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Some Limit Theorems of Donsker-Varadhan Type for Markov Process Expectations

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Introduction

In the present paper we shall be concerned with generalization of those results by M.D. Donsker and S.R.S. Varadhan [2, 3]. They have given in [3] the solution of the sausage problem for symmetric stable processes. Our goal is to extend this result to the case of symmetric Lévy processes which are close to a symmetric stable process (Theorem 4.1).

The contents of this paper are as follows. Let $0<\alpha\leqq 2$ and let $X^{(\alpha)}=(X_t, P_x^{(\alpha)})$ be a symmetric stable process on R^d of order α. Let $X=(X_t, P_x)$ be another symmetric Lévy process on R^d. We shall assume that the process X is close to the process $X^{(\alpha)}$ in the sense that conditions (Q_1) and (Q_2) in Lemma 3.1 hold. Theorem 4.1 asserts that the solution of the sausage problem for the process X is given by the asymptotic formulas (4.5) and (4.6), which are reduced to the solution by Donsker and Varadhan [3] when $X=X^{(\alpha)}$. It should be noted that the limiting constant $k(v, L^{(\alpha)})$ is common for the processes satisfying conditions (Q_1) and (Q_2).

The proof of the upper estimate (4.5) goes along the same idea as in [3]. For this purpose we first define $X_t^s=X_0+s^{-1}(X_{s^\alpha t}-X_0)$, $t\geqq 0$, for any path X_t, $t\geqq 0$ and any $s>0$, and then have to treat the one-parameter family $\{(\pi(X_t^s), P_x);\ s>0\}$ of Lévy processes on a torus T in R^d, where π denotes the canonical map of R^d onto T. In the special case of $X=X^{(\alpha)}$ the law of $(\pi(X_t^s), P_x^{(\alpha)})$ is identical with that of $(\pi(X_t), P_x^{(\alpha)})$ for any $s>0$ by virtue of the scaling property of $X^{(\alpha)}$. Donsker and Varadhan [3] have proved the upper estimate in the special case of $X=X^{(\alpha)}$ by applying to the process $(\pi(X_t), P_x^{(\alpha)})$ the general theorem on the asymptotic evaluation of certain expectations with respect to a Markov process on a compact space. The last theorem has been obtained by Donsker and Varadhan [2]. Thus in order to use the method of [3] for our general case we have to extend the results of [2] in such a manner that they apply to a one-parameter family of Markov processes on a compact space. This extension will be done in Sects. 1–3; Theorem 1.1 extends the first half of Theorem 1.2 of [2] and in case of Lévy processes on a torus Theorem 3.1 extends the first half of Theorem 5.1 of [2] and its corollary.

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The proof of the lower estimate (4.6) of Theorem 4.1 is quite different. We shall not use any results of Sects. 1-3, but use the method essentially due to L.A. Pastur [7] in which some related problems are treated. We further note that the author [6; Theorem 6.2′] has proved a similar result to (4.6) for the case of the pinned processes of the process X.

In Sect. 5 we shall give necessary and sufficient conditions for (Q_1), a sufficient condition for (Q_2) and some examples.

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1. A One-Parameter Family of Markov Processes on a Compact Space

Let X be a compact metric space and $\mathfrak{B}_X$ its topological Borel field. Let $M(X)$ denote the set of all signed measures of bounded variation defined on X. The norm $\|\mu\|$ of $\mu \in M(X)$ is defined by the total variation $\|\mu\| = \sup_{A \in \mathfrak{B}_X} (\mu(A) - \mu(A^c))$. Let $B(X)$ (resp. $C(X)$) denote the space of all bounded Borel (resp. continuous) functions on X with the supremum norm $\|\cdot\|_\infty$. Let $\langle \mu, f \rangle = \int_X \mu(dx) f(x)$ for $\mu \in M(X)$ and $f \in B(X)$.

Let $p(t, x, dy)$ be a Feller transition probability on X, T_t the corresponding semigroup on $C(X)$ and L the infinitesimal generator of T_t with domain $\mathscr{D}(L) \subset C(X)$. Let Ω be the set of all X-valued right continuous functions $\omega = x(\cdot)$ on $[0, \infty)$ having left hand limits on $(0, \infty)$. It is well known that there exists a Hunt process $(\Omega, x(t), P_x\colon t \geqq 0, x \in X)$ having $p(t, x, dy)$ as its transition probability.

Let $\mathscr{M}$ denote the space of all probability measures on X. We shall endow $\mathscr{M}$ with the weak topology so that $\mathscr{M}$ is a compact metric space. For any $t > 0$, $\omega = x(\cdot) \in \Omega$ and $A \in \mathfrak{B}_X$, let

$$L_t(\omega, A) = \frac{1}{t} \int_0^t \chi_A(x(\sigma))\, d\sigma. \tag{1.1}$$

Note that $L_t(\omega, \cdot) \in \mathscr{M}$ for each $t > 0$ and $\omega \in \Omega$. For each $x \in X$ and $t > 0$, let $Q_{x,t}$ be the probability measure on $\mathscr{M}$ induced by the map $\omega \to L_t(\omega, \cdot)$ of Ω into $\mathscr{M}$ from P_x, i.e., for any Borel subset B of $\mathscr{M}$,

$$Q_{x,t}(B) = P_x(\omega \in \Omega;\ L_t(\omega, \cdot) \in B).$$

Following Donsker and Varadhan we define the I-functional $I(\mu)$, $\mu \in \mathscr{M}$ corresponding to the transition probability $p(t, x, dy)$ by

$$I(\mu) = -\inf_{\substack{u > 0 \\ u \in \mathscr{D}(L)}} \langle \mu, Lu/u \rangle. \tag{1.2}$$

$I(\mu)$ is a non-negative, lower semicontinuous functional on $\mathscr{M}$.

