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Ergodic Theorems for Operator Sequences

A. BRUNEL and M. KEANE

Summary. Let T be a measure preserving transformation on a finite measure space. Then for certain increasing sequences $k_1, k_2, \ldots$ of positive integers, called uniform sequences, the average

$$\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f$$

converges in the mean and almost everywhere. For strongly mixing transformations and any sequence of powers, an individual ergodic theorem with weights is valid.

1. Introduction

Our goal is to prove individual ergodic theorems for sequences of powers of a measure preserving transformation on a finite measure space. Let $(\Omega, \mathscr{B}, m)$ be a probability space and T a measure preserving transformation on $(\Omega, \mathscr{B}, m)$. Suppose that $k_1, k_2, \ldots$ is an increasing sequence of positive integers, and let f be an integrable function on Ω. Investigating the almost everywhere existence of the limit

$$\bar{f}(\omega) = \lim_n \frac{1}{n}\sum_{i=1}^{n} f(T^{k_i}\omega),$$

we present the following results. We define a certain class of sequences, called uniform sequences, such that if $k_1, k_2, \ldots$ is a sequence of this type, then $\bar{f}$ exists almost everywhere for each integrable f. With a condition on T a bit stronger than ergodicity (and depending on the uniform sequence involved), we can show that

$$\bar{f} = \int f\, dm$$

almost everywhere. In particular, $\bar{f} = \int f\, dm$ for each uniform sequence if T is weakly mixing.

A by-product of these results is the mean ergodic theorem for uniform sequences. We also prove the mean ergodic theorem for strongly mixing transformations and any increasing sequence of positive integers, which is due to Blum and Hanson [1]; this result is included for completeness.

Finally we prove an individual ergodic theorem for strongly mixing transformations and increasing sequences. This theorem states that the following condition is necessary and sufficient for T to be strongly mixing: for any integrable f and any increasing sequence $k_1, k_2, \ldots$ of positive integers, there exists a decreasing sequence $c_1, c_2, \ldots$ of positive reals such that $\sum_{i=1}^{\infty} c_i$ diverges and

$$\lim_n \frac{\sum_{i=1}^{n} c_i T^{k_i} f}{\sum_{i=1}^{n} c_i} = \int f\, dm$$

almost everywhere. The almost everywhere convergence of

$$\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f$$

in the strongly mixing case for any increasing sequence remains an open question.

2. Definitions

Let $(\Omega, \mathscr{B}, m)$ be a probability space and T a measure preserving transformation on $(\Omega, \mathscr{B}, m)$. We shall also denote by T the operators induced by T in the (real) Banach spaces $L^p = L^p(\Omega, \mathscr{B}, m)$ $(1 \leqq p < \infty)$. T is called

1. *ergodic*, if the constants are the only almost T-invariant functions in L^1;
2. *weakly mixing*, if

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} |m(T^{-k}E \cap F) - m(E)m(F)| = 0$$

for each pair $E, F \in \mathscr{B}$;

3. *strongly mixing*, if

$$\lim_k m(T^{-k}E \cap F) = m(E)m(F)$$

for all $E, F \in \mathscr{B}$, or, equivalently, if $T^k f$ converges weakly to the constant $\int f\,dm$ for each $f \in L^2$.

The reader is referred to [3, 4] for equivalent formulations and discussions of these properties.

