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On the Possibility of an Unusual Extension of the Minimax Theorem*

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Summary. A natural extension of the minimax theorem for cooperative games is considered.

1. Introduction

The famous minimax theorem of von Neumann has motivated a large number of generalizations which are concerned with weaker conditions on the payoff function and the strategy sets.

In this note, we consider an examination of the possibility of extending the minimax theorem for *finite zero-sum two-person games.* The treatment will be based on an examination of the difference between the product of the mixed strategy sets and the cooperative strategy set. Certain intuitive considerations are related with this analysis. We will demonstrate the possibility of obtaining the minimax theorem in situations involving correlation among the behavior of both players, that is with *cooperation.* This cooperation must not be understood as a joint act on which the positions of both players increase, since the game is zero-sum.

The subject presented here possesses somewhat unusual characteristics in the actual theory of games.

2. Basic Facts

Consider a zero-sum finite two-person game $\Gamma=\{\Sigma_1, \Sigma_2; A\}$ whose mixed extension is $\tilde{\Gamma}=\{\tilde{\Sigma}_1, \tilde{\Sigma}_2; E\}$. Then, the set of *non-cooperative* actions $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ belongs to a euclidean space $R^{m_1+m_2}$ of m_1+m_2 dimensions, where m_i indicates the number of pure strategies of Σ_i, with $i\in\{1, 2\}$. Intuitively, this set *should* be a subset of the set $\widetilde{\Sigma_1 \times \Sigma_2}$ of *correlated* or *cooperative* strategies, which is a region of $m_1 \times m_2$ dimensions. Formally, such an embedding is obtained in a natural way by the continuous function nat which assigns to each pair of vectors $x\in\tilde{\Sigma}_1$ and $y\in\tilde{\Sigma}_2$ the corresponding tensor product $x\otimes y$ belonging to $\widetilde{\Sigma_1 \times \Sigma_2}$ which is defined by $x\otimes y(\sigma_1, \sigma_2)=x(\sigma_1)\cdot y(\sigma_2)$ for all the pure strategies $\sigma_1\in\Sigma_1$ and $\sigma_2\in\Sigma_2$.

Immediately we derive the following result:

Lemma 1. *The set* $\operatorname{nat}(\tilde{\Sigma}_1 \times \tilde{\Sigma}_2)=I$ *is contractible.*

Proof. For any arbitrary element $\bar{x}\otimes\bar{y}$ of I, consider a function α on $[0, 1]\times I$ with values in I defined by

$$(t, x\otimes y)\mapsto (t\,\bar{x}+(1-t)\,x)\otimes(t\,\bar{y}+(1-t)\,y).$$

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This function is well defined, since the strategy sets $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are both convex. It is continuous because it is a composition of continuous functions, namely: $\alpha = \text{nat} \circ \beta \circ (i \times (\text{nat})^{-1})$. Here, i indicates the identity function on the unit interval $[0,1]$ and β assigns for each point (t, x, y) of $[0,1] \times \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ the element $(t\bar{x} + (1-t)x, t\bar{y} + (1-t)y)$ belonging to $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$. For $t=0$, we have $\alpha(0, x \otimes y) = x \otimes y$ and for $t=1$, $\alpha(1, x \otimes y) = \bar{x} \otimes \bar{y}$ and therefore the set I is contractible. (q.e.d.)

It is remarkable that the set of joint statistical stategies $\widetilde{\Sigma_1 \times \Sigma_2}$ is the convex hull of the set I. Indeed, the vertices of the polyhedron $\widetilde{\Sigma_1 \times \Sigma_2}$ are those points $e_{ij} = e_i^1 \otimes e_j^2 \in I$ with $i \in \Sigma_1$ and $j \in \Sigma_2$ where $e_i^1 \in \tilde{\Sigma}_1$ and $e_j^2 \in \tilde{\Sigma}_2$ are the probabilities distributions having the i-th and j-th components equal to one, respectively.