We assume that there exists a finite reference measure λ on X such that $p(t, x, dy)$ is absolutely continuous relative to λ for each $t > 0$ and $x \in X$. Let γ

denote the space of all $\mu\in\mathscr{M}$ which are absolutely continuous relative to λ. We shall endow γ with the norm topology. Note that if $\mu\in M(X)$ is absolutely continuous relative to λ, then $f=d\mu/d\lambda\in L^1(\lambda)$ and $\|\mu\|=\|f\|_{L^1(\lambda)}$. Thus one can identify γ with the subset of $L^1(\lambda)$ with the $L^1(\lambda)$-norm topology. Let $\{k_\varepsilon(x,y);\ \varepsilon>0\}$ be a family of measurable functions on $X\times X$ such that $k_\varepsilon(x,\cdot)\in\gamma$ for each $\varepsilon>0$ and $x\in X$. Define, for any $\varepsilon>0$, $t>0$, $\omega=x(\cdot)\in\Omega$ and $y\in X$,

$$l_t^\varepsilon(\omega,y)=\int_X k_\varepsilon(x,y)\,L_t(\omega,dx)$$
$$=\frac{1}{t}\int_0^t k_\varepsilon(x(\sigma),y)\,d\sigma. \tag{1.3}$$

Note that $l_t^\varepsilon(\omega,\cdot)\in\gamma$ for each $\varepsilon>0$, $t>0$ and $\omega\in\Omega$. Let $\varepsilon(t)$ be a positive function of $t>0$ tending to zero as $t\to\infty$ and let

$$g_t(\omega,y)=l_t^{\varepsilon(t)}(\omega,y). \tag{1.4}$$

The map $\omega\to g_t(\omega,\cdot)$ of Ω into γ is measurable for each $t>0$ so that the probability measure $R_{x,t}$ on γ is defined by

$$R_{x,t}(A)=P_x(\omega\in\Omega;\ g_t(\omega,\cdot)\in A),$$

where A is any Borel subset of γ.

In the first half of Theorem 1.2 in [2], Donsker and Varadhan have proved the following relation under their Assumptions A, B, C and D: If C is any closed subset of γ, then

$$\limsup_{t\to\infty}\frac{1}{t}\log R_{x,t}(C)\leqq-\inf_{\mu\in C}I(\mu). \tag{1.5}$$

In this paper we shall consider a one-parameter family $\{p^s(t,x,dy);\ s\in(0,\infty]\}$ of Feller transition probabilities instead of a single $p(t,x,dy)$. Let $s(t)$ be any positive function increasing to infinity with t. Generalizing (1.5), we claim that $R_{x,t}^{s(t)}$ obeys the following relation: For every closed subset C of γ,

$$\limsup_{t\to\infty}\frac{1}{t}\log R_{x,t}^{s(t)}(C)\leqq-\inf_{\mu\in C}I^\infty(\mu).$$

Here and after the semigroup, generator, I-functional, P_x-measure, $Q_{x,t}$-measure and $R_{x,t}$-measure corresponding to $p^s(t,x,dy)$ are denoted by T_t^s, L^s, $I^s(\mu)$, P_x^s, $Q_{x,t}^s$ and $R_{x,t}^s$, respectively.

We now state the assumption for the one-parameter family $\{p^s(t,x,dy);\ s\in(0,\infty]\}$.

Assumption A. (i) *There exists a subset* $\mathscr{D}_0$ *of* $\bigcap_{s\in(0,\infty]}\mathscr{D}(L^s)$ *such that* $\mathscr{D}_0$ *is uniformly dense in* $C(X)$, $T_t^\infty\mathscr{D}_0\subset\mathscr{D}_0$ *for all* $t>0$, *and* $L^s u$ *tends to* $L^\infty u$ *uniformly as* $s\to\infty$ *for each* $u\in\mathscr{D}_0$.

(ii) *For each* $s \in (0, \infty]$, $t>0$ *and* $x \in X$, $p^s(t, x, dy)$ *is absolutely continuous relative to* λ *with the density* $p^s(t, x, y)$ *and, moreover,* $a^s(t) \equiv \inf_{x,y} p^s(t, x, y) > 0$ *and* $A^s(t) \equiv \sup_{x,y} p^s(t, x, y) < \infty$ *hold for each* $t>0$ *and* $s \in (0, \infty]$.

(iii) *For each* $t>0$, $p^s(t, x, y)$ *tends to* $p^\infty(t, x, y)$ *uniformly for* x *and* y *as* $s \to \infty$.

(iv) *For each* $t>0$, *the map* $x \to p^\infty(t, x, \cdot)$ *of* X *into* $\gamma \subset L^1(\lambda)$ *is continuous.*

Remark. If $p^s(t, x, dy)$ is independent of s, that is, the family $\{p^s(t, x, dy); s \in (0, \infty]\}$ consists only of a single transition probability $p(t, x, dy)$, then the above Assumption A is reduced to Assumptions A and D in [2].

