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Asymptotic Normality Under Contiguity in a Dependence Case ★

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Summary. Let $X_{\nu i}=(X_{\nu i1}, X_{\nu i2}, \ldots, X_{\nu iK_i})$ $1\leqq i\leqq n_\nu$ be a sequence of independent random vectors following the regression model $X_{\nu ij}=\alpha+\beta C_{\nu ij}+\sigma Y_{\nu ij}$, with $-\infty<\alpha, \beta, C_{\nu ij}<\infty, \sigma>0$, and where $Y_{\nu i}=(Y_{\nu i1}, \ldots, Y_{\nu iK_i})$, $1\leqq i\leqq n_\nu$, are independent random vectors with absolutely continuous distributions $F^{(i)}(x^{(i)})$ and with densities $f^{(i)}(x^{(i)})$ $(x^{(i)}=(x_1, x_2, \ldots, x_{K_i}))$. Define $S_\nu=\sum_i\sum_j d_{\nu ij}\,\xi_{\nu R_{\nu ij}}$ where $\{\xi_{\nu k}\colon 1\leqq k\leqq N_\nu\}$ $\left(N_\nu=\sum_{i=1}^{n_\nu} K_i\right)$ is a double sequence of real numbers, $-\infty<d_{\nu ij}<\infty$ and $R_{\nu ij}=$rank of $X_{\nu ij}$ in a combined ranking of N_ν components $X_{\nu ij}$, $1\leqq j\leqq K_i$, $1\leqq i\leqq n_\nu$. Under certain assumptions on the densities $f^{(i)}(x^{(i)})$ and the sequences $\{\xi_{\nu ij}\}$, $\{d_{\nu ij}\}$ and $\{C_{\nu ij}\}$, the asymptotic normality of the sequence $\{S_\nu\}$, as $n_\nu\to\infty$, is proved. The results extend similar results of Hájek [3] and [4], from independently distributed components to the above pattern of dependence. An extension of the main theorem also covers the case when some of the distributions $F^{(i)}(x^{(i)})$ are singular. The connection between the Hájek condition (1.8) of [4] and the present condition (6.1) on the multivariate densities $f^{(i)}(x^{(i)})$ is also discussed.

1. Introduction

Let $X_{\nu i}=(X_{\nu i1}, X_{\nu i2}, \ldots, X_{\nu iK_i})$, $i=1,2,\ldots,n_\nu$, with $n_\nu\to\infty$ as $\nu\to\infty$, be a sequence of n_ν independent random vectors, where

$$X_{\nu i}=\alpha^{(i)}+\beta C_{\nu i}+\sigma Y_{\nu i}, \tag{1.1}$$

$\alpha^{(i)}=(\alpha, \alpha, \ldots, \alpha)$ (with K_i components), $C_{\nu i}=(C_{\nu i1}, C_{\nu i2}, \ldots, C_{\nu iK_i})$, $-\infty<\alpha$, $\beta<\infty$, $\sigma>0$ are constants and $Y_{\nu i}=(Y_{\nu i1}, Y_{\nu i2}, \ldots, Y_{\nu iK_i})$ are n independent random vectors distributed according to absolutely continuous distribution functions (d.f.'s) $F(x)=F^{(i)}(x)=F(x_1, x_2, \ldots, x_{K_i})$ $\left(x=x^{(i)}=(x_1, x_2, \ldots, x_{K_i})\right)$, $i=1,2,\ldots,n_\nu$; (although $F(x)$ depends on i, i is being suppressed for convenience). Denote $N_\nu=\sum_{i=1}^{n_\nu} K_i$ and let $V_{\nu 1}<V_{\nu 2}, \ldots, <V_{\nu N_\nu}$ stand for the N_ν ordered random components $X_{\nu ij}$, $1\leqq j\leqq K_i$, $1\leqq i\leqq n_\nu$; then if $R_{\nu ij}$ is the rank of $X_{\nu ij}$ in the above ordering, $X_{\nu ij}=V_{\nu R_{\nu ij}}$[1]. Consider now a double sequence of real numbers $\{\xi_{\nu k}\colon k=1,2,\ldots,N_\nu\}$ and define

$$S_\nu=\sum_{i=1}^{n_\nu}\sum_{j=1}^{K_i} d_{\nu ij}\,\xi_{\nu R_{\nu ij}}, \tag{1.2}$$

where $d_{\nu ij}$ are constants (not all zero) with $\sum_{i=1}^{n_\nu}\sum_{j=1}^{K_i} d_{\nu ij}=0$ and satisfying the condition (2.3).

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1. On account of the absolute continuity of $F(x)$, we may assume with probability one that no two of the X's are equal.

The purpose of this paper is to prove, under a set of sufficient conditions (stated in Section 2; see also the concluding remarks) on the d.f.'s $F^{(i)}(x)$ and the sequences $\{C_{\nu ij}\}$ and $\{\xi_{\nu,k}\colon k=1,2,\ldots,N_\nu\}$, the asymptotic normality, as $\nu\to\infty$, of the sums S_ν defined by (1.2). The present results extend those of Hâjek [4] to a case where the random components $X_{\nu ij}$ may not all be independent. Also in the proofs the constants $d_{\nu ij}$ are different, in general, from the constants $C_{\nu ij}$ and may not satisfy the condition (2.4)(ii). We are thus proving a Central Limit Theorem where the summands are functions of the random variables (components) which follow a certain pattern of dependence. The methods of proof follow the same line as in [4] based on the notion of "contiguous" measures introduced by Lecam [6] and Hâjek [4]. In the process, certain results of Hâjek [3] are also extended in Section 3. Sections 4 and 5 contain, respectively, the main theorem and an extension of the case when some $F^{(i)}(x)$, $i=1,2,\ldots,n_\nu$, are singular distributions. In the concluding Section 6, an application of the main theorem useful in nonparametric satistical theory is also considered.

2. Assumptions

Assume that the d.f.'s $F^{(i)}(x)$, $1\leqq i\leqq n_\nu$, satisfy the following conditions:

(i) *For any subset* $A_\nu=\{i\}\subset\{1,2,\ldots,n_\nu\}$, *the marginal joint distribution of any m components* $Y_{\nu ij_k}$ $1\leqq k\leqq m$, *where* $m\leqq\min_{i\in A_\nu} K_i$ *is the same for all* $i\in A_\nu$.

