, A\rangle$ is given by

$$F = \{x\colon\ P_x\{\gamma(0)=0\}=1\},$$

here γ_t is again the inverse of $\langle I, A\rangle$. Then Getoor [2], establishes the following results.

(i) F is a finely closed nearly Borel set.

(ii) Each x in F is regular for F.

(iii) If $\langle I, A\rangle_{t+\varepsilon} - \langle I, A\rangle_t > 0$ for each $\varepsilon>0$, then $X_t \in F$ a.s.

(iv) If $X_t \in F$, then $\langle I, A\rangle_{t+\varepsilon} - \langle I, A\rangle_{t-\varepsilon} > 0$ for each $\varepsilon > 0$ a.s.

From this it follows that $X_{\gamma(t)} \in F$ almost surely for each $t \geqq 0$, and that F is the natural state space of $X_{\gamma(t)}$. In Section 3 we show that for 1-dimensional diffusions F may be an arbitrary closed set. This simple description of F is however an exception, and no general description of F is available. McKean and Tanaka [5] give relevant information when (X_t) has Brownian hitting distributions in dimensions $\geqq 2$.

The technical assumption, made in Proposition 2, that γ has no fixed discontinuities is also related to the structure of F. The only example I know where γ has a fixed discontinuity is given in Section 4. It is highly degenerate and the discontinuity there only causes trivial problems. The following simple condition is satisfactory for applications to 1-dimensional diffusions.

Lemma 1. *If the fine support F of $\langle I, A\rangle$ is closed, γ has no fixed discontinuities.*

Proof. Since γ is right continuous we must only show for each fixed $t>0$, that if $\gamma(t-)$ is finite then $\gamma(t-)=\gamma(t)$ a.s. Now $\langle I, A\rangle$ is increasing from the left at $\gamma(t-)$ so that $\gamma(t-)=\gamma(t)<\infty$ iff $\langle I, A\rangle_{\gamma(t-)+\varepsilon}-\langle I, A\rangle_{\gamma(t-)}>0$ for each $\varepsilon>0$.

Let $t_n \uparrow t$ be a strictly increasing sequence. Then on the set $\{\gamma(t-)<\infty\}$ we have that $X_{\gamma(t_n)}\in F$ a.s. and that $X_{\gamma(t_n)}\to X_{\gamma(t-)}$ a.s. since (X_t) is a Hunt process. Thus $X_{\gamma(t-)}\in F$ a.s. because F is assumed closed. Getoor's description of F now implies that $X_{\gamma(t-)}$ is almost surely a regular point for F and this in turn implies that for $\varepsilon>0$, $\langle I, A\rangle_{\gamma(t-)+\varepsilon}-\langle I, A\rangle_{\gamma(t-)}>0$. The result follows.

If (X_t) is a non-singular 1-dimensional diffusion, a set is finely closed iff it is a closed set in the usual topology. Since F is always finely closed, in this case it is also closed and by the lemma γ has no fixed discontinuities.

§ 3. Products of 1-Dimensional Diffusions

When restricted to the case of non-singular diffusions on $E=\mathbb{R}^1$ the results in Section 2 simplify and become more explicit. We treat this example here.

Let m and n be two Borel measures on the real line $\mathbb{R}^1$ which attribute positive masses to open sets. If (b_t) is the standard Brownian motion on $\mathbb{R}^1$ with the local time at x denoted by $\mathfrak{t}(t, x)$ we may define a pair $f(t)$ and $g(t)$ of continuous additive functionals of (b_t) as

$$f(t)=\int_{\mathbb{R}^1} \mathfrak{t}(t,x)\,dm(x), \qquad g(t)=\int_{\mathbb{R}^1} \mathfrak{t}(t,x)\,dn(x); \tag{3.1}$$