Theorem 1.1. *Let* $\{p^s(t, x, dy);\ s \in (0, \infty]\}$ *be a one-parameter family of Feller transition probabilities on* X *satisfying Assumption* A. *Let* $\{k_\varepsilon(x, y);\ \varepsilon>0\}$ *be a family of functions on* $X \times X$ *satisfying Assumption* B *of* [2] *and* $\varepsilon(t)$ *a positive function satisfying Assumption* C *of* [2]. *Then for each closed subset* C *of* γ *(in the norm topology) and each* $x \in X$,

$$\limsup_{t \to \infty} \frac{1}{t} \log R_{x,t}^{s(t)}(C) \leqq - \inf_{\mu \in C} I^\infty(\mu). \tag{1.6}$$

The next theorem is a corollary of Theorem 1.1, which follows from the lower semicontinuity of $I^\infty(\mu)$, and the compactness of $\{\mu;\ I^\infty(\mu) \leqq l\}$, $l<\infty$ (see [2; p. 285]) by the arguments in Varadhan [9; Sect. 3].

Theorem 1.2. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Let* $\{\Phi_t(f);\ t>0\}$ *be a family of measurable functionals on* γ *and* Φ *any functional on* γ *such that, for each* $f \in \gamma$ *with* $I^\infty(f)<\infty$ *and each family* $\{f_t\} \subset \gamma$ *converging to* f *in norm,* $\liminf_{t \to \infty} \Phi_t(f_t) \geqq \Phi(f)$. *We assume* $\Phi_t(f) \geqq 0$ *for all* $t>0$ *and* $f \in \gamma$. *Then*

$$\limsup_{t \to \infty} \frac{1}{t} \log \int_\gamma \exp\{-t\Phi_t(f)\} R_{x,t}^{s(t)}(df) \leqq - \inf_{f \in \gamma} [\Phi(f) + I^\infty(f)], \tag{1.7}$$

where $I^\infty(f) = I^\infty(\mu)$ *with* $\mu = f \cdot \lambda \in \gamma$.

We shall prove Theorem 1.1 in Sect. 2. In Sect. 3 we shall give a class of examples for Theorem 1.2 which will be used in Sect. 4 for the sausage problem.

2. The Proof of Theorem 1.1

In this section we shall give the proof of Theorem 1.1. We first give some preliminary results. Recall that

$$I^s(\mu) = - \inf_{\substack{u>0 \\ u \in \mathscr{D}(L^s)}} \langle \mu, L^s u/u \rangle, \qquad \mu \in \mathscr{M},\ s \in (0, \infty].$$

We have the following lemma.

Lemma 2.1. *Suppose that Assumption* A(i) *is satisfied. Then, for each* $\mu \in \mathcal{M}$,

$$I^{\infty}(\mu) = -\inf_{\substack{u>0 \\ u \in \mathcal{D}_0}} \langle \mu, L^{\infty} u/u \rangle. \tag{2.1}$$

The proof is carried out along the same line as in [4; Lemma 2.1]. We omit the detail.

The following theorem generalizes the first half of Theorem 3 of [1].

Theorem 2.1. *Suppose that Assumption* A(i) *is satisfied. Let* $s(t)$ *be any positive function increasing to infinity with* t*. Then, for each closed subset* C *of* $\mathcal{M}$ *(in the weak topology) and each* $x \in X$,

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{x,t}^{s(t)}(C) \leqq -\inf_{\mu \in C} I^{\infty}(\mu). \tag{2.2}$$

Proof. The proof is similar to that of the first half of Theorem 3 in [1]. Let $s \in (0, \infty]$ be fixed. Then one can prove that, for each $u \in \mathcal{D}(L^s)$ with $u>0$ and each Borel subset B of $\mathcal{M}$,

$$Q_{x,t}^{s}(B) \leqq \frac{u(x)}{\min_y u(y)} \exp\{t \sup_{\mu \in B} \langle \mu, L^s u/u \rangle\}$$

(see [1; p. 40]). Since $\min_y u(y) > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{x,t}^{s(t)}(B) \leqq \limsup_{t\to\infty} \sup_{\mu \in B} \langle \mu, L^{s(t)} u/u \rangle. \tag{2.3}$$

Let $u \in \mathcal{D}_0$ and $u>0$. By Assumption A(i), $L^s u/u$ tends to $L^{\infty} u/u$ uniformly as $s \to \infty$ and thus $\langle \mu, L^s u/u \rangle$ tends to $\langle \mu, L^{\infty} u/u \rangle$ uniformly for $\mu \in \mathcal{M}$ as $s \to \infty$ so that the right hand side of (2.3) is equal to $\sup_{\mu \in B} \langle \mu, L^{\infty} u/u \rangle$. Hence we have, for any Borel subset B of $\mathcal{M}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{x,t}^{s(t)}(B) \leqq \inf_{\substack{u>0 \\ u \in \mathcal{D}_0}} \sup_{\mu \in B} \langle \mu, L^{\infty} u/u \rangle. \tag{2.4}$$

This relation implies that, for each closed (compact) subset C of $\mathcal{M}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{x,t}^{s(t)}(C) \leqq \sup_{\mu \in C} \inf_{\substack{u>0 \\ u \in \mathcal{D}_0}} \langle \mu, L^{\infty} u/u \rangle \tag{2.5}$$

(see [1; p. 40]). Since the right hand side of (2.5) is equal to $-\inf_{\mu \in C} I^{\infty}(\mu)$ by Lemma 2.1, the proof of Theorem 2.1 is complete.