(ii) $F(x)=F^{(i)}(x^{(i)})$ *is absolutely continuous and possesses a continuous density* $f(x)=f(x^{(i)})$, $1\leqq i\leqq n_\nu$. (2.1)

(iii) *The derivative* $f^{(j)}=\partial f/\partial x_j$ *exists and is finite for each x, except perhaps for a countable number of them, and satisfies for each* $j=1,2,\ldots,K_i$, *whatever* K_i *may be, the condition* $0<\int[\{f^{(j)}(x)\}^2/f(x)]\,dx<\infty$.

(Define the integrand to be zero when $f(x)=0$.)

If F_1 denotes the marginal distribution of a single component and F_2 denotes the bivariate marginal distribution of $(Y_{\nu ij}, Y_{\nu ij'})$ and so on, then the condition (2.1)(i) states that $F_1, F_2, \ldots$ do not depend on i, $1\leqq i\leqq n_\nu$; nor on j, (j,j'), ... if, in addition, $F^{(i)}(x_1,\ldots,x_{K_i})$ are symmetric in the arguments. The condition (2.1)(iii) is not too stringent a condition: It is satisfied when $F(x)$ is multivariate normal and for $K_i=1$, it is satisfied by several well known unvariate distributions (see [4]). It is also satisfied when the components $Y_{\nu ij}$, $1\leqq j\leqq K_i$, are independent and the marginal distributions satisfy this condition. It is satisfied by the multivariate symmetric Cauchy (see Feller [1], p. 69) and logistic distributions (see Gumbel [2]), and so on (see also the concluding remarks). Also, in case of the symmetry of $F(x)$, the condition (2.1)(iii) is essentially a single condition and may be stated for any fixed j, say $j=1$. Further for $K_i=1$, it reduces to (cf. (6.2) of [4]).

$$\int_0^1 \eta^2(u)\,du<\infty \tag{2.2}$$

where $\eta(u)=\{-f_1'(F_1^{-1}(u))/f_1(F_1^{-1}(u))\}$, $0<u<1$, $f_1'=df_1/dx$ and $F_1^{-1}(u)=\operatorname{Inf}\{x\colon F_1(x)=u\}$.

For the sequences $\{C_{\nu ij}\}$ and $\{d_{\nu ij}\}$, we impose the conditions

$$\lim_{\nu\to\infty}\left\{\max_{(i,j)}|d_{\nu ij}|^2\Big/\sum_{i=1}^{n_\nu}\sum_{j=1}^{K_i}d_{\nu ij}^2\right\}=0, \tag{2.3}$$

and

$$\begin{aligned}&\text{(i)}\quad \lim_{\nu\to\infty}\left\{\max_{(i,j)}|C_{\nu ij}|^2\Big/\sum_{i=1}^{n_\nu}\sum_{j=1}^{K_i}C_{\nu ij}^2\right\}=0\\ &\text{(ii)}\quad \sup_{\nu}\sum_{i=1}^{n_\nu}\sum_{j=1}^{K_i}C_{\nu ij}^2<\infty,\end{aligned} \tag{2.4}$$

where (without loss of generality) we assume $\sum_i\sum_j C_{\nu ij}=0$. For the sequence of vectors $\{\xi_{\nu k}\colon k=1,2,\dots,N_\nu\}$, assume that $\xi_{\nu 1}\leqq\xi_{\nu 2}\leqq\cdots\leqq\xi_{\nu N_\nu}$ and that there exists a nonconstant, nondecreasing function $\xi(u)$ on $(0,1)$ with $\int_0^1\xi^2(u)\,du<\infty$, such that the step function $\xi_\nu(u)$ defined on $(0,1)$ by $\xi_\nu(u)=\xi_{\nu,k}=\xi_\nu(k/(N_{\nu+1}))$ for $(k-1/N_\nu)<u\leqq(k/N_\nu)$, $k=1,2,\dots,N_\nu$, satisfies the condition

$$\lim_{\nu\to\infty}\int_0^1\{\xi_\nu(u)-\xi(u)\}^2\,du=0. \tag{2.5}$$

3. Preliminary Results

Consider a sequence $\{U_{\nu i}\colon i=1,2,\dots,n_\nu\}$, with $U_{\nu i}=(U_{\nu i1},U_{\nu i2},\dots,U_{\nu iK_i})$, of independent random vectors such that

(3.1)

(i) Each component $U_{\nu ij}$ $(1\leqq i\leqq n_\nu, 1\leqq j\leqq K_i)$ is distributed marginally as a uniform random variable over $(0,1)$.

(ii) Random vector $U_{\nu i}$ is distributed according to a continuous d.f. $G(x)=G^{(i)}(x)=G(x_1,x_2,\dots,x_{K_i})$ which is jointly symmetric in the arguments and such that $P[U_{\nu ij}=U_{\nu ij'}]=0$ for every pair (j,j'), with $j\neq j'$.

(iii) The condition (2.1)(i) is satisfied for the distributions $G^{(i)}(x)$ in place of $F^{(i)}(x)$.

Let as before $R_{\nu ij}$ denote the rank of $U_{\nu ij}$ in a combined ordering of the $N_\nu=\sum_{i=1}^{n_\nu}K_i$ components $U_{\nu ij}$'s. The second condition in (3.1)(ii) has been assumed to ensure that, with probability 1, all components have distinct ranks. We shall concern ourselves in this section with the asymptotic behaviour of the sums S_ν under the assumptions (3.1). The results in Theorem 3.1 below are analogues of Theorems 3.1 and 3.2 of Hájek [3] and are based on the following extension of Lemma 2.1 of [3]: Let $Z_{\nu 1}<Z_{\nu 2}<\cdots<Z_{\nu N_\nu}$ denote the ordered components $U_{\nu ij}$'s i.e., $U_{\nu ij}=Z_{\nu R_{\nu ij}}$, $K_\nu^*=\max_{1\leqq i\leqq n_\nu}K_i$, $E_{\nu l}=\{i\colon K_i=l\}$, $l=1,2,\dots,K_\nu^*$, $n_{\nu l}=$ number of elements in $E_{\nu l}$ and $N_{\nu l}=\sum_{i\in E_{\nu l}}K_i$.