Since we have imbedded the game in the cooperative set of strategies, it is now interesting to find the set of joint acts of both players having a same given marginal distribution of probability of one player. For any arbitrary mixed strategy of the first player $x \in \tilde{\Sigma}_1$, we define such a set

$$\Sigma_x = \{z \in \widetilde{\Sigma_1 \times \Sigma_2} : x(\sigma_1) = \sum_{\sigma_2 \in \Sigma_2} z(\sigma_1, \sigma_2) \text{ for all } \sigma_1 \in \Sigma_1\},$$

as the set of cooperative strategies *generated* by $x \in \tilde{\Sigma}_1$, that is, all those correlated acts reachable by the first player's mixed strategy $x \in \tilde{\Sigma}_1$. Similarly one can define with respect to the second player the set Σ_y.

A property of these regions is considered below:

Lemma 2. *For any $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$ the non empty sets Σ_x and Σ_y are convex and therefore $L(x, y) = \Sigma_x \cap \Sigma_y$ is non empty and convex.*

Proof. For a given mixed strategy $x \in \tilde{\Sigma}_1$, let z_1 and z_2 be two joint strategies belonging to Σ_x. Thus for each pure strategy $\sigma_1 \in \Sigma_1$,

$$\sum_{\sigma_2 \in \Sigma_2} z_1(\sigma_1, \sigma_2) = \sum_{\sigma_2 \in \Sigma_2} z_2(\sigma_1, \sigma_2) = x(\sigma_1),$$

and therefore for each real λ in the unit interval, it follows:

$$\sum_{\sigma_2 \in \Sigma_2} [\lambda z_1(\sigma_1, \sigma_2) + (1-\lambda) z_2(\sigma_1, \sigma_2)] = x(\sigma_1)$$

for all the $\sigma_1 \in \Sigma_1$, which implies the convexity of the non empty set Σ_x. Similarly, one could show the same for Σ_y. Finally, because $x \otimes y \in L(x, y)$, the intersection is non empty. (q.e.d.)

We will now try to characterize analytically the sets of joint strategies Σ_x, Σ_y and therefore also their intersection $L(x, y)$. Because of the similarity between this problem and the determination of the set of double stochastic matrices as the convex hull of the set of permutation matrices, we will use as Mirsky has done in [4] for such a type of problems, the following basic result given in Bonnesen and Fenchel [1]:

Lemma 3. *If $v, u_1, \ldots, u_r \in R^p$, then v belongs to the convex hull $K(u_1, \ldots, u_r)$ of the vectors $u_1, \ldots, u_r$ if and only if for each $a \in R^p$ there is a u_k $(k = 1, \ldots, r)$ such that the scalar product $a \cdot (u_k - v) \geqq 0$.*

For an arbitrary mixed strategy of the first player $x \in \tilde{\Sigma}_1$ and any function $\alpha\colon \Sigma_1 \to \Sigma_2$, let us define the joint statistical strategy $w(x, \alpha) \in \Sigma_x$ by

$$w(x,\alpha)(\sigma_1,\sigma_2)=x(\sigma_1)\delta(\alpha(\sigma_1),\sigma_2) \qquad \text{for all } \sigma_1 \in \Sigma_1 \text{ and } \sigma_2 \in \Sigma_2,$$

where δ is the Kronecker's delta function. Let W_x be the set $\bigcup_{\alpha \in \Sigma_2^1} \{w(x,\alpha)\}$ where Σ_2^1 indicates the set of functions $\alpha\colon \Sigma_1 \to \Sigma_2$, which has $m_2^{m_1}$ elements. Analogously, for any function $\beta\colon \Sigma_2 \to \Sigma_1$ and any strategy $y \in \Sigma_2$ one defines the strategy $w(y,\beta) \in \Sigma_y$ and the set $W_y = \bigcup_{\beta \in \Sigma_1^2} \{w(y,\beta)\}$ where Σ_1^2 indicates the set of all the functions $\beta\colon \Sigma_2 \to \Sigma_1$.