In the remainder of this section we shall assume that the three assumptions of Theorem 1.1 are satisfied. The map T_t^s of $C(X)$ into $C(X)$ is given by

$$(T_t^s \phi)(x) = \int_X p^s(t, x, y) \phi(y) \lambda(dy), \qquad \phi \in C(X).$$

We also think of T_t^s as the dual map on $M(X)$ defined by

$$(\mu T_t^s)(dy) = (\int_X p^s(t, x, y)\,\mu(dx))\,\lambda(dy), \qquad \mu \in M(X).$$

Note that T_t^s maps $\mathscr{M}$ into γ. Similarly, by K_ε we denote two maps in duality defined by

$$(K_\varepsilon \phi)(x) = \int_X k_\varepsilon(x, y)\,\phi(y)\,\lambda(dy), \qquad \phi \in C(X)$$

and

$$(\mu K_\varepsilon)(dy) = (\int_X k_\varepsilon(x, y)\,\mu(dx))\,\lambda(dy), \qquad \mu \in M(X).$$

Assumptions B(iii) and (iv) of [2; p. 281] assure that K_ε forms a compact operator of $C(X)$ into itself and it also maps $\mathscr{M}$ into γ.

We need some lemmas. First, by Assumption A(iv), we have the following lemma.

Lemma 2.2 ([2; p. 293]). *For each $\delta > 0$, the map $\mu \to \mu T_\delta^\infty$ of $\mathscr{M}$ (with the weak topology) into γ (with the norm topology) is continuous.*

Next we prepare two lemmas involving the family $\{I^s(\mu);\ s \in (0, \infty]\}$ of I-functionals.

Lemma 2.3. *Suppose that a sequence $\{\mu_n\} \subset \mathscr{M}$ converges weakly to $\mu \in \mathscr{M}$ and $s_n \to \infty$ as $n \to \infty$, then*

$$I^\infty(\mu) \leqq \liminf_{n \to \infty} I^{s_n}(\mu_n).$$

Proof. It follows from Assumption A(i) that if $u \in \mathscr{D}_0$ and $u > 0$, then $\langle \mu_n, L^{s_n} u/u \rangle$ tends to $\langle \mu, L^\infty u/u \rangle$ as $n \to \infty$. Thus the lemma is an immediate consequence of Lemma 2.1.

Lemma 2.4. *Let $\{\mu_n\}$ be a sequence in γ and $\{s_n\}$ a sequence tending to infinity. Suppose that $\sup_n I^{s_n}(\mu_n) < \infty$. Then $\{\mu_n\}$ is totally bounded in γ in the norm topology.*

Proof. One can assume that μ_n converges weakly to an element $\mu \in \mathscr{M}$ since $\mathscr{M}$ is compact. Then, by Lemma 2.3, we have

$$I^\infty(\mu) \leqq \liminf_{n \to \infty} I^{s_n}(\mu_n) \leqq l, \tag{2.6}$$

where $l = \sup_n I^{s_n}(\mu_n) < \infty$. We have only to show that

$$\|\mu_n - \mu\| \to 0 \qquad \text{as } n \to \infty. \tag{2.7}$$

To this end observe that, for each $t > 0$,

$$\|\mu_n - \mu\| \leqq \|\mu_n - \mu_n T_t^{s_n}\| + \|\mu_n (T_t^{s_n} - T_t^\infty)\| + \|(\mu_n - \mu) T_t^\infty\| + \|\mu T_t^\infty - \mu\|.$$

The third term on the right tends to zero as $n\to\infty$ by Lemma 2.2. One can show that the second term also tends to zero as follows:

$$\begin{aligned}\|\mu_n(T_t^{s_n}-T_t^{\infty})\| &= \|\int_X \mu_n(dx)(p^{s_n}(t,x,\cdot)-p^{\infty}(t,x,\cdot))\|_{L^1(\lambda)}\\ &\leqq \int_X \lambda(dy)\int_X \mu_n(dx)|p^{s_n}(t,x,y)-p^{\infty}(t,x,y)|\\ &\leqq \lambda(X)\cdot\sup_{x,y}|p^{s_n}(t,x,y)-p^{\infty}(t,x,y)|;\end{aligned}$$

the last term tends to zero as $n\to\infty$ by Assumption A(iii). Thus, by (2.6), we have

$$\limsup_{n\to\infty}\|\mu_n-\mu\|\leqq\sup_n\sup_{I^{s_n}(\mu)\leqq l}\|\mu-\mu T_t^{s_n}\|+\sup_{I^{\infty}(\mu)\leqq l}\|\mu-\mu T_t^{\infty}\| \tag{2.8}$$

for each $t>0$. It follows from Corollary in p. 44 of [1] that each term on the right hand side of (2.8) tends to zero as $t\to0$. Thus we have (2.7), which proves the lemma.

In the remainder of this section we denote by ϕ any function in $C(X)$ and by $s(t)$ any function increasing to infinity with $t>0$. For any $\varepsilon>0$, $\delta>0$ and ϕ, we define

$$\phi_{\varepsilon,\delta}=K_{\varepsilon}(T_{\delta}^{\infty}-I)\phi. \tag{2.9}$$

Here and after we denote by I the identity operators on $C(X)$ and $M(X)$. For each $s\in(0,\infty]$, we define a functional $\lambda^s(\phi)$ on $C(X)$ by

$$\lambda^s(\phi)=\sup_{\mu\in\mathcal{M}}[\langle\mu,\phi\rangle-I^s(\mu)]. \tag{2.10}$$

The following lemma corresponds to Lemma 2.1 of [2].