Lemma 3.1. *Let $E(\cdot)$ stand for the expectation operation under the assumptions (3.1). Then for every $i \in E_{\nu l}$,*

$$E[\xi_\nu(U_{\nu ij}) - \xi_\nu(R_{\nu ij}/(N_\nu+1))]^2 \leqq \frac{(8K_\nu^*)^{\frac{1}{2}}}{N_{\nu l}} \max_{1 \leqq k \leqq N_\nu} |\xi_{\nu k} - \bar{\xi}_\nu| \left(\sum_{k=1}^{N_\nu} (\xi_{\nu k} - \bar{\xi}_\nu)^2\right)^{\frac{1}{2}},$$

where $\bar{\xi}_\nu = (\sum_k \xi_{\nu k})/N_\nu$.

Proof. First we note that for every pair (i, j), $1 \leqq j \leqq K_i$ and $i \in E_{\nu l}$, with $E_{\nu l}$ non-empty,

$$P[R_{\nu ij} = k/(Z_{\nu 1}, Z_{\nu 2}, \ldots, Z_{\nu N_\nu})] \leqq (1/N_{\nu l}) \quad \text{a.s.} \tag{3.2}$$

To see this, let $p_{ij}^{(k)}$ denote the L.H.S. of (3.2). Then on account of the symmetry condition (3.1)(ii), $p_{ij}^{(k)} = p_{ij'}^{(k)}$ for every pair (j, j') and on account of (3.1)(iii), $p_{ij}^{(k)} = p_{i'j'}^{(k)}$ for every pair (j, j') and $i, i' \in E_{\nu l}$. Denoting the common value by $p_l^{(k)}$, we have

$$\sum_{l=1}^{K_\nu^*} N_{\nu l} p_l^{(k)} = \sum_{l=1}^{K_\nu^*} \sum_{i \in E_{\nu i}} \sum_{j=1}^{K_i} p_{ij}^{(k)} = \sum_{i=1}^{n_\nu} \sum_{j=1}^{K_i} p_{ij}^{(k)} = 1 \quad \text{a.s.},$$

from which (3.2) follows. Now let ν be surpressed for convenience while writing $U_{\nu ij}$, $R_{\nu ij}$, $Z_{\nu ij}$, n_ν and N_ν's. On account of (3.2) we have for every $i \in E_{\nu l}$, with $E_{\nu l}$ non-empty,

$$\begin{aligned} &E[\xi_\nu(U_{ij}) - \xi_\nu(R_{ij}/(N+1))]^2 \\ &= E[E(\{\xi_\nu(U_{ij}) - \xi_\nu(R_{ij}/(N+1))\}^2/(Z_1, Z_2, \ldots, Z_N))] \\ &= E\left[\sum_{k=1}^{N} P[R_{ij} = k/(Z_1, \ldots, Z_N)] \{\xi_\nu(Z_k) - \xi_\nu(k/(N+1))\}^2\right] \\ &\leqq \frac{1}{N_{\nu l}} E\left[\sum_{k=1}^{N} \{\xi_\nu(Z_k) - \xi_\nu(k/(N+1))\}^2\right]. \end{aligned} \tag{3.3}$$

Now let M denote the number of U_{ij}'s less than (m/N) $(m \leqq N)$ and I_A the indicator function of the set A, then $M = \sum_{i=1}^{n} \sum_{j=1}^{K_i} I_{[U_{ij} < (m/N)]}$ and it is easily verified that

$$\begin{aligned} \operatorname{Var}(M) &\leqq \sum_{i=1}^{n} K_i \sum_{j=1}^{K_i} \operatorname{Var}(I_{[U_{ij} < (m/N)]}) \\ &\leqq K_\nu^* m[1 - (m/N)]. \end{aligned} \tag{3.4}$$

From (3.3), (3.4) and the remaining arguments of Lemma 2.1 of [3], the proof follows.

We need the following notation for the statement of Theorem 3.1: Let

$$\begin{aligned} \lambda &= \frac{\left[\int_0^1 \int_0^1 \xi(u)\xi(v)\, dG(u,v) - \left(\int_0^1 \xi(u)\, du\right)^2\right]}{\left[\int_0^1 \xi^2(u)\, du - \left(\int_0^1 \xi(u)\, du\right)^2\right]} \qquad (|\lambda| \leqq 1) \\ \mu_\nu &= \left[-\sum_{i=1}^{n_\nu} \sum_{j \neq j'} d_{\nu ij} d_{\nu ij'}\right] \Big/ \left[\sum_{i=1}^{n_\nu} \sum_{j=1}^{K_i} d_{\nu ij}^2\right], \qquad (|\mu_\nu| \leqq 1), \end{aligned} \tag{3.5}$$

where $G(u, v)$ is the marginal d.f. of $(U_{vij}, U_{vij'})$ and denote

$$T_v^* = \sum_{i=1}^{n_v} \sum_{j=1}^{K_i} d_{vij} \xi(U_{vij})$$
$$T_v = \sum_{i=1}^{n_v} \sum_{j=1}^{K_i} d_{vij} \xi_v(U_{vij}). \tag{3.6}$$

Assume further that

$$K_i < t \quad \text{for all } i. \tag{3.7}$$

Theorem 3.1. *Assume that either* (i) $K_i = K$ *and* $\operatorname{Lim}_v \inf(1-\mu_v \lambda) > 0$ *or* (ii) $\sum_j d_{vij} = 0$, $1 \leqq i \leqq n_v$, *and* $\operatorname{Lim}_v (n_{vl}/n_v) = \zeta_l > 0$ *for every l for which E_{vl} is non-empty. Then under the conditions* (2.3), (2.4), (2.5), (3.1) *and* (3.7),

(a) $\operatorname{Lim}_{v\to\infty} \{E(S_v - T_v)^2/\operatorname{Var}(T_v)\} = 0$