By using the previous lemma, one easily derives the following:

Theorem 4. *For any $x \in \tilde{\Sigma}_1$ the set Σ_x coincides with the convex hull of W_x: $\Sigma_x = K(W_x)$. Analogously $\Sigma_y = K(W_y)$.*

Proof. Consider a point $z \in K(W_x)$, then there is an $\lambda \geqq 0$ with $\sum_{\alpha \in \Sigma_2^1} \lambda(\alpha) = 1$ in the simplex $\tilde{\Sigma}_2^1$ of $R^{m_2^{m_1}}$ such that for all $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$:

$$z(\sigma_1,\sigma_2) = \sum_{\alpha \in \Sigma_2^1} \lambda(\alpha)\, w(x,\alpha)(\sigma_1,\sigma_2)$$

and therefore

$$\begin{aligned}\sum_{\sigma_2 \in \Sigma_2} \sum_{\alpha \in \Sigma_2^1} \lambda(\alpha) w(x,\alpha)(\sigma_1,\sigma_2) &= \sum_{\sigma_2 \in \Sigma_2} \sum_{\alpha \in \Sigma_2^1} \lambda(\alpha) x(\sigma_1) \delta(\alpha(\sigma_1),\sigma_2) \\ &= x(\sigma_1) \sum_{\alpha \in \Sigma_2^1} \lambda(\alpha) \sum_{\sigma_2 \in \Sigma_2} \delta(\alpha(\sigma_1),\alpha_2) = x(\sigma_1)\end{aligned}$$

which implies $K(W_x) \subset \Sigma_x$. Conversely, let us consider a point $z \in \Sigma_x$ and define for each matrix $a\colon \Sigma_1 \times \Sigma_2 \to R$ a function $\alpha_m\colon \Sigma_1 \to \Sigma_2$ such that

$$\max_{s_2 \in \Sigma_2} a(\sigma_1, s_2) = a(\sigma_1, \alpha_m(\sigma_1)) \qquad \text{for all } \sigma_1 \in \Sigma_1,$$

thus, for the point $z \in \Sigma_x$ under consideration, one has

$$\begin{aligned} a \cdot z &= \sum_{\sigma_1 \in \Sigma_1} \sum_{\sigma_2 \in \Sigma_2} a(\sigma_1,\sigma_2) z(\sigma_1,\sigma_2) \leqq \sum_{\sigma_1 \in \Sigma_1} \max_{s_2 \in \Sigma_2} a(\sigma_1, s_2) \sum_{\sigma_2 \in \Sigma_2} z(\sigma_1,\sigma_2) \\ &= \sum_{\sigma_1 \in \Sigma_1} a(\sigma_1, \alpha_m(\sigma_1)) x(\sigma_1) = a \cdot w(x, \alpha_m). \end{aligned}$$

For each vector $a \in R^{m_1 \times m_2}$ there is a $w(x, \alpha_m)$ such that $a \cdot (w(x,\alpha_m) - z) \geqq 0$. Thus, from Lemma 3 it follows that the point z belongs to $K(W_x)$. Similarly, one could prove the remaining equality. (q.e.d.)

As an immediate consequence of this result, one obtains that the intersection set $L(x, y)$ is a polyhedron.

3. An Example

Let us demonstrate geometrically and analytically by an example, which is the easiest non trivial one, the nature of joint strategy set $L(x, y)$.

Let the strategy sets Σ_1 and Σ_2 be the set $\{1, 2\}$. Then, the cooperative strategy set $\widetilde{\Sigma_1 \times \Sigma_2}$ is spanned by its vertices

$$e_{11} = e_1^1 \otimes e_1^2 = (1,0) \otimes (1,0), \qquad e_{12} = e_1^1 \otimes e_2^2 = (1,0) \otimes (0,1)$$
$$e_{21} = e_2^1 \otimes e_1^2 = (0,1) \otimes (1,0) \quad \text{and} \quad e_{22} = e_2^1 \otimes e_2^2 = (0,1) \otimes (0,1).$$