Lemma 2.5. *For each* $0<\rho<\infty$,

$$\limsup_{\delta\to0}\limsup_{t\to\infty}\sup_{\|\phi\|_{\infty}\leqq\rho}\lambda^{s(t)}(\phi_{\varepsilon(t),\delta})\leqq0. \tag{2.11}$$

Proof. By the argument of [2; pp. 284–285] we have only to show that

$$\lim_{\delta\to0}\limsup_{t\to\infty}\sup_{I^{s(t)}(\mu)\leqq2\rho}\|\mu K_{\varepsilon(t)}(T_{\delta}^{\infty}-I)\|=0. \tag{2.12}$$

To this end let

$$\eta(\delta)=\limsup_{t\to\infty}\sup_{I^{s(t)}(\mu)\leqq2\rho}\|\mu K_{\varepsilon(t)}(T_{\delta}^{\infty}-I)\|,\qquad\delta>0.$$

For each $\delta>0$, we can choose sequences $t_n>0$ and $\mu_n\in\gamma$ such that t_n tends to infinity, $I^{s_n}(\mu_n)\leqq2\rho$ $(s_n=s(t_n))$ and

$$\lim_{n\to\infty}\|\mu_nK_{\varepsilon_n}(T_{\delta}^{\infty}-I)\|=\eta(\delta)\qquad(\varepsilon_n=\varepsilon(t_n)). \tag{2.13}$$

By Lemma 2.4, we can assume that there exists an element $\mu\in\gamma$ such that $\lim_{n\to\infty}\|\mu_n-\mu\|=0$. Then it follows from Lemma 2.3 that

$$I^{\infty}(\mu)\leqq\liminf_{n\to\infty}I^{s_n}(\mu_n)\leqq2\rho. \tag{2.14}$$

Noting the contraction properties of the operators T_δ^∞ and K_ε on $M(X)$, we have

$$\begin{aligned}\|\mu_n K_{\varepsilon_n}(T_\delta^\infty - I)\| &\leqq \|(\mu_n-\mu)K_{\varepsilon_n}(T_\delta^\infty - I)\| + \|\mu K_{\varepsilon_n}(T_\delta^\infty - I)\| \\ &\leqq 2\|\mu_n-\mu\| + \|(\mu K_{\varepsilon_n}-\mu)T_\delta^\infty\| + \|\mu T_\delta^\infty - \mu\| + \|\mu - \mu K_{\varepsilon_n}\| \\ &\leqq 2\|\mu_n-\mu\| + 2\|\mu K_{\varepsilon_n}-\mu\| + \|\mu T_\delta^\infty - \mu\|.\end{aligned}$$

The first term on the right side tends to zero as $n\to\infty$ and the second tends to zero by Assumption B(v) of [2]. Thus, by (2.13) and (2.14), we have $\eta(\delta)\leqq \sup_{I^\infty(\mu)\leqq 2\rho}\|\mu T_\delta^\infty-\mu\|$. The right hand side tends to zero as $\delta\to 0$ by Corollary in p. 44 of [1], which completes the proof of Lemma 2.5.

Let $g_t(\omega, y)$ be the function defined by (1.4). For any $\delta>0$, we define

$$g_t^\delta(\omega, y) = \int_X p^\infty(\delta, x, y)\, g_t(\omega, x)\,\lambda(dx) \tag{2.15}$$

and

$$\Delta_t^\delta(\omega) = \|g_t^\delta(\omega, \cdot) - g_t(\omega, \cdot)\|_{L^1(\lambda)}. \tag{2.16}$$

Now we prove the main estimate corresponding to Theorem 3.1 of [2].

Theorem 2.2. *For each* $\theta>0$ *and* $x\in X$,

$$\limsup_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log P_x^{s(t)}(\Delta_t^\delta(\omega)\geqq\theta) = -\infty. \tag{2.17}$$

Proof. Let $s\in(0,\infty]$ be fixed. Then Lemma 2.2 of [2] holds for the process $(\Omega, x(t), P_x^s\colon t\geqq 0, x\in X)$ by Assumption A(ii). Thus, by the argument of [2; pp. 289–290], we have, for any $0<\rho<\infty$,

$$P_x^s(\Delta_t^\delta\geqq\theta)\leqq N_{\theta/8}(\varepsilon(t))\,e^{-t\rho\theta/4}\,C_\rho^s\exp(t\lambda_{t,\delta,\rho}^s),$$

where $N_{\theta/8}(\varepsilon(t))$ denotes the smallest number of $\frac{\theta}{8}$-covering of the image of the unit ball in $C(X)$ under the compact operator $K_{\varepsilon(t)}$,

$$\lambda_{t,\delta,\rho}^s = \sup_{\|\phi\|_\infty\leqq 1/2}\lambda^s(\rho\,\phi_{\varepsilon(t),\delta}) \quad\text{and}\quad C_\rho^s = e^{2\rho}A^s(1)/a^s(1);$$

the concrete form of C_ρ^s was given in the proof of Lemma 2.2 of [2].

Assumption C of [2; p. 283] asserts that, for each $\theta>0$,

$$\alpha(\theta)\equiv\limsup_{t\to\infty}\frac{1}{t}\log N_{\theta/8}(\varepsilon(t))<\infty.$$

Since Assumption A(iii) implies that $C_\rho^s\to C_\rho^\infty\,(<\infty)$ as $s\to\infty$, we have $\limsup_{t\to\infty}\frac{1}{t}\log C_\rho^{s(t)}=0$. Note that $\lambda_{t,\delta,\rho}^s=\sup_{\|\phi\|_\infty\leqq\rho/2}\lambda^s(\phi_{\varepsilon(t),\delta})$ by the linearity of the map $\phi \to \phi_{\varepsilon,\delta}$. Thus, we have, by Lemma 2.5, $\limsup_{\delta\to 0} \limsup_{t\to\infty} \lambda_{t,\delta,\rho}^{s(t)} \leqq 0$. Hence we have

$$\limsup_{\delta\to 0} \limsup_{t\to\infty} \frac{1}{t} \log P_x^{s(t)}(\Delta_t^\delta \geqq \theta) \leqq \alpha(\theta) - \rho\,\theta/4.$$

Letting $\rho \to \infty$, we have the theorem.