(b) $\operatorname{Lim}_{v\to\infty} \{E(T_v - T_v^*)^2/\operatorname{Var} T_v\} = 0.$

Proof. We prove part (a); part (b) is easily proved through similar arguments. For convenience we shall again suppress v as in the proof of Lemma 3.1. First consider the case (ii). Setting $t_{vij} = \xi_v(U_{ij}) - \xi_v(R_{ij}/(N+1))$ we note that, on account of the symmetry condition (3.1)(ii), $E(t_{vij})$ and $\sigma_i^2 = \operatorname{Var}(t_{vij})$ do not depend on j and $\tau_{ii'} = \operatorname{Cov}(t_{vij}, t_{vi'j'})$ does not depend on (j, j'). Consequently $E(S_v - T_v) = 0$, so that denoting $t_{vi} = \sum_j d_{vij} t_{vij}$ we obtain

$$\begin{aligned} E(S_v - T_v)^2 &= \operatorname{Var}(S_v - T_v) \\ &= \sum_i \operatorname{Var}(t_{vi}) + \sum_{i \neq i'} \operatorname{Cov}(t_{vi}, t_{vi'}) \\ &= \sum_i \Big(\sum_j d_{vij}^2\Big) \sigma_i^2 + \sum_i \Big(\sum_{j \neq j'} d_{vij} d_{vij'}\Big) \tau_{ii} \\ &\quad + \sum_{i \neq i'} \operatorname{Cov}(t_{vi}, t_{vi'}). \end{aligned}$$

But since $\sum_j d_{vij} = 0$, $\sum_j d_{vij}^2 = -\sum_{j \neq j'} d_{vij} d_{vij'}$, *and*

$$\begin{aligned} \operatorname{Cov}(t_{vi}, t_{vi'}) &= \left(\sum_{j=1}^{K_i} \sum_{j'=1}^{K_{i'}} d_{vij} d_{vi'j'}\right) \tau_{ii'} \\ &= \left(\sum_{j=1}^{K_i} d_{vij}\right)\left(\sum_{j'=1}^{K_{i'}} d_{vi'j'}\right) \tau_{ii'} \\ &= 0, \end{aligned}$$

so that using the easily verifiable relation $-\tau_{ii} \leqq \sigma_i^2/(K_i - 1)$, $(K_i \geqq 2)$, we obtain

$$\begin{aligned} E(S_v - T_v)^2 &= \sum_i \Big(\sum_j d_{vij}^2\Big)(\sigma_i^2 - \tau_{ii}) \\ &\leqq \sum_i \Big(\sum_j d_{vij}^2\Big) K_i \sigma_i^2/(K_i - 1) \\ &\leqq 2 \sum_i \Big(\sum_j d_{vij}^2\Big) E[\xi_v(U_{ij}) - \xi_v(R_{ij}/(N+1))]^2. \end{aligned} \tag{3.8}$$

On account of the condition (2.5), it easily follows that $\operatorname{Var}(T_\nu) \sim \operatorname{Var}(T_\nu^*)$, so that from (3.8) and Lemma 3.1, we obtain

$$\frac{E(S_\nu - T_\nu)^2}{\operatorname{Var}(T_\nu)} \leqq 4\sqrt{2}\, t^{\frac{1}{2}} \left\{ \sum_l \frac{N}{N_{\nu l}} \sum_{i \in E_{\nu l}} \sum_j d_{\nu ij}^2 \Big/ \left(\sum_i \sum_j d_{\nu ij}^2 \right) \right\} \cdot \frac{\max_{1 \leqq k \leqq N} |\xi_{\nu k} - \bar{\xi}_\nu| \left(\sum_{k=1}^{N_\nu} (\xi_{\nu k} - \bar{\xi}_\nu)^2 \right)^{\frac{1}{2}}}{N \left[\int_0^1 \xi^2(u)\,du - \left(\int_0^1 \xi(u)\,du \right)^2 \right] (1 - \mu_\nu \lambda)}.$$

Using the fact that $\mu_\nu = 1$ and (3.16) and (3.17) of Hájek [3] (with a's replaced by ξ's), the proof of part (a)(ii) follows. The proof of part (a)(i) can be accomplished similarly by using $\sum_{i=1}^{n_\nu} \sum_{j=1}^{K} d_{\nu ij} = 0$, the symmetry properties and deriving an inequality similar to (3.8). The proof is complete.

Remark 3.1. It is important to observe that the symmetry condition in (3.1)(ii) is not an essential condition: For example, if $K_i = K$ and $\sum_i d_{\nu ij} = 0$ or all $1 \leqq j \leqq K$, it is easily seen that the conclusions of Theorem 3.1 continue to hold under the same conditions, but without the condition of symmetry of $G(x_1, x_2, \ldots, x_K)$ in the arguments.

4. The Main Theorem

The proof of the main theorem below, concerning the asymptotic normality of the sequence S_ν under the model (1.1), is based on the notion of sequences of "contiguous" measures: Consider two sequences $\{P_\nu\}$ and $\{Q_\nu\}$ of probability measures defined on a sequence of measurable spaces $(\mathscr{X}_\nu, \mathscr{A}_\nu)$, $1 \leqq \nu < \infty$. If $P_\nu(A_\nu) \to 0$ implies $Q_\nu(A_\nu) \to 0$, as $\nu \to \infty$, for any sequences of events of $A_\nu \in \mathscr{A}_\nu$, then the measures Q_ν are said to be contiguous to the measures P_ν. Contiguity implies that the P_ν-singular part in the decomposition of Q_ν (with respect to P_ν) tends to zero, as $\nu \to \infty$.