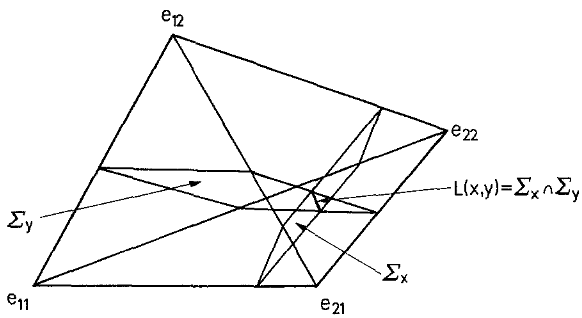
Now, for a given mixed strategy of the first player $x \in \tilde{\Sigma}_1$, Theorem 4 provides that the vertices of the convex polyhedron Σ_x are

$$x \otimes (1,0) = \begin{vmatrix} x(1) & 0 \\ x(2) & 0 \end{vmatrix}, \quad x \otimes (0,1) = \begin{vmatrix} 0 & x(1) \\ 0 & x(2) \end{vmatrix}, \begin{vmatrix} 0 & x(1) \\ x(2) & 0 \end{vmatrix}, \begin{vmatrix} x(1) & 0 \\ 0 & x(2) \end{vmatrix}.$$

Analogously, the convex polyhedron Σ_y has the following four vertices

$$(1,0) \otimes y = \begin{vmatrix} y(1) & y(2) \\ 0 & 0 \end{vmatrix}, \quad (0,1) \otimes y = \begin{vmatrix} 0 & 0 \\ y(1) & y(2) \end{vmatrix}, \begin{vmatrix} y(1) & 0 \\ 0 & y(2) \end{vmatrix}, \begin{vmatrix} 0 & y(2) \\ y(1) & 0 \end{vmatrix}$$

as it is shown in the figure.



Actually, the intersection set $L(x, y)$ as it is seen in the illustration is a segment which can be expressed as the convex hull of two points. But the analytical expression of these two points depends on the values of $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$, since they are located in different faces for different values.

Introducing the extremal points

$$P = \begin{vmatrix} 0 & x(1) \\ y(1) & x(2) - y(1) \end{vmatrix}, \qquad Q = \begin{vmatrix} x(1) & 0 \\ x(2) - y(2) & y(2) \end{vmatrix},$$
$$R = \begin{vmatrix} x(1) - y(2) & y(2) \\ x(2) & 0 \end{vmatrix} \quad \text{and} \quad S = \begin{vmatrix} y(1) & x(1) - y(1) \\ 0 & x(2) \end{vmatrix},$$

which belong to the corresponding faces of the simplex just considered for appropriated values of $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$, we have, by simple computation that

$$L(x, y) = \begin{cases} K(S, P) & \text{when} \quad x(2) \geqq y(1) \quad \text{and} \quad x(1) \geqq y(1), \quad (\pi_1) \\ K(Q, P) & \text{when} \quad x(2) \geqq y(1) \quad \text{and} \quad x(2) \geqq y(2), \quad (\pi_2) \\ K(Q, R) & \text{when} \quad x(2) \geqq y(2) \quad \text{and} \quad x(1) \geqq y(2), \quad (\pi_3) \\ K(S, R) & \text{when} \quad x(1) \geqq y(2) \quad \text{and} \quad x(1) \geqq y(1), \quad (\pi_4). \end{cases}$$

Now, one can express the point $x \otimes y \in L(x, y)$ in terms of the extremal points. Indeed, let us consider the continuous function λ_{nat} which assigns for each pair of non-cooperative strategy in $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ a value in the unit interval defined by

$$\lambda_{nat} = \begin{cases} x(1) & \text{when} \quad \pi_1 \\ y(1) & \text{when} \quad \pi_2 \\ x(2) & \text{when} \quad \pi_3 \\ y(2) & \text{when} \quad \pi_4 . \end{cases}$$

Then, it results that for any pair $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$:

$$x \otimes y = \lambda_{nat}(x, y)\, T + (1 - \lambda_{nat}(x, y))\, U$$

where T and U are the corresponding points P, Q, R and S depending on the values of $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$ as before.