Now suppose that X is a compact metric space, $\mathscr{X}$ the collection of Borel subsets of X, and φ a homeomorphism of X such that φ^n, n integral, is an equicontinuous set of mappings (i.e. for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $x, y \in X$, $|x, y| < \delta$ implies $|\varphi^n x, \varphi^n y| < \varepsilon$ for any integer n, where $|\cdot, \cdot|$ denotes the metric in X). The system (X, φ) is then called *uniformly L-stable* (stable in the sense of Liapounov, see [2, 6]). We assume henceforth that X possesses a dense orbit, that is, there exists an $x \in X$ such that $\{\varphi^n x \mid n \text{ integral}\}$ is dense in X. It follows (see Oxtoby [6]) that the system (X, φ) is strictly ergodic; i.e. there exists a unique φ-invariant (and thus ergodic) probability measure on $(X, \mathscr{X})$ which we denote by μ, and for any $x \in X$ and any continuous function f on X,

$$\int f\,d\mu = \lim_n \frac{1}{n}\sum_{t=0}^{n-1} f(\varphi^t x).$$

In particular, each open set of X has positive μ-measure. Such a system $(X, \mathscr{X}, \mu, \varphi)$ will be called *strictly L-stable*.

If $Y \in \mathscr{X}$ and $y \in X$, then we define the i^{th} *entry time* $k_i(y, Y)$ of y into Y recursively as:

$$k_1(y, Y) = \min\{i \geqq 1 \mid \varphi_i y \in Y\}$$

$$k_i(y, Y) = \min\{j > k_{i-1}(y, Y) \mid \varphi^j y \in Y\} \qquad (i > 1),$$

allowing infinity as a value.

Definition. A sequence $k_1, k_2, \ldots$ of natural numbers will be called uniform if there exist

1) a strictly L-stable system $(X, \mathscr{X}, \mu, \varphi)$,

2) a $Y \in \mathscr{X}$ such that $\mu(Y) > 0 = \mu(\partial Y)$ (where ∂Y denotes the boundary of Y), and

3) a point $y \in X$ such that $k_i = k_i(y, Y)$ for each $i \geqq 1$.

It is easily seen that if $k_1, k_2, \ldots$ is a uniform sequence generated by y and Y as above, then

$$\lim_i \frac{i}{k_i} = \mu(Y).$$

However, the proof of Theorem 1 will deliver this statement also.

Examples of strictly L-stable systems are provided by ergodic rotations of compact abelian groups; in fact, every strictly L-stable system is homeomorphic to such a rotation. If we take X to be the circle group, φ the rotation through an angle incommensurable with 2π, μ the Haar measure on X, Y an interval on X, and y any point of X, then the sequence of entry times of y into Y under φ is a uniform sequence.

3. The Individual Ergodic Theorem for Uniform Sequences

Let $(\Omega, \mathscr{B}, m)$ be a probability space and T a measure-preserving transformation on Ω. In this paragraph we shall make repeated use of the individual ergodic theorem of Birkhoff (see [3, 4]):

For each $f \in L^1$,

$$\bar{f}(\omega) = \lim_n \frac{1}{n} \sum_{k=1}^{n} f(T^k \omega)$$

exists almost everywhere. Furthermore, $\bar{f}$ belongs to L^1, is T-invariant, and

$$\int \bar{f}\, dm = \int f\, dm.$$

If T is ergodic, then $\bar{f}$ is obviously almost everywhere equal to the constant $\int f\, dm$.

Theorem 1. *If $f \in L^1$ and if $k_1, k_2, \ldots$ is a uniform sequence, then*

$$\tilde{f}(\omega) = \lim_n \frac{1}{n} \sum_{i=1}^{n} f(T^{k_i} \omega)$$

exists almost everywhere and $\tilde{f} \in L^1$.

Proof. Let $(X, \mathscr{X}, \mu, \varphi)$ and y, Y be the apparatus connected with the uniform sequence $k_1, k_2, \ldots$ and choose any $\varepsilon > 0$. Then there exist open subsets Y', Y'', and W of X such that:

1) $Y' \subseteq Y \subseteq Y''$,

2) $\mu(Y'' - Y') < \varepsilon$,

3) $y \in W$,

4) for any $x\in W$ and any $n\geqq 1$,

$$1_{Y'}(\varphi^n x)\leqq 1_Y(\varphi^n y)\leqq 1_{Y''}(\varphi^n x).$$

For instance, we can take

$$Y'=\{x\in Y \mid |x, \partial Y|>\delta\}$$

$$Y''=\{x\in X \mid |x, Y|<\delta\}$$

$$W=\{x\in X \mid |x, y|<\delta'\}$$

for suitable δ and δ'.