(see [4], Chapter 5). Corresponding diffusions may then be introduced as

$$X_t=b_{\tau_1(t)}, \qquad Y_t=b_{\tau_2(t)}, \tag{3.2}$$

where τ_1 and τ_2 are the inverses of f and g respectively. The generators of (X_t) and (Y_t) are then given by two of Feller's generalized differential operators; namely

$$\mathbf{A}=D_m D_x \quad \text{and} \quad \mathbf{B}=D_n D_x.$$

Write (P^t) and (Q^t) for the corresponding transition semi-groups. If we formally attempt to apply Trotter's theorem and try writing the operator $\mathbf{A}+\mathbf{B}$ in the form $D_\mu D_x$, it is seen that μ must be given by

$$d\mu=\left(\frac{dm}{dv}\cdot\frac{dn}{dv}\right)dv, \tag{3.3}$$

where $dv=dm+dn$. We might then predict that the product semi-group corresponds to the process (Z_t) given by

$$Z_t=b_{\tau_3(t)}, \quad \text{where } \tau_3 \text{ is the inverse of} \quad h(t)=\int_{\mathbb{R}^1} \mathfrak{t}(t,x)\,d\mu(x). \tag{3.4}$$

This is in fact correct and in this case Proposition 2 of Section 2 becomes:

Proposition 2′. *Suppose (P^t) and (Q^t) have generators $D_m D_x$ and $D_n D_x$, and that m and n are not orthogonal measures. Then for each bounded continuous f*

$$\lim_{h\to 0}(P^h Q^h)^{[t/h]} f(x)=E_x f(Z_t),$$

where (Z_t) is given by (3.4).

This is easily obtained from Proposition 2. We first write $Y_t = X_{\alpha(t)}$ where α is the inverse of $A_t = g(\tau_1(t))$. In the notation of Section 2 we then have

$$I_t = t = \int_{\mathbb{R}^1} \mathfrak{t}(\tau_1(t), x)\, dm(x),$$

$$A_t = \int_{\mathbb{R}^1} \mathfrak{t}(\tau_1(t), x)\, dn(x),$$

$$B_t = \int_{\mathbb{R}^1} \mathfrak{t}(\tau_1(t), x)\, dv(x)$$

and

$$\langle I, A\rangle_t = \int_{\mathbb{R}^1} \mathfrak{t}(\tau_1(t), x)\, d\mu(x), \\ = h(\tau_1(t)).$$

If γ is the inverse of $\langle I, A\rangle$ then

$$X_{\gamma(t)} = X_{f(\tau_3(t))} = b_{\tau_3(t)}.$$

In order to apply Proposition 2 we need only observe that $\mathfrak{t}(t, x) \to \infty$ a.s. as $t \to \infty$, and that by Lemma 1 of Section 2, γ has no fixed discontinuities. The proof is complete.

Note that if m and n are orthogonal then $\mu \equiv 0$ and $\gamma_{h,[t/h]} \to +\infty$ as $h \to \infty$.

As mentioned earlier the natural state space F of the limit process $(X_{\gamma(t)})$ may be any closed set F and in our case coincides with the support of the measure μ.

Example. Let m and n both attribute to a unit mass to the points 0 and 1, and assume they are orthogonal on the set $\mathbb{R}^1 - \{0, 1\}$. The function $h(t)$ in Eq. (3.4) reduces to

$$h(t) = \tfrac{1}{2}[\mathfrak{t}(t, 0) + \mathfrak{t}(t, 1)].$$

F is then equal to $\{0, 1\}$ and $X_{\gamma(t)}$ is a Markov chain on the state space F.

It is easily seen that the processes (Z_t) which arise as products of such diffusions are characterized by the right continuity of (Z_t) for $t > 0$, and the fact that their sample paths have the intermediate value property.

Such processes form a natural extension of diffusion processes and may be analysed by the same techniques. Dynkin's discussion of the generators is applicable and the processes are obtainable from Brownian motion via time changes based on speed measure integrals. The possibility of such a treatment is suggested by Feller's discussion [1] of birth and death processes.

We also note that $(P \circ Q)^t$ maps $C(\mathbb{R}^1)$ (continuous functions on $\mathbb{R}^1$) into $C(F)$ which is the natural Banach space for $(P \circ Q)^t$ to act on. This is a special and transparent case of a sequence of semi-groups acting on one space while the "limit" acts on a distinct space.