One can extend this example in the general case by means of a continuous function λ, thus one has generated the point

$$z(x, y) = \lambda(x, y)\, T + (1 - \lambda(x, y))\, U \in L(x, y),$$

which represents the global action of both players when they choose $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$ respectively. Of course, in most of the cases they are correlated.

In the first instance, this fact might seem rather strange, since by the usual assumption in the game theory, the players act independently when they are choosing their own strategies. Now, this assumption concerning the rules of the game is generally not completely satisfied from an intuitive point of view in many games, for example when the players are physically present. Indeed, one might say that in such situations the *independence is broken down by the disturbance of psychological mechanisms*. Thus, in some sense the *distortion* from the independence is measured by the function λ. Of course, without any doubt this very delicate point has some connection with the intuitive and usual concepts of experience and spying.

4. On the Possibility of an Extention

In the simplest case just considered in the previous section, one could express the different *correlated surfaces* in the joint mixed strategy set by taking into account the function λ. This has been possible because we have known the vertices of the set $L(x, y)$, but in general we do not know such kind of points in an explicit form. Therefore the *distortion* from the independence should be measured in a different way.

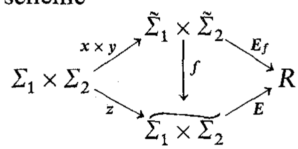
Before going into this in detail let us mention that the computation of the vertices of $L(x, y)$ is not a simple matter and it will not be treated here.

For a given game Γ, a continuous function

$$f\colon \tilde{\Sigma}_1 \times \tilde{\Sigma}_2 \to \widetilde{\Sigma_1 \times \Sigma_2}$$

such that for all the $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$ satisfies: $f(x, y) \in L(x, y)$ is called a *coupling function* of the game Γ. Intuitively such a function determines the cooperative surface $f(\tilde{\Sigma}_1 \times \tilde{\Sigma}_2)$ of strategies. When f coincides with the imbedding nat, then we have complete independence on the behavior of the players. The function f

is assumed given as a part of the game. It is related with the rules of the game and the characteristic of both players. Thus, the new components of the game Γ are now given in the following scheme



where E_f denotes the composition $E \circ f$.

Consider for each $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$ the sets of the maxima and minima values,

$$X_f(y) = \{x \in \tilde{\Sigma}_1 : \; E_f(x, y) = \max_{u \in \tilde{\Sigma}_1} E_f(u, y)\}$$

and

$$Y_f(x) = \{y \in \tilde{\Sigma}_2 : \; E_f(x, y) = \min_{v \in \tilde{\Sigma}_2} E_f(x, v)\}.$$

Let $\mathfrak{C}_1(A)$ be the set of coupling functions f such that $X_f(y)$ is contractible for every $y \in \tilde{\Sigma}_2$ and similarly $\mathfrak{C}_2(A)$ that set of f such that $Y_f(x)$ is contractible for every $x \in \tilde{\Sigma}_1$, then by using the following general result due to Debreu [2], which can be easily derived from the theorem of fixed point due to Eilenberg and Montgomery [3], we will derive the Theorem 6 which indicates that the minimax property holds for a wide class of cooperative surfaces.

Theorem 5. *Let $\Gamma^* = \{\Sigma_1^*, \Sigma_2^*; A^*\}$ be a zero-sum two-person game such that the strategy sets Σ_1^* and Σ_2^* are contractible polyhedra, and for each $\sigma_1^* \in \Sigma_1^*$ and $\sigma_2^* \in \Sigma_2^*$ the sets*

$$\{\sigma_2^* \in \Sigma_2^* : \; A^*(\sigma_1^*, \sigma_2^*) = \min_{t \in \Sigma_2^*} A(\sigma_1^*, t)\}$$

and

$$\{\sigma_1^* \in \Sigma_1^* : \; A^*(\sigma_1^*, \sigma_2^*) = \max_{s \in \Sigma_1^*} A(s, \sigma_2^*)\}$$

are contractible. Then, if A is continuous the game Γ^* has a saddle point.