Finally we shall give the proof of Theorem 1.1. Recall that $R_{x,t}^s$ is the measure on γ induced by the map $\omega \to g_t(\omega, \cdot)$ of Ω into γ from P_x^s. We shall also consider the measure $R_{x,t}^{s,\delta}$ on γ induced by the map $\omega \to g_t^\delta(\omega, \cdot)$. Note that

$$R_{x,t}^{s,\delta}(A) = R_{x,t}^s(\mu \in \gamma;\ \mu T_\delta^\infty \in A)$$

for any measurable subset A of γ.

Proof of Theorem 1.1. Theorem 2.1 implies that for each $C \subset \gamma$ which is a closed subset of $\mathcal{M}$ in the *weak* topology,

$$\limsup_{t\to\infty} \frac{1}{t} \log R_{x,t}^{s(t)}(C) \leqq -\inf_{\mu\in C} I^\infty(\mu) \tag{2.18}$$

(see the proof of Theorem 4.1 of [2]). By Lemma 2.2, this inequality implies that

$$\limsup_{t\to\infty} \frac{1}{t} \log R_{x,t}^{s(t),\delta}(C) \leqq -\inf_{\mu T_\delta^\infty \in C} I^\infty(\mu) \tag{2.19}$$

for each $\delta > 0$ and each closed subset C of γ in the *norm* topology (see the proof of Theorem 4.2 of [2]). Let C be any closed subset of γ in the norm topology and let $C_\theta = \{\beta \in \gamma;\ \|\beta - \alpha\| < \theta \text{ for some } \alpha \in C\}$, $\theta > 0$. Then we have

$$R_{x,t}^s(C) \leqq R_{x,t}^{s,\delta}(\bar{C}_\theta) + P_x^s(\Delta_t^\delta(\omega) \geqq \theta),$$

where $\bar{C}_\theta$ denotes the norm closure of C_θ (see [2; p. 293]). Thus it follows from Theorem 2.2 and (2.19) that

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \log R_{x,t}^{s(t)}(C) &\leqq \liminf_{\theta\to 0} \liminf_{\delta\to 0} \limsup_{t\to\infty} \frac{1}{t} \log R_{x,t}^{s(t),\delta}(\bar{C}_\theta) \\ &\leqq -\limsup_{\theta\to 0} \limsup_{\delta\to 0} \inf_{\mu T_\delta^\infty \in \bar{C}_\theta} I^\infty(\mu) \end{aligned} \tag{2.20}$$

(see [2; p. 294]). By the relation

$$\liminf_{\substack{\theta\to 0\\ \delta\to 0}} \inf_{\mu T_\delta^\infty \in \bar{C}_\theta} I^\infty(\mu) \geqq \inf_{\mu\in C} I^\infty(\mu)$$

((4.5) of [2]), we have the theorem.

3. The One-Parameter Family of Lévy Processes on a Torus

In this section we shall consider a class of examples for Theorem 1.2 which will be used in § 4 for the sausage problem.

Let R^d be the d-dimensional Euclidean space. Let $M>0$ be fixed and let $G=(MZ)^d$, where $(MZ)^d$ denotes the discrete subgroup of R^d consisting of vectors having for each coordinate an integral multiple of M. We take as the compact metric space X the d-dimensional torus $T=R^d/G$ of size M. Let π denote the canonical map of R^d onto T. We may identify T with the subset $\{x=(x^1,\dots,x^d);\ 0\leqq x^i<M,\ i=1,\dots,d\}$ of R^d.

Let $X=(X_t, P_x\colon t\geqq 0,\ x\in R^d)$ be a symmetric Lévy process on R^d; here by a Lévy process we mean a Hunt process with stationary independent increments. It is well known that the process $(\pi(X_t), P_x\colon t\geqq 0,\ x\in T)$ is a Lévy process on the torus T, which will be denoted by $\pi(X)$. In the following we shall make a one-parameter family of Lévy processes on the torus T satisfying Assumption A and apply Theorem 1.2.

Let $Q(\xi)$ be the exponent of the Lévy process X on R^d, i.e.,

$$E_0[\exp(i\langle\xi, X_t\rangle)]=\exp\{-tQ(\xi)\},\qquad t>0,\ \xi\in R^d; \tag{3.1}$$

here and after E_x denotes the expectation with respect to P_x for each $x\in R^d$. $Q(\xi)$ is a non-negative, symmetric, continuous function. Let $0<\alpha\leqq 2$. A symmetric Lévy process on R^d is said to be a symmetric stable process of order α and denoted by $X^{(\alpha)}$ if the exponent $Q^{(\alpha)}(\xi)$ has the property that $Q^{(\alpha)}(\lambda\xi)=\lambda^\alpha Q^{(\alpha)}(\xi)$ for $\lambda>0$. For the concrete forms $Q(\xi)$ and $Q^{(\alpha)}(\xi)$ see (5.1) and (5.2), respectively.