§ 4. Translation Semi-Groups

This section contains two examples of products of translation semi-groups. Both cases exhibit types of behavior that is quite different from that encountered in the previous sections.

Example (a). *Fixed discontinuities of γ:* Processes corresponding to uniform translation with respect to singular scales can be used to construct fixed discontinuities in γ. For example, let $X_t = X_0 + t$, and assume that $s(x)$ is a strictly increasing function. Define a caf A of (X_t) by $A_t = s(X_t) - s(X_0)$. Then $Y_t = X_{\alpha(t)}$, where α is the inverse of A, corresponds to uniform translation with respect to the coordinate s.

Now choose s so that $s(x) = x$ for $x \leqq 0$ and $x \geqq 1$, but with s singular with respect to Lebesgue measure on the interval $[0, 1]$. If $X_0 = -1$, then $\langle I, A\rangle_t = t/2$ for $t < 1$ and is constant on the interval $(1, 2)$. Thus γ has a fixed discontinuity at $t_0 = \frac{1}{2}$. Here $\gamma_{h, [t_0/h]}$ converges to $\gamma(t_0 -)$ rather than $\gamma(t_0)$.

Example (b). *Uniform translations in opposite directions:* A very different type of behavior occurs when considering translations in opposite directions. For the processes that are related by time changes, which we have been considering, the product semi-group $(P \circ Q)^t$ was seen to exist provided that the time change was not too singular. In the case of uniform translations in opposite directions with respect to singular scales a different behavior occurs. The product semi-group always exists and is equal to the identity operator **I**.

We again assume that s is a continuous strictly increasing function. Let $X_t = X_0 + t$ be as before and let Y_t be the process of uniform translation to the left with respect to s. Then $(P^h Q^h)^k (x, dy)$ is a unit point mass at the point $Z_{h,k}(x)$ given by the recursion

$$Z_{h,1}(x) = y \qquad \text{where } s(x+h) - s(y) = h,$$
$$Z_{h,k+1}(x) = Z_{h,1}(Z_{h,k}(x)).$$

Now assume that $s' = ds/dx$ exists and is different from 1 at x_0. If $s'(x_0) < 1$ and h is sufficiently small we have $Z_{h,k}(x_0) < x_0$, while $Z_{h,k}(x_0) > x_0$ if $s'(x_0) > 1$. If s is also assumed to be singular then both of the sets

$$\{x \colon s'(x) = 0\} \quad \text{and} \quad \{x \colon s(x) = +\infty\}$$

are dense and it follows that $\lim Z_{h,[t/h]}(x) = x$ for all x.

If one considers the transition semi-groups as operating on the space of continuous functions which vanish at infinity then the intersection of the domains of the generators contains only the zero function, but the strong $\lim (P^h Q^h)^{[t/h]} = \mathbf{I}$. The two semi-groups are highly singular with respect to each other, but in the product semi-group these singularities somehow cancel. Similar behavior does not occur when the processes have identical hitting distributions.

§ 5. Proof of Theorem 1

We recall the set up of Section 2. A non-zero caf A of the Hunt process (X_t) was given. We set $I_t = t \wedge \xi$ and $B = I + A$. Then

$$I_t = \int_0^t i(s)\, dB_s \quad \text{and} \quad A_t = \int_0^t a(s)\, dB_s,$$

where a and i are two non-negative measurable functions with $i+a\equiv 1$. Writing $\alpha(t)$ for the inverse of A we introduced the stopping times

$$\gamma_{h,1}=h+\alpha(h)\circ\Theta_h \quad \text{and} \quad \gamma_{h,k+1}=\gamma_{h,k}+\gamma_{h,1}\circ\Theta_{\gamma_{h,k}},$$

and the sequence of functionals

$$C_h(t)\begin{array}{ll} =2kh+(I_t-I_{\gamma_{h,k}}) & \text{if } \gamma_{h,k}\leqq t<\gamma_{h,k}+h \\ =(2k+1)h+(A_t-A_{\gamma_{h,k}+h}) & \text{if } \gamma_{h,k}+h\leqq t<\gamma_{h,k+1}. \end{array}$$

Theorem 1 then stated that

$$\lim_{h\to 0} C_h(t)=2\langle I,A\rangle_t \quad \text{a.s.,} \tag{5.1}$$

where $\langle I,A\rangle_t=\int_0^t i(s)\,a(s)\,dB_s$.