Theorem 6. *For each $f \in \mathfrak{C}(A) = \mathfrak{C}_1(A) \cap \mathfrak{C}_2(A)$, the minimax property for E_f holds true, that is, there exist points $\bar{x} \in \tilde{\Sigma}_1$ and $\bar{y} \in \tilde{\Sigma}_2$, such that*

$$E_f(x, \bar{y}) \leqq E_f(\bar{x}, \bar{y}) \leqq E_f(\bar{x}, y)$$

for any $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$.

Proof. Consider the *f-mixed extension* given by $\tilde{\Gamma}_f = \{\tilde{\Sigma}_1, \tilde{\Sigma}_2; E_f\}$ which by construction satisfies all the requirements of Theorem 5, since E_f is continuous. Thus, the validity of the assertion is completely guaranteed. (q.e.d.)

Of course, the set for functions $\mathfrak{C}(A)$ always contains the embedding nat.

We will now show by an example that it can indeed have more than one element.

Again, let us consider the simplest case of two-person games treated in Section 3, and let λ_α be a continuous function defined by $\lambda_\alpha(x, y) = \alpha \lambda_{\text{nat}}(x, y) \in [0, 1]$ where $0 \leqq \alpha \leqq 1$. Thus, it induces a function f_α defined by

$$f_\alpha(x, y) = \lambda_\alpha(x, y)\, T + \big(1 - \lambda_\alpha(x, y)\big)\, U \in L(x, y)$$

where T and U as before are the extremal points of $L(x, y)$. When $\alpha = 1$, then $f_\alpha = \text{nat}$.

We want to see that under the condition

$$\Delta(1-\alpha)=0$$

where

$$\Delta=A(1,1)+A(2,2)-A(1,2)-A(2,1),$$

the payoff function E_{f_α} is bilinear. Then $f_\alpha \in \mathfrak{C}(A)$ and therefore Theorem 6 guarantees the existence of a saddle point for E_{f_α}.

In order to obtain it, let us compute explicitely the values of the expectation function. On the region π_1, we have

$$\begin{aligned} f_\alpha(x,y) &= \alpha\,\lambda_\alpha(x,y)\,S+(1-\alpha\,\lambda_\alpha(x,y))\,P \\ &= \alpha\,x(1)\begin{vmatrix} y(1) & x(1)-y(1) \\ 0 & x(2) \end{vmatrix} + (1-\alpha\,x(1))\begin{vmatrix} 0 & x(1) \\ y(1) & x(2)-y(1) \end{vmatrix} \\ &= \begin{vmatrix} \alpha\,x(1)\,y(1) & x(1)-\alpha\,x(1)\,y(1) \\ y(1)-\alpha\,x(1)\,y(1) & x(2)-y(1)+\alpha\,x(1)\,y(1) \end{vmatrix}. \end{aligned}$$

Thus, the value of the payoff function in this region π_1 is

$$\begin{aligned} E_{f_\alpha}(x,y)=E_{S,P}(x,y) &= \alpha\Delta\,x(1)\,y(1)+(A(1,2)-A(2,2))\,x(1) \\ &\quad +(A(2,1)-A(2,2))\,y(1)+A(2,2). \end{aligned}$$

Similarly, one can obtain that in the region π_2 the expectation which we indicate by $E_{S,P}(x,y)$ has the same expression as in π_1, and on the remaining regions π_3 and π_4 the expression of the expectation is

$$E_{f_\alpha}(x,y)=\alpha\Delta\,x(2)\,y(2)+(A(2,1)-A(1,1))\,x(2)+(A(1,2)-A(1,1))\,y(2)+A(1,1).$$

One can immediately verify that in the diagonal $x(1)+y(1)=1$ the first and seconds expressions coincide. Thus, the function E_{f_α} is continuous.