We fix a symmetric stable process $X^{(\alpha)}$ with exponent $Q^{(\alpha)}(\xi)$. Let X be another symmetric Lévy process with exponent $Q(\xi)$. For any sample path X_t, $t\geqq 0$, of X and any $s\in(0,\infty)$, let $X_t^s=X_0+s^{-1}(X_{s^\alpha t}-X_0)$, $t\geqq 0$. It is easy to see that, for each $s\in(0,\infty)$, the process $X^s=(X_t^s, P_x\colon t\geqq 0,\ x\in R^d)$ is a symmetric Lévy process with the exponent $Q^s(\xi)$ defined by

$$Q^s(\xi)=s^\alpha Q(s^{-1}\xi). \tag{3.2}$$

We now write X^∞ for $X^{(\alpha)}$. Thus we have a one-parameter family $\{\pi(X^s);\ s\in(0,\infty]\}$ of Lévy processes on T. Let $p^s(t,x,dy)$ be the transition probability of $\pi(X^s)$ for each $s\in(0,\infty]$. We define

$$Q_*(\xi)=\inf_{s\geqq 1} Q^s(\xi).$$

Lemma 3.1. *The one-parameter family $\{p^s(t,x,dy);\ s\in(0,\infty]\}$ of transition probabilities on T defined above satisfies Assumption* A *under the following conditions on the processes X and $X^{(\alpha)}$, or rather on the exponents $Q(\xi)$ and $Q^{(\alpha)}(\xi)$:*

(Q_1) $Q(\xi)=Q^{(\alpha)}(\xi)+o(|\xi|^\alpha)$ $(|\xi|\downarrow 0)$.

(Q_2) *For any $t>0$ and $r>0$,* $\sum_{\xi\in(rZ)^d}\exp\{-tQ_*(\xi)\}<\infty$.

For the proof we shall introduce the Fourier transform on T. Let λ be the Lebesgue measure on the torus T and let $\tilde G=\left(\frac{2\pi}{M}Z\right)^d$. For any function f in $L^1(\lambda)$, the Fourier transform $\hat f$ of f is the function defined by

$$\hat f(\xi)=M^{-d/2}\int_T e^{i\langle\xi,x\rangle}f(x)\,dx,\qquad \xi\in\tilde G. \tag{3.3}$$

Moreover, if $f \in C(T)$ (the space of continuous functions) and $\sum_{\xi \in \tilde{G}} |\hat{f}(\xi)| < \infty$, then we have the inversion formula

$$f(x) = M^{-d/2} \sum_{\xi \in \tilde{G}} e^{-i\langle \xi, x\rangle} \hat{f}(\xi), \qquad x \in T. \tag{3.4}$$

Remark. One can replace condition (Q_2) in Lemma 3.1 by the following weaker condition:

($Q_{2,M}$) *For any* $t > 0$, $\sum_{\xi \in \tilde{G}} \exp\{-t Q_*(\xi)\} < \infty$.

In fact we shall prove Lemma 3.1 under the conditions (Q_1) and ($Q_{2,M}$).

Proof of Lemma 3.1. We first observe that, for each $\xi \in \tilde{G}$,

$$E_0[\exp\{i\langle \xi, \pi(X_t^s)\rangle\}] = \exp\{-t Q^s(\xi)\}, \qquad t \geqq 0,\ s \in (0, \infty). \tag{3.5}$$

In the following we shall write $Q^\infty(\xi)$ for $Q^{(\alpha)}(\xi)$. We have $Q^s(\xi) \geqq Q_*(\xi)$ for $s \in (0, \infty]$ by condition (Q_1). Thus it follows from condition ($Q_{2,M}$) that $\sum_{\xi \in \tilde{G}} \exp\{-t Q^s(\xi)\} < \infty$ for any $t > 0$ and $s \in (0, \infty]$. Hence, for each $s \in (0, \infty]$, we can define a function

$$p^s(t, x) = M^{-d} \sum_{\xi \in \tilde{G}} \exp\{-i\langle \xi, x\rangle - t Q^s(\xi)\}, \tag{3.6}$$

which is continuous in $x \in T$ and analytic in $t > 0$. Let $p^s(t, x, y) = p^s(t, y - x)$ for $t > 0$, $x \in T$, $y \in T$ and $s \in (0, \infty]$. Then $p^s(t, x, y)$ is the density of $p^s(t, x, dy)$ relative to $\lambda(dy)$ for each $s \in (0, \infty]$, $t > 0$ and $x \in T$. Assumptions A(ii) and (iv) are easily verified except for the condition

$$a^s(t) \equiv \inf_{x, y} p^s(t, x, y) > 0, \qquad t > 0. \tag{3.7}$$

To prove (3.7) it suffices to show that $p^s(t, x) > 0$ for any $t > 0$ and $x \in T$. One can show this by an elementary argument as in [6; Proposition 3.1].

We next check Assumption A(iii). It suffices to show that, for each $t > 0$, $p^s(t, x)$ converges to $p^\infty(t, x)$ uniformly for $x \in T$ as $s \to \infty$. Since condition (Q_1) implies that $\exp\{-t Q^s(\xi)\}$ tends to $\exp\{-t Q^\infty(\xi)\}$ as $s \to \infty$, the desired assertion follows from the expression (3.6) and condition ($Q_{2,M}$).

Finally we check Assumption A(i). Let T_t^s be the semigroup on $C(T)$ corresponding to $p^s(t, x, dy)$. Then we have, for any $f \in C(T)$,

$$T_t^s f(x) = \int_T p^s(t, y) f(x + y)\, dy, \qquad x \in T,\ t > 0,\ s \in (0, \infty]. \tag{3.8}$$

One can easily see that T_t^s is a strongly continuous Feller semigroup. Let L^s be the infinitesimal generator of T_t^s with domain $\mathscr{D}(L^s)$. Let $C^\infty(T)$ denote the space of all C^∞-functions on T. We shall check Assumption A(i) with $\mathscr{D}_0 = C^\infty(T)$, that is, the following four assertions: (a) $C^\infty(T) \subset \mathscr{D}(L^s)$ *for each* $s \in (0, \infty]$, (b) $C^\infty(T)$ *is uniformly dense in* $C(T)$, (c) $T_t^\infty C^\infty(T) \subset C^\infty(T)$ *for all* $t > 0$, *and* (d) $L^s u$ *tends to* $L^\infty u$ *uniformly as* $s \to \infty$ *for each* $u \in C^\infty(T)$.