The idea of the proof is to approximate A by piecewise linear functions for which the result is easy. The details follow.

We will prove that the convergence of (5.1) occurs for each ω such that $A_t(\omega)$ is continuous and non-decreasing. The result then becomes a theorem about continuous monotonic functions and we will drop the ω in our notations. Also note that without loss of generality we may assume that $I_t\equiv t$.

The functions A, B, etc., are monotonic and as such they induce measures on $[0,\infty)$ which we will denote with the same symbols. We write l for Lebesgue measure.

Proof. Let $\varepsilon>0$ be fixed. We set $H=\{s\leqq t\colon i(s)=0\}$, and for each positive integer N we define the sets

$$F_{N,k}=\{s\leqq t\colon (k-1)N^{-1}\leqq i(s)<kN^{-1}\}, \qquad \text{for } 1\leqq k\leqq N-1,$$
$$F_{N,N}=\{s\leqq t\colon 1-N^{-1}\leqq i(s)\leqq 1\}.$$

Then $F_{N,1}\downarrow H$ as $N\to\infty$ and we choose an N so large that $B(F_{N,1}-H)<\varepsilon$. Having fixed such an N we drop the subscript N in our notations. The proof now proceeds by noting

$$|C_h(t)-2\langle I,A\rangle_t|\leqq\sum_{k=1}^{N}|C_h(F_k)-2\langle I,A\rangle(F_k)|$$

and then estimating each term in the sum.

Part 1: $\limsup_{h\to 0} C_h(t)\leqq 2\langle I,A\rangle_t$.

We first consider F_1 and write $G=F_1-H$. Then $B(G)<\varepsilon$, and H is of Lebesque measure zero. We cover H with a sequence $\{I_j\}$ of open intervals with $\sum l(I_j)<\varepsilon$ and $B((\bigcup I_j)-H)<\varepsilon$, and then choose an M so large that $B(\bigcup\{I_j\colon j>M\})<\varepsilon$. For $j=1,\ldots,M$ we write $I_j=(a_j,b_j)$ and $\varepsilon_j=b_j-a_j$. By assumption we have $\sum_{j=1}^{M}\varepsilon_j<\varepsilon$.

Now consider the $C_h(I_j)$. From the definition of C_h it follows that $C_h(J)<2h$ for any interval J of length h. Thus for h we have the estimate $C_h(I_j)\leqq 3\varepsilon_j$. Combining this with the above remarks we have

$$\begin{aligned} C_h(F_1) &\leqq B(G)+\sum_{j=1}^{M} C_h(I_j)+B\Big(\bigcup_{j\geqq M} I_j\Big) \\ &\leqq 5\varepsilon. \end{aligned}$$

Also,

$$0\leqq \langle I,A\rangle(F_1)=\langle I,A\rangle(G)\leqq B(G)<\varepsilon$$

so that

$$|C_h(F_1)-2\langle I,A\rangle(F_1)|<5\varepsilon \qquad \text{for small } h. \tag{5.2}$$

To estimate the $\{C_h(F_k)\}$ for $k\neq 1$, we cover F_k with a sequence $\{I_j\}$ of disjoint open intervals with $B\big((\bigcup I_j)-F_k\big)<\varepsilon/N$, and then choose an M such that $B\big(\bigcup\{I_j\colon j\geqq M\}\big)<\varepsilon/N$. We have

$$C_h(F_k)<\sum_{j=1}^{M} C_h(I_j)+\varepsilon/N.$$

For $j=1,\dots,M$ we set $\varepsilon_j=B(I_j-F_k)$. Then $\sum_{j=1}^{M}\varepsilon_j<\varepsilon/N$.