Now, for a given $x\in\tilde{\Sigma}_1$, the derivative with respect to $y(1)\in[0,1]$ is

$$\frac{\partial}{\partial y(1)}E_{f_\alpha}(x,y)=\begin{cases} \alpha\Delta\,x(1)+A(2,1)-A(2,2) & \text{if } (x,y)\in\pi_1\cup\pi_2 \\ -\alpha\Delta\,x(2)+A(1,1)-A(1,2) & \text{if } (x,y)\in\pi_3\cup\pi_4 \end{cases}$$

and then the payoff function is concave in $y\in\tilde{\Sigma}_2$ if and only if

$$\alpha\Delta\leqq\Delta.$$

On the other hand,

$$\frac{\partial}{\partial x(1)}E_{f_\alpha}(x,y)=\begin{cases} \alpha\Delta\,y(1)+A(1,2)-A(2,2) & \text{if } (x,y)\in\pi_1\cup\pi_2 \\ -\alpha\Delta\,y(2)+A(1,1)-A(2,1) & \text{if } (x,y)\in\pi_3\cup\pi_4 \end{cases}$$

is the respective derivative with respect to $x(1)\in[0,1]$. Thus, E_{f_α} is convex in $x\in\tilde{\Sigma}_1$ if and only if

$$\alpha\Delta\geqq\Delta.$$

From both inequalities, we get the condition $\Delta(1-\alpha)=0$. If this equality holds, then the payoff function E_f is bilinear.

The case $\alpha=1$ corresponds to the classic case and when $\Delta=0$, then for each $\alpha\in[0,1]$ the function E_{f_α} has a saddle point.

In the general case there is a subclass of coupling functions for which we will prove an interesting property. Let $\mathfrak{D}_1(A) \subset \mathfrak{C}_1(A)$ be the set of coupling functions for which the payoff function E_f is concave in $x \in \tilde{\Sigma}_1$ for each $y \in \tilde{\Sigma}_2$ and $\mathfrak{D}_2(A)$ those which are convex in $y \in \tilde{\Sigma}_2$ for each $x \in \tilde{\Sigma}_1$. Then one derives the following result:

Lemma 7. *If*

$$f, g \in \mathfrak{D}(A) = \mathfrak{D}_1(A) \cap \mathfrak{D}_2(A) \subset \mathfrak{C}(A),$$

then for each real number $\mu \in [0, 1]$, *the function* $\mu f + (1-\mu) g$ *belongs to* $\mathfrak{D}(A)$.

Proof. For any real number $\mu \in [0, 1]$, one has that

$$\mu f(x, y) + (1-\mu) g(x, y) \in L(x, y)$$

which indicates that the convex combination function is well defined. On the other hand, by construction,

$$E_{\mu f + (1-\mu) g}(x, y) = \mu E_f(x, y) + (1-\mu) E_g(x, y).$$

From the convexity and concavity of E_f and E_g, we see that the same conditions are satisfied for $E_{\mu f + (1-\mu) g}$. (q.e.d.)

This result is nothing more than the convexity of $\mathfrak{D}(A)$. Since this set belongs to the linear space of all the functions over $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ with values in the dual of $\Sigma_1 \times \Sigma_2$, the operation of convexity is consistent.

As an immediate consequence appears the next fact:

Corollary 8. *If there is a coupling function* $f \neq \text{nat}$ *in* $\mathfrak{D}(A)$, *then there are infinitely many functions for which the minimax theorem holds true.*

Proof. Given a function f in $\mathfrak{D}(A)$ which is not the natural embedding, consider the function $f_\mu = \mu f + (1-\mu)\,\text{nat}$ which by virtue of Lemma 7 belongs to $\mathfrak{D}(A)$. Thus, by Theorem 6, the assertion is guaranteed. (q.e.d.)

The examination just made was concerned explicitely on the matrix. Now, the same study can be realized for a whole class of games having the same strategy sets. Thus, if $\mathfrak{A}$ indicates the set of all the matrices with a fixed number of rows and columns, then from the above results one can inmediately derive that the non empty set

$$\mathfrak{D} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}_1(A) \cap \bigcap_{A \in \mathfrak{A}} \mathfrak{D}_2(A)$$

is convex.

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