Now set

$$(\Omega', \mathscr{B}', m')=(\Omega, \mathscr{B}, m)\times(X, \mathscr{X}, \mu)$$

and define the transformation $T'. \Omega'\to\Omega'$ by

$$T'(\omega, x):=(T\omega, \varphi x) \qquad (\omega\in\Omega,\ x\in X).$$

Choose $f\in L^1$ with $f\geqq 0$. We consider the functions

$$g(\omega, x)=f(\omega)\,1_Y(x),$$

$$g'(\omega, x)=f(\omega)\,1_{Y'}(x),$$

$$g''(\omega, x)=f(\omega)\,1_{Y''}(x)$$

which all belong to $L^1(\Omega', \mathscr{B}', m')$. Now 4) implies

$$g'(\omega, \varphi^n x)\leqq g(\omega, \varphi^n y)\leqq g''(\omega, \varphi^n x) \tag{1}$$

for all $x\in W$, $\omega\in\Omega$, and $n\geqq 1$. Since T' preserves the measure m', Birkhoff's theorem says that

$$\bar{g}'(\omega, x)=\lim_n \frac{1}{n}\sum_{k=1}^n g'(T^k\omega, \varphi^k x)$$

and

$$\bar{g}''(\omega, x)=\lim_n \frac{1}{n}\sum_{k=1}^n g''(T^k\omega, \varphi^k x)$$

exist almost everywhere (with respect to m'). We set

$$\bar{S}(\omega)=\bar{S}(\omega, y)=\limsup_n \frac{1}{n}\sum_{k=1}^n g(T^k\omega, \varphi^k y),$$

$$\underline{S}(\omega)=\underline{S}(\omega, y)=\liminf_n \frac{1}{n}\sum_{k=1}^n g(T^k\omega, \varphi^k y).$$

In view of the inequalities (1) (note that $\mu(W)=m'(\Omega\times W)>0$) we have

$$\bar{g}'(\omega, x)\leqq\underline{S}(\omega)\leqq\bar{S}(\omega)\leqq\bar{g}''(\omega, x)$$

for almost all $(\omega, x) \in \Omega \times W$ (with respect to m'). We now calculate

$$\begin{aligned}
\int_{\Omega\times W} \bar{g}'\,dm' &= \int_{\Omega\times W} \left[\lim_n \frac{1}{n}\sum_{k=1}^n f(T^k\omega)\,1_{Y'}(\varphi^k x)\right] m'(d(\omega\times x))\\
&= \lim_n \frac{1}{n}\sum_{k=1}^n \iint_{\Omega\times W} f(T^k\omega)\,1_{Y'}(\varphi^k x)\,m(d\omega)\,\mu(dx)\\
&= \int f\,dm\cdot \lim_n \int_W \frac{1}{n}\sum_{k=1}^n 1_{Y'}(\varphi^k x)\,\mu(dx)\\
&= \int f\,dm\cdot \int_W \mu(Y')\,d\mu = \mu(W)\,\mu(Y')\int f\,dm,
\end{aligned}$$

making use of the uniform integrability of the corresponding Césaro sums and of the ergodicity of $(X, \mathscr{X}, \mu, \varphi)$ in particular, which allows us to get rid of the last limit.