Now let $Q_{\nu i}$, $i \leqq i \leqq n_\nu$, and Q_ν stand, respectively, for the probability distributions of the vectors $X_{\nu i} = (X_{\nu i1}, X_{\nu i2}, \ldots, X_{\nu iK_i})$, $1 \leqq i \leqq n_\nu$, and $X_\nu = (X_{\nu 1}, X_{\nu 2}, \ldots, X_{\nu n_\nu})$ under the model (1.1), and $P_{\nu i}$, $1 \leqq i \leqq n_\nu$, and P_ν correspond to the model (1.1) with $\beta = 0$. The proof of the main theorem below is accomplished, as in [4], by showing that under the assumed conditions Q_ν are contiguous to P_ν and that the conclusions of Lemma 4.2 of Hájek [4] apply irrespective of whether $X_{\nu i}$ are univariate or multivariate. We need the following

Lemma 4.1. *Let $s(x) = s(x_1, x_2, \ldots, x_K)$ be absolutely continuous in each argument x_α, $1 \leqq \alpha \leqq K$, for almost all (w.r. to the Lebesgue measure $\mu^{(K-1)}$ on $E_\alpha^{(K-1)}$) $x^{(\alpha)} \in E_\alpha^{(K-1)} = \{y^{(\alpha)} = (y_1, \ldots, y_{\alpha-1}, y_{\alpha+1}, \ldots, y_K): -\infty < y_i < \infty\}$ such that for each, $1 \leqq \alpha \leqq K$, $\int_{E^{(K)}} (\partial s/\partial x_j)^2\, dx < \infty$. Then if $h = (h_1, h_2, \ldots, h_K)$ and $\|h\|^2 = \sum_{j=1}^{K} h_j^2$,*

$$\lim_{\|h\| \to 0} \int_{E^{(K)}} \left\{ \frac{s(x) - s(x-h)}{\|h\|} - \sum_{j=1}^{K} \frac{h}{\|h\|} \frac{\partial s}{\partial x_j} \right\}^2 dx = 0,$$

where $E^{(K)}$ = K-dimensional Cartesian space.

Proof. First we prove that

$$\underset{\|h\|\to 0}{\text{Lim}} \int_{E^{(K)}} \left\{ \frac{s(x-h^{(j+1)})-s(x-h^{(j)})}{h_j} - \frac{\partial s}{\partial x_j} \right\}^2 dx = 0 \tag{4.1}$$

where $h^{(j)}=(0,0,\dots,h_j,h_{j+1},\dots,h_K)$. To see this let $h_j^*=(0,0,\dots,0,h_j,0,\dots,0)$ and $y=x-h^{(j)}$, where $y=(y_1,y_2,\dots,y_K)$. Then

$$\begin{aligned}
\text{L.H.S.} &= \underset{\|h\|\to 0}{\text{Lim}} \int_{E^{(K)}} \left\{ \frac{s(y+h_j^*)-s(y)}{h_j} - \frac{\partial s(y+h^{(j)})}{\partial y_j} \right\}^2 dy \\
&\leqq 2 \underset{\|h\|\to 0}{\text{Lim}} \int_{E^{(K)}} \left\{ \frac{s(y+h_j^*)-s(y)}{h_j} - \frac{\partial s(y)}{\partial y_j} \right\}^2 dy \\
&\quad + 2 \underset{\|h\|\to 0}{\text{Lim}} \int_{E^{(K)}} \left\{ \frac{\partial s(y)}{\partial y_j} - \frac{\partial s(y+h^{(j)})}{\partial y_j} \right\}^2 dy.
\end{aligned}$$

The second term on the right equals zero by the well-known property of "continuity of translation in the L_2-norm" for all quadratically integrable functions (see, for example, Hewitt and Stromberg [5], p. 199). The first term on the right can be proved to be equal to zero by using Lemma 4.3 of [4] and the Lebesgue dominated convergence theorem. To see this let

$$E_j^{(K-1)} = \{y^{(j)} = (y_1,\dots,y_{j-1},y_{j+1},\dots,y_K):\ -\infty < y_i < \infty\}$$

and note that since $\int_{E^{(K)}} \{\partial s(y)/\partial y_j\}^2 dy < \infty$, we have for almost all $y^{(j)} \in E_j^{(K-1)}$

$$h(y^{(j)}) = \int_{-\infty}^{\infty} \left(\frac{\partial s(y)}{\partial y_j} \right)^2 dy_j < \infty; \tag{4.2}$$

since $s(y)$ is absolutely continuous in each argument x_j for almost all $y^{(j)} \in E_j^{(K-1)}$, it follows from (4.2) and Lemma 4.3 of [4] that for almost all $y^{(j)} \in E_j^{(K-1)}$

$$\begin{aligned}
g_{h_j}(y^{(j)}) &= \int_{-\infty}^{\infty} \left\{ \frac{s(y+h_j^*)-s(y)}{h_j} - \frac{\partial s}{\partial y_j} \right\}^2 dy_j \\
&\to 0,
\end{aligned}$$

as $h_j \to 0$, and from (4.23) of [4] that $|g_{h_j}(y^{(j)})| \leqq 4h(y^{(j)})$ with $\int h(y^{(j)}) dy^{(j)} < \infty$, so that the Lebesgue dominated convergence theorem applies and (4.1) is proved. The proof of the Lemma now follows from (4.1) and

$$\begin{aligned}
0 &\leqq \underset{\|h\|\to 0}{\text{Lim}} \int_{E^{(K)}} \left\{ \frac{s(x)-s(x-h)}{\|h\|} - \sum_{j=1}^{K} \frac{h_j}{\|h\|} \frac{\partial s}{\partial x_j} \right\}^2 dx \\
&\leqq \underset{\|h\|\to 0}{\text{Lim}} \int_{E^{(K)}} \left[\sum_{j=1}^{K} \frac{h_j}{\|h\|} \left\{ \frac{s(x-h^{(j+1)})-s(x-h^{(j)})}{h_j} - \frac{\partial s}{\partial x_j} \right\} \right]^2 dx \\
&\leqq K \sum_{j=1}^{K} \underset{\|h\|\to 0}{\text{Lim}} \int_{E^{(K)}} \left\{ \frac{s(x-h^{(j+1)})-s(x-h^{(j)})}{h_j} - \frac{\partial s}{\partial x_j} \right\}^2 dx \\
&= 0;
\end{aligned}$$

the proof is complete.

Let $\mathscr{L}(Y_\nu/P_\nu)\to N(a_\nu, b_\nu^2)$ denote that $(Y_\nu - a_\nu)/b_\nu$ converges in distribution to the normal $N(0,1)$ distribution, under P_ν.