To estimate $C_h(I_j)$ we first observe that $dB_s=ds/i(s)$ on the set $\{s\colon i(s)\neq 0\}$.

Introducing the sets

$$\begin{aligned} G_h &=\{s\in[0,t]\colon s\in[\gamma_{h,k},\gamma_{h,k}+h) \text{ for some } k\}, \\ H_h &=[0,t]-G_h, \end{aligned}$$

we observe that both G_h and H_h are finite unions of disjoint intervals and that $C_h(I_j)$ can be estimated simply by counting the number of disjoint intervals in $G_h\cap I_j$. In fact, if

$$\theta_h=\text{the number of disjoint intervals in } G_h\cap I_j,$$

then

$$C_h(I_j)=2\theta_h\cdot h+O(h).$$

But we also have the two immediate relations

$$\begin{aligned} C_h(I_j\cap G_h) &= C_h(I_j\cap H_h)+O(h) \\ C_h(I_j\cap G_h) &= l(I_j\cap G_h)=\theta_h\cdot h+O(h). \end{aligned} \tag{5.3}$$

Moreover,

$$\begin{aligned} C_h(I_j\cap H_h) &\leqq \int_{I_j\cap H_h\cap F_k} \frac{a(s)}{i(s)}\,ds+\varepsilon_j \\ &\leqq \left(\frac{N+1-k}{k-1}\right) l(I_j\cap H_h)+\varepsilon_j, \end{aligned}$$

since $\frac{a(s)}{i(s)} \leqq \frac{N+1-k}{k-1}$ on F_k. The right side may be estimated as

$$\left(\frac{N+1-k}{k-1}\right)[b_j - a_j - \theta_h \cdot h + O(h)] + \varepsilon_j,$$

and combining this with (5.3) we get

$$\theta_h \cdot h \leqq \left(\frac{N+1-k}{k-1}\right)[b_j - a_j - \theta_h \cdot h + O(h)] + \varepsilon_j.$$

In turn, this gives

$$\theta_h \cdot h \leqq (b_j - a_j)\left(\frac{N+1-k}{N}\right) + O(h) + \varepsilon_j$$

or

(5.4) $$C_h(I_j) \leqq 2(b_j - a_j)\left(\frac{N+1-k}{N}\right) + 2\varepsilon_j + O(h).$$

One now checks easily that

(5.5) $$\langle I, A\rangle (I_j) \geqq (b_j - a_j)\left(\frac{N-k}{N}\right) - \varepsilon_j.$$

After summing over j, (5.4) and (5.5) give the inequality

(5.6) $$\limsup_{h \to 0} C_h(F_k) \leqq 2\langle I, A\rangle (F_k) + N^{-1}[7\varepsilon + 2l(F_k)],$$

which is valid for $k > 1$. The proof of part 1 is completed by summing this over $2 \leqq k \leqq N$ and using (5.2) to obtain

$$\limsup_{h \to 0} C_h(t) \leqq 2\langle I, A\rangle_t + 2t \cdot N^{-1} + 12\varepsilon.$$

Part 2: $\liminf_{h \to 0} C_h(t) \geqq 2\langle I, A\rangle_t$.

This inequality is proved in the same manner. One uses the inequality $\frac{a(s)}{i(s)} \geqq \frac{N-k}{k}$ for s in F_k to obtain the estimate

$$C_h(I_j \cap H_h) \geqq \frac{N-k}{N}[b_j - a_j - \theta_h \cdot h + O(h) - \varepsilon_j].$$

This gives

$$\theta_h h \geqq \frac{N-k}{N}(b_j - a_j) + O(h) - \varepsilon_j.$$

Similarly one has

$$\langle I, A\rangle (I_j) \leqq \left(\frac{N+1-k}{N}\right)(b_j - a_j) + \varepsilon_j$$

and the result follows as before.

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