Similarly,

$$\int_{\Omega\times W} \bar{g}''\,dm' = \mu(W)\,\mu(Y'')\int f\,dm.$$

Now $\bar{S}(\cdot) \geqq \underline{S}(\cdot)$ and

$$\bar{S}(\omega) - \underline{S}(\omega) \leqq \bar{g}''(\omega, x) - \bar{g}'(\omega, x) \qquad (\omega\in\Omega,\ x\in W),$$

and therefore

$$\begin{aligned}
\int [\bar{S}(\omega) - \underline{S}(\omega)]\,dm &= \frac{1}{\mu(W)} \int_{\Omega\times W} [\bar{S}(\omega) - \underline{S}(\omega)]\,dm'\\
&\leqq \frac{1}{\mu(W)} \int_{\Omega\times W} (\bar{g}'' - \bar{g}')\,dm'\\
&= (\mu(Y'') - \mu(Y'))\int f\,dm < \varepsilon \int f\,dm.
\end{aligned}$$

Since ε was arbitrary, this implies

$$\int [\bar{S}(\omega) - \underline{S}(\omega)]\,m(d\omega) = 0$$

and thus

$$S(\omega) = \lim_n \frac{1}{n}\sum_{k=1}^n g(T^k\omega, \varphi^k y)$$

exists for almost every $\omega\in\Omega$ (with respect to m). But

$$\begin{aligned}
S(\omega) &= \lim_n \frac{1}{n}\sum_{k=1}^n f(T^k\omega)\,1_Y(\varphi^k y)\\
&= \lim_n \frac{1}{n}\sum_{\{i\mid k_i\leqq n\}} f(T^{k_i}\omega).
\end{aligned}$$

Setting $f \equiv 1$, we see that

$$\lim_n \frac{|\{i \mid k_i \leqq n\}|}{n}$$

exists and is greater than zero, since in this case $\bar{g}'(\omega, x) \equiv \mu(Y')$ because of the ergodicity of $(X, \mathscr{X}, \mu, \varphi)$.

Therefore for almost every $\omega \in \Omega$

$$\tilde{f}(\omega) = \lim_n \frac{1}{n} \sum_{i=1}^{n} f(T^{k_i}\omega) = \lim_n \frac{n}{|\{i \mid k_i \leqq n\}|} \cdot \frac{1}{n} \sum_{\{i \mid k_i \leqq n\}} f(T^{k_i}\omega)$$

exists, and since $\tilde{f}(\omega) \leqq \bar{g}''(\omega, x)$ for almost every $x \in W$, it follows that $\tilde{f} \in L^1$.

Corollary. *If T is ergodic and if T and φ have no eigenvalues (other than 1) in common, then*

$$\tilde{f} = \int f\, dm \tag{2}$$

almost everywhere. In particular if T is weakly mixing, then (2) *is valid almost everywhere.*

Proof. The given condition implies that T' is ergodic (see [4]), therefore in the proof to Theorem 1

$$\bar{g}'(\omega, x) = \mu(Y') \int f\, dm$$

and

$$\bar{g}''(\omega, x) = \mu(Y'') \int f\, dm$$

almost everywhere (with respect to m'), and the conclusion follows.

4. Mean Ergodic Theorems

Theorem 2. *If $k_1, k_2, \ldots$ is a uniform sequence, $1 \leqq p < \infty$, and $f \in L^p$, then*

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f = \tilde{f}$$

in the L^p-norm.

Proof. Since $L^p \subseteq L^1$ and since the sequence of functions in question is uniformly integrable, the theorem follows from Theorem 1.

Theorem 3 (Blum-Hanson [1]). *If T is strongly mixing and $k_1, k_2, \ldots$ is any strictly increasing sequence of positive integers, then*

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f = \int f\, dm \tag{3}$$

in the L^p-norm for every $f \in L^p$ $(1 \leqq p < \infty)$. Conversely, if (3) *is valid in the L^2-norm for all indicator functions f and every strictly increasing sequence of positive integers, then T is strongly mixing.*