Theorem 4.1. *Suppose that the model* (1.1), *together with conditions* (2.1), (2.4) *and the symmetry of* $F(x_1,\dots,x_{K_i})$ *in the arguments, holds and that either the conditions* (i) *or the conditions* (ii) *of Theorem* 3.1 *are satisfied. Then under the assumptions* (2.3), (2.5) *and* (3.7), *the sequence* S_ν, *defined by* (1.2), *satisfies*

$$\mathscr{L}(S_\nu)\to N(m_\nu, b_\nu^2),$$

with

$$m_\nu = \Big(\sum_i\sum_j d_{\nu ij}\, C_{\nu ij}\Big)\int_0^1 \eta(u)\,\xi(u)\,du,$$
$$b_\nu^2 = \Big(\sum_i\sum_j d_{\nu ij}^2\Big)\left[\int_0^1 \xi^2(u)\,du - \Big(\int_0^1 \xi(u)\,du\Big)^2\right](1-\mu_\nu\lambda), \tag{4.3}$$

where λ *and* μ_ν *are given by* (3.5), $\eta(u)$ *by* (2.2) *and* $G(u,v)$ *is the d.f. of the vector* $(F_1(Y_{\nu ij}), F_1(Y_{\nu ij'}))$.

Remark. Under the conditions (ii) of Theorem 3.1, viz., $\sum_{j=1}^{K_i} d_{\nu ij}=0$ for each i, $\mu_\nu=1$ so that b_ν^2 reduces to

$$b_\nu^2 = \Big(\sum_i\sum_j d_{\nu ij}^2\Big)\left[\int_0^1 \xi^2(u)\,du - \int_0^1\int_0^1 \xi(u)\,\xi(v)\,dG(u,v)\right].$$

Proof. Let $s_i(x)=[f(x^{(i)})]^{\frac{1}{2}}$; then on account of the condition (2.1)(iii) each $(\partial s_i/\partial x_j)$, $j=1,2,\dots,K_i$, is quadratically integrable and consequently the conclusion of Lemma 4.1 holds for each $s_i(x)$, $i=1,2,\dots,n_\nu$. Also (4.15) of [4] takes the form

$$W_\nu = 2\sum_{i=1}^{n_\nu}\left\{\frac{s_i(Y_{\nu i}-\gamma C_{\nu i})}{s_i(Y_{\nu i})}-1\right\}$$

where $\gamma=(\beta/\sigma)$ and $Y_{\nu i}=X_{\nu i}-\alpha^{(i)}-\beta C_{\nu i}$. Letting $h_{\nu ij}=\gamma C_{\nu ij}$, $h_{\nu i}=\gamma C_{\nu i}$, $E(\cdot)$ stand for the expectation under P_ν and $T_\nu'=\sum_i\sum_j C_{\nu ij}[f^{(j)}(Y_{\nu i})/f(Y_{\nu i})]$, it follows that

$$\begin{aligned} &E\{W_\nu - E(W_\nu)-\gamma T_\nu'\}^2\\ &\leqq 4\gamma^2\sum_i\Big(\sum_j C_{\nu ij}^2\Big)\int\left\{\frac{s(x)-s(s-h_{\nu i})}{\|h_{\nu i}\|}-\sum_j\frac{h_{\nu ij}}{\|h_{\nu i}\|}\frac{\partial s_i}{\partial x_j}\right\}^2 dx \\ &\to 0, \end{aligned} \tag{4.4}$$

on account of Lemma 4.1, $K_i<t$ and (2.4)(ii). Further it is easily seen that $E(W_\nu)\sim -\gamma^2\tau_\nu^2/4$ and that for the sums $T_{\nu i}'$ Lindeberg-Feller condition (see Loeve [7] p. 280) is satisfied due to $K_i<t$ and (2.4)(i), so that from (4.4) we obtain $\mathscr{L}(W_\nu/P_\nu)\to N(-\gamma^2\tau_\nu^2/4, \gamma^2\tau_\nu^2)$, where

$$\tau_\nu^2=\sum_i\int\Big\{\sum_j C_{\nu ij}[f^{(j)}(x)/\sqrt{f(x)}]\Big\}^2 dx.$$

The condition (4.17) of [4] is also satisfied on account the same conditions. Since τ_ν^2 remains bounded, as $\nu\to\infty$, assuming without loss of generality that $\tau_\nu^2\to\tau^2<\infty$, it follows from Lemma 4.2 of [4] that $\{Q_\nu\}$ is contiguous to $\{P_\nu\}$ and that

$$P_\nu - \operatorname*{Lim}_{\nu\to\infty} \{L_\nu + \tfrac{1}{2}\gamma^2\tau^2 - \gamma T_\nu'\} = 0, \tag{4.5}$$

where L_ν is given by (4.16) of [4].

We will now use part (iii) of this lemma to establish the asymptotic normality of S_ν under Q_ν, as $\nu\to\infty$. Let $U_{\nu ij}=F_1(Y_{\nu ij})$, then the vectors $U_{\nu i}$, $1\leq i\leq n_\nu$, satisfy the conditions (3.1). Consequently, if we set $T_\nu^* = \sum_i \sum_j d_{\nu ij}\xi(F_1(Y_{\nu ij}))$ it follows using Lindeberg-Feller theorem again that

$$\mathscr{L}(T_\nu^*/P_\nu) \to N(0, b_\nu^2), \tag{4.6}$$

and from Theorem (3.1) and the contiguity of $\{Q_\nu\}$ to $\{P_\nu\}$ that if, after proper normalization, one of the limits exists

$$\operatorname*{Lim}_{\nu\to\infty} \mathscr{L}(S_\nu/Q_\nu) = \operatorname*{Lim}_{\nu\to\infty} \mathscr{L}(T_\nu^*/Q_\nu). \tag{4.7}$$