Proof. Let T be strongly mixing, $0<k_1<k_2\ldots$, and $f\in L^2$ with $\int f\,dm=0$. Then, denoting the scalar product in L^2 by $\langle\cdot,\cdot\rangle$, we have

$$\begin{aligned}\left\|\frac{1}{n}\sum_{i=1}^{n}T^{k_i}f\right\|_2^2&=\left\langle\frac{1}{n}\sum_{i=1}^{n}T^{k_i}f,\frac{1}{n}\sum_{j=1}^{n}T^{k_j}f\right\rangle\\&=\frac{1}{n^2}\sum_{i,j=1}^{n}\langle T^{k_i}f,T^{k_j}f\rangle=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n}\langle f,T^{k_i-k_j}f\rangle\right).\end{aligned}\tag{4}$$

Since strong mixing implies

$$\langle f,T^n f\rangle\xrightarrow[n]{}\left(\int f\,dm\right)^2=0,$$

each of the inner sums in the last member of (4) will be small if n is large enough, and thus also their mean. We conclude that

$$\lim_n\left\|\frac{1}{n}\sum_{i=1}^{n}T^{k_i}f\right\|_2=0.$$

The rest follows by approximation. The converse is easy, since Césaro norm convergence implies Césaro weak convergence, and Césaro weak convergence along all strictly increasing sequences implies in turn usual weak convergence.

5. An Individual Ergodic Theorem for Strictly Increasing Sequences

Let $f_1,f_2,\ldots\in L^1$ and define

$$\begin{aligned}F_n&=\sup_{1\leqq m\leqq n}\left(\sum_{i=1}^{m}f_i\right)^+\qquad(n\geqq 1),\\F_0&=0\\g_n&=F_n-F_{n-1}\qquad(n\geqq 1).\end{aligned}$$

Then

$$F_n=\sum_{i=1}^{n}g_i$$

and

$$g_n=\inf_{m\leqq n}\left(\sum_{i=m}^{n}f_i\right)^+.$$

We set

$$D=\left\{c\,|\,c=(c_n)_{n=1,2,\ldots},\ c_n\geqq c_{n+1}>0,\ \sum_{n=1}^{\infty}c_n\ \text{diverges}\right\}.$$

We shall need the following lemmas, which are special cases of Lemmas 6 and 7 in Krengel [5]; we omit the easy proofs.

Lemma 1. *For any $c\in D$ we have*

$$\limsup_n\frac{\sum_{i=1}^{n}c_i f_i}{\sum_{i=1}^{n}c_i}\leqq\limsup_n\frac{\sum_{i=1}^{n}c_i g_i}{\sum_{i=1}^{n}c_i}.$$

Lemma 2. *If $(a_{mn})_{m,n=1,2,\dots}$ is an infinite matrix such that*

1. *for fixed n, a_{mn} increases with m, and*

2. *for fixed m, a_{mn} decreases to 0 with increasing n, then there exists a $c\in D$ such that*

$$\sum_{i=1}^{\infty} c_i\, a_{mi}$$

converges for each m.

Theorem 4. *Let T be a measure preserving transformation on $(\Omega, \mathscr{B}, m)$. Then the following statements are equivalents:*

a) *T is strongly mixing.*

b) *For each increasing sequence of positive integers $k_1, k_2, \dots$ and each $f\in L^1$ there exists a $c\in D$ such that*

$$\lim_n \frac{\sum_{i=1}^{n} c_i\, T^{k_i} f}{\sum_{i=1}^{n} c_i} = \int f\, dm$$

almost everywhere.

Proof. 1. We show that a) implies b). Let $k_1, k_2, \dots$ be an increasing sequence and $f\in L^1$. Whithout loss of generality we may assume that $\int f\, dm = 0$. Set

$$f_n^{(m)} = T^{k_n}(f - 2^{-m})$$

and construct the sequence $g_1^{(m)}, g_2^{(m)}, \dots$ corresponding to the sequence $f_1^{(m)}, f_2^{(m)}, \dots$ as above. Now T is strongly mixing by assumption and