To apply part (iii) of Lemma 4.2 of [4], we need to prove the asymptotic bivariate normality of L_ν and T_ν^* or equivalently (cf. Section 7 of Wald-Wolfowitz [9]) that of an arbitrary linear combination of L_ν and (T_ν^*/d_ν), say, $aL_\nu + b(T_\nu^*/d_\nu)$, where $d_\nu = (\sum_i \sum_j d_{\nu ij}^2)^{\frac{1}{2}}$. For this it suffices on account of (4.5) to prove the asymptotic normality of $H_\nu = a\gamma T_\nu' + b(T_\nu^*/d_\nu) - (\gamma^2\tau^2 a/2)$. Now

$$H_\nu^* = H_\nu - E(H_\nu) = \sum_i \sum_j \{r_{\nu ij}^{(1)} + r_{\nu ij}^{(2)}\}$$

where $r_{\nu ij}^{(1)} = -a\,C_{\nu ij}\{f^{(j)}(Y_{\nu i})/f(Y_{\nu i})\}$ and $r_{\nu ij}^{(2)} = b(d_{\nu ij}/d_\nu)\,\xi(F_1(Y_{\nu ij}))$. Clearly we may assume that $\operatorname*{Lim\,inf}_\nu \operatorname{Var}(H_\nu^*) > 0$. Now note that, if $H_{\nu i}^* = \sum_j (r_{\nu ij}^{(1)} + r_{\nu ij}^{(2)})$,

$$\begin{aligned}
&\frac{1}{\operatorname{Var}(H_\nu^*)} \sum_i E\{I_{[|H_{\nu i}^*|\geq \varepsilon(\operatorname{Var}(H_\nu^*))^{\frac{1}{2}}]}(H_{\nu i}^*)^2\}\\
&\quad\leq \frac{2t}{\operatorname{Var}(H_\nu^*)} \sum_i \sum_j \sum_{j'} \sum_{l=1}^{2} \sum_{l'=1}^{2} E\{I_{[|r_{\nu ij}^{(l)}|\geq(\varepsilon/2K_i)(\operatorname{Var}(H_\nu^*))^{\frac{1}{2}}]}(r_{\nu ij'}^{(l')2})\},
\end{aligned}$$

and that $\operatorname{Var}(H_\nu^*)$ remains bounded, as $\nu\to\infty$. It easily follows that the Lindeberg-Feller condition is satisfied by H_ν^* under the assumed conditions. It follows that $\mathscr{L}(L_\nu, T_\nu^*/P_\nu)$ converges to a bivariate normal distribution which coupled with (4.6), (4.7) and an application of part (iii) of Lemma 4.2 of [4] shows that $\mathscr{L}(s_\nu/Q_\nu)\to N(m_\nu, b_\nu^2)$, as $\nu\to\infty$, where $m_\nu \sim \operatorname{cov}(T_\nu^*, T_\nu')$, under P_ν, and b_ν is given by (4.3). Now

$$\begin{aligned}
m_\nu &\sim \operatorname{cov}(T_\nu^*, T_\nu')\\
&= \sum_i \operatorname{Cov}\Big(-\sum_j C_{\nu ij}[f^{(j)}(Y_{\nu i})/f(Y_{\nu i})], \sum_j d_{\nu ij}\xi(F_1(Y_{\nu ij}))\Big)\\
&= \sum_i \sum_j C_{\nu ij} d_{\nu ij} \int [-f^{(j)}(x)]\,\xi(F_1(x_j))\,dx\\
&\quad + \sum_i \sum_{j\neq j'} C_{\nu ij} d_{\nu ij'} \int (-f^{(j)}(x))\,\xi(F_1(x_{j'}))\,dx\\
&= \Big(\sum_i \sum_j C_{\nu ij} d_{\nu ij}\Big) \int_0^1 \eta(u)\,\xi(u)\,du.
\end{aligned}$$

The last equality follows by using the notation (2.2) and the fact that the second term in the preceeding expression vanishes identically on account of (2.1)(iii); the proof is complete.

Remark 4.1. In view of the remark 3.1, clearly if $K_i=K$ and $\sum_i d_{ij}=0$ for each $1\leqq j\leqq K$, one can similarly prove the asymptotic normality (with appropriate constants m_ν and b_ν^2) of the sequence $\{S_\nu\}$ under the same conditions as in Theorem 4.1, but without the condition of symmetry of $F(x_1,\ldots,x_K)$ in the arguments. This remark applies to Theorem 5.1 also.

5. Extension: When Some $F^{(i)}(x)$ are Singular

Let the rank of a distribution $F(x)$ in $E^{(K)}$ denote the smallest integer r, such that the total mass of the distribution $F(x)$ is contained in a linear subspace of dimensions r. Let

(i) Each $F^{(i)}(x)$ has rank $r_i\leqq K_i$.

(ii) $P[Y_{ij}=Y_{ij'}]=0$ for every pair (j,j') $j\neq j'$ and $1\leqq i\leqq n_\nu$. (5.1)

(i) The marginal distribution of any r_i components of

$$Y_{\nu i}=(Y_{\nu i1},Y_{\nu i2},\ldots,Y_{\nu iK_i})$$

is absolutely continuous with density $f_{(r_i)}(x)=f_{(r_i)}(x_1,x_2,\ldots,x_{r_i})$, $1\leqq i\leqq n_\nu$.

(ii) The conditions (2.1) are satisfied, with K_i replaced by r_i in (2.1)(ii) and (iii). (5.2)

Theorem 5.1. *Let the model* (1.1) *with conditions* (2.4), (5.1) *and* (5.2) *hold. Then under the same conditions as for Theorem* 4.1, $\mathscr{L}(S_\nu)\to N(m_\nu,b_\nu^2)$, *with* m_ν *and* b_ν^2 *given by* (4.3).

Proof. Let $U_{\nu ij}=F_1(Y_{\nu ij})$. Since $F_1(x)$ is absolutely continuous (in fact, only continuity is enough) $P[U_{\nu ij}=U_{\nu ij'}]=0$ is equivalent to $P[Y_{\nu ij}=Y_{\nu ij'}]=0$ for every fixed i and pair (j,j'). Clearly then the conditions (3.1) are all satisfied for the $U_{\nu i}$'s defined above and consequently the conclusions of Theorem 3.1 are applicable.

Let Q_ν^* and P_ν^* be defined as Q_ν and P_ν in Section 4 with

$$F_{(r_i)}(x)=F_{(r_i)}(x_1,x_2,\ldots,x_{r_i})$$

in place of $F^{(i)}(x)=(x_1,x_2,\ldots,x_{K_i})$, and note that if the distribution $F^{(i)}(x)$ is singular with rank $r_i<K_i$, then any (K_i-r_i) components of the vector $X_{\nu i}$ can be expressed, with probability 1, as linear combinations of the remaining r_i components. Consequently, the distributions $P_{\nu i}^*$ and $Q_{\nu i}^*$ are respectively equivalent to the distributions $P_{\nu i}$ and $Q_{\nu i}$, $1\leqq i\leqq n_\nu$. Proceeding as in the proof of Theorem 4.1 with $f_{(r_i)}(x)$ in place of $f(x)=f_{(K_i)}(x)$ it follows that $\{Q_\nu^*\}$ is contiguous to $\{P_\nu^*\}$, or in other words, $\{Q_\nu\}$ is contiguous to $\{P_\nu\}$. The rest of the arguments of Theorem 4.1 apply verbatim and the proof follows.

6. An Application

In this section we give an application to Theorem 5.1 useful in the non-parametric statistical theory. Let $K_i = K$ and

$$T_\nu^{(j)} = \Big(\sum_i \xi_{\nu, R_{\nu ij}}\Big)/\sqrt{n_\nu} \qquad 1 \leqq j \leqq K,$$

and consider the sequence $S_\nu = (S_\nu^{(1)}, S_\nu^{(2)}, \ldots, S_\nu^{(K)})$ with $S_\nu^{(j)} = T_\nu^{(j)} - E(T_\nu^{(j)})$, where $E(\cdot)$ denotes as before the expectation under P_ν. Clearly then

$$S_\nu^{(j)} = \sum_i \sum_{j'} d_{\nu ij'}^{(j)} \xi_{\nu, R_{\nu ij'}}$$

where

$$d_{\nu ij'}^{(j)} = -1/(K\sqrt{n_\nu}) \qquad \text{for} \quad j' \neq j$$

and

$$d_{\nu ij'}^{(j)} = \{(K-1)/K\sqrt{n_\nu}\} \quad \text{for} \quad j' = j.$$

For the d's above, the condition $\sum_j d_{\nu ij'}^{(j)} = 0$, $1 \leqq i \leqq n_\nu$, is satisfied. We can thus deduce

Theorem 6.1. *Suppose that the model* (1.1) *with conditions* (2.4), (5.1) *and* (5.2) *holds. Then under the assumptions* (2.3) *and* (2.5), *the sequence* $\{S_\nu\}$ *converges in distribution to a normal random vector with the mean* $m_\nu = (m_\nu^{(1)}, m_\nu^{(2)}, \ldots, m_\nu^{(K)})$ *and the covariance matrix* $\sum = \|\delta_{jj'} - (1/K)\| A^2$, *where*

$$m_\nu^{(j)} = \Big(\sum_i C_{\nu ij}/\sqrt{n_\nu}\Big) \int_0^1 \xi(u)\eta(u)\,du,$$

$$A^2 = \int_0^1 \xi^2(u)\,du - \int_0^1\int_0^1 \xi(u)\xi(v)\,dG(u,v).$$

Proof. According to the arguments of Section 7 of Wald and Wolfowitz [9], it suffices to prove the asymptotic normality of for an arbitrary linear combination of $S_\nu^{(j)}$, $1 \leqq j \leqq K$, viz.,

$$H_\nu = \sum_j \lambda^{(j)} S_\nu^{(j)} = \sum_i \sum_{j'} \Big(\sum_j d_{\nu ij'}^{(j)} \lambda^{(j)}\Big) \xi_{\nu R_{\nu ij'}}.$$

If we set $d_{\nu ij'}^{*} = \big(\sum_j d_{\nu ij'}^{(j)} \lambda^{(j)}\big)$, then the condition $\sum_{j'} d_{\nu ij'}^{*} = 0$, $i = 1, 2, \ldots, n_\nu$, are satisfied whatever be the λ's. By applying Theorem 5.1 the proof follows after some computations.

Concluding Remarks. Theorem 6.1 has been employed for investigating the asymptotic properties of certain testing and estimation procedures (e.g. in [8]). It is worth observing that in view of the statement of Lemma 4.1, the condition 2.1 (iii) can be replaced by a weaker condition, viz., for each $1 \leqq i \leqq n_\nu$ (i is suppressed below for convenience),

$f(x) = f(x_1, x_2, \ldots, x_K)$ is absolutely continuous in each argument x_α, $1 \leqq \alpha \leqq K$, for almost all (w.r. to the Lebesgue measure $\mu^{(K-1)}$ on $E^{(K-1)}$) $x^{(\alpha)} \in E^{(K-1)} = \{y^{(\alpha)} = (y_1, \ldots, y_{\alpha-1}, y_{\alpha+1}, \ldots, y_K)\colon -\infty < y_i < \infty\}$ and for each α, $1 \leqq \alpha \leqq K$, $0 < \int_{E^{(K)}} [\{f^{(\alpha)}(x)\}^2/f(x)]\,dx < \infty$ where $f^{(\alpha)}(x) = \partial f/\partial x_\alpha$. (6.1)

It is proved in [8] (Theorem 4.2) that if a random vector $X=(X_1, X_2, \ldots, X_K)$ possesses a density $f(x_1, x_2, \ldots, x_K)$ satisfying (6.1), the random vector $Y=(Y_1, \ldots, Y_t)=A.\ X$, where A is a $t \times K$ matrix of rank $t \leqq K$, also satisfies (6.1). Accordingly if the distribution of X satisfies this condition, so will every marginal distribution. The above result together with the remarks of section 2 establishes a connection between the Hájek condition (1.8) in [4] and the present condition (6.1). We conclude with the following remark: The assumption $P[Y_{vij}=Y_{vij'}]=0$ in Theorem 5.1 has been made to obviate, with probability 1, the problem of ties. In statistical applications, in case of a tie either the mean rank or randomization is employed in assigning ranks to the tied observations. On can, indeed, extend the above arguments to cover these cases also.

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