$$\int f_n^{(m)}\, dm = -2^{-m} < 0;$$

thus it follows from Theorem 3 that

$$\lim_n \int g_n^{(m)}\, dm = 0.$$

Now we put

$$a_{mn} = \sup_{j\geqq n} \int g_j^{(m)}\, dm.$$

Then for fixed n, a_{mn} increases with m, and for fixed m, a_{mn} decreases to zero. Applying Lemma 2, we obtain a $c\in D$ such that

$$\sum_{i=1}^{\infty} c_i\, a_{mi} < +\infty$$

for each m. It follows that for each m,

$$\sum_{n=1}^{\infty} c_n\, g_n^{(m)} < +\infty$$

almost everywhere. Therefore, using Lemma 1,

$$\limsup_n \frac{\sum_{i=1}^n c_i T^{k_i} f}{\sum_{i=1}^n c_i} = \limsup_n \frac{\sum_{i=1}^n c_i f_i^{(m)}}{\sum_{i=1}^n c_i} + 2^{-m} \leqq \lim_n \frac{\sum_{i=1}^n c_i g_i^{(m)}}{\sum_{i=1}^n c_i} + 2^{-m} = 2^{-m}. \qquad \text{a.e.}$$

Replacing f by $-f$, we obtain a.e.

$$\liminf_n \frac{\sum_{i=1}^n c_i T^{k_i} f}{\sum_{i=1}^n c_i} \geqq -2^{-m};$$

since these inequalities hold for each m, we conclude that

$$\lim_n \frac{\sum_{i=1}^n c_i T^{k_i} f}{\sum_{i=1}^n c_i} = 0$$

almost everywhere.

2. To show that b) implies a), suppose that b) is valid but T is not strongly mixing. Then there exist $E, F \in \mathscr{B}$ and an increasing sequence $k_i, k_2, \dots$ such that

$$\lim_n m(T^{-k_n} E \cap F)$$

exists but is not equal to $m(E)\, m(F)$. For the function $1_E \in L^1$ and the sequence $k_1, k_2, \dots$ choose a c as in b). Then

$$\lim_n \frac{\sum_{i=1}^n c_i 1_{T^{-k_i} E}}{\sum_{i=1}^n c_i} = m(E)$$

almost everywhere. Integrating over F and interchanging the limit with the integral (the sequence in question being uniformly integrable), we obtain

$$m(E)\, m(F) = \int_F m(E)\, dm = \lim_n \frac{\sum_{i=1}^n c_i \int_F 1_{T^{-k_i} E}\, dm}{\sum_{i=1}^n c_i}$$

$$= \lim_n \frac{\sum_{i=1}^n c_i\, m(T^{-k_i} E \cap F)}{\sum_{i=1}^n c_i} = \lim_n m(T^{-k_n} E \cap F),$$

since $\sum_{i=1}^{\infty} c_i$ diverges. This contradicts our assumption, and T must be strongly mixing.

Literature

1. Blum, J. R., and D. L. Hanson: On the mean ergodic theorem for subsequences. Bull. Amer. math. Soc. **66**, 308–311 (1960).
2. Fomin, S.: On dynamical systems with pure point spectrum. Doklady Akad. Nauk SSSR **77**, 29–32 (1951).
3. Halmos, P.R.: Lectures on ergodic theory. Math. Soc. Japan (1953).
4. Jacobs, K.: Neuere Methoden und Ergebnisse der Ergoden theorie. Springer Erg. d. Math. (1960).
5. Krengel, U.: Classification of states for operators. Proc. V Berkeley Sympos math. Statist. Probab. II, 2 (1967).
6. Oxtoby, J.C.: Ergodic sets. Bull. Amer. math. Soc. **58**, 116–136 (1952).

Dr. A. Brunel
Faculté des Sciences
Université de Rennes
Avenue du Général Leclerc
Rennes, France

Dr. M. Keane
Yale University
Department of Mathematics
Box 2155, Yale Station
New Haven, Conn. 06520, USA

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