Assertion (b) is obvious and assertion (c) is immediate from (3.8). To prove (a) and (d) we note the following bound:

$$Q^s(\xi) \leqq c|\xi|^2 \quad \text{for } |\xi| \geqq 1,\ \xi \in \tilde{G} \text{ and } s \in [1, \infty]; \tag{3.9}$$

this follows from the relation $Q(\xi) \leqq c'(|\xi|^\alpha + |\xi|^2)$, which is obtained from condition (Q_1). Note that $\widehat{(T_t^s u)}(\xi) = \exp\{-tQ^s(\xi)\} \times \hat{u}(\xi)$, $\xi \in \tilde{G}$ for each $u \in C(T)$ and $s \in (0, \infty]$. By an elementary calculation we have $|t^{-1}[\widehat{(T_t^s u)}(\xi) - \hat{u}(\xi)]| \leqq Q^s(\xi)|\hat{u}(\xi)|$ for any $t > 0$ and $t^{-1}[\widehat{(T_t^s u)}(\xi) - \hat{u}(\xi)] \to -Q^s(\xi)\hat{u}(\xi)$ as $t \to 0$. Thus, by the inversion formula (3.4) and the bound (3.9), one can show that if $u \in C(T)$ satisfies

$$\sum_{\xi \in \tilde{G}} |\xi|^2 |\hat{u}(\xi)| < \infty, \tag{3.10}$$

then $t^{-1}[T_t^s u - u]$ converges uniformly as $t \to 0$, that is, $u \in \mathscr{D}(L^s)$ and moreover

$$\widehat{(L^s u)}(\xi) = -Q^s(\xi)\hat{u}(\xi), \qquad \xi \in \tilde{G}. \tag{3.11}$$

Thus assertion (a) follows from the fact that $u \in C^\infty(T)$ satisfies (3.10). To see (d) it suffices to show that $\sum_{\xi \in \tilde{G}} |\widehat{(L^s u)}(\xi) - \widehat{(L^\infty u)}(\xi)|$ tends to zero as $s \to \infty$. This follows from (3.9), (3.10), (3.11) and the fact that $Q^s(\xi) \to Q^\infty(\xi)$ as $s \to \infty$ for each $\xi \in \tilde{G}$. This completes the proof of Lemma 3.1.

Let $\tilde{k}(x)$ be an arbitrary probability density on R^d relative to the Lebesgue measure. For $\varepsilon > 0$, define

$$\begin{aligned} \tilde{k}_\varepsilon(x) &= \varepsilon^{-d}\tilde{k}(\varepsilon^{-1}x), \\ k_\varepsilon(x) &= \sum_{g \in G} \tilde{k}_\varepsilon(x+g), \qquad x \in T. \end{aligned} \tag{3.12}$$

It is known [2] that Assumptions B and C in [2] are satisfied by

$$k_\varepsilon(x, y) = k_\varepsilon(x-y) \quad \text{and} \quad \varepsilon(t) = t^{-1/d}. \tag{3.13}$$

We have seen that Theorems 1.1 and 1.2 are applicable to the present case. For the convenience of reference for the sausage problem in Sect. 4 we shall restate Theorem 1.2 as it applies to this case.

For a given $M > 0$, let T_M denote the d-dimensional torus of size M and π the projection of R^d onto T_M. Let $k_\varepsilon(x-y)$ and $\varepsilon(t)$ be defined by (3.12) and (3.13). For a path $\omega = x(\cdot)$ on T_M, define

$$g_t(\omega, y) = \frac{1}{t}\int_0^t k_{\varepsilon(t)}(x(\sigma) - y)\, d\sigma, \qquad y \in T_M. \tag{3.14}$$

Let γ_M be the space of all probability densities on T_M relative to the Lebesgue measure λ endowed with the $L^1(\lambda)$-norm topology. Note that $g_t(\omega, \cdot) \in \gamma_M$. Let $I_M^{(\alpha)}(f)$, $f \in \gamma_M$, be the I-functional corresponding to the projection $\pi(X^{(\alpha)})$ of $X^{(\alpha)}$ onto T_M. We then have the following theorem.

Theorem 3.1. *Let $X^{(\alpha)}$ and $X=(X_t, P_x: t \geqq 0, x \in R^d)$ satisfy conditions* (Q_1) *and* (Q_2) *in Lemma* 3.1. *Let $\Phi_t(f)$, $t>0$ and $\Phi(f)$ be the functionals on γ_M satisfying the conditions in Theorem* 1.2. *Then, for any $s(t)$ increasing to infinity with $t>0$ and any $x \in R^d$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log E_x[\exp\{-t\Phi_t(g_t(\pi(X_\cdot^{s(t)}), \cdot))\}] \leqq - \inf_{f\in\gamma_M} [\Phi(f) + I_M^{(\alpha)}(f)]. \tag{3.15}$$

Here $\pi(X_\cdot^s)$ denotes the path $\{\pi(X_t^s),\ t \geqq 0\}$ on T_M and $X_t^s = X_0 + s^{-1}(X_{s^\alpha t} - X_0)$, $t \geqq 0$, $s>0$.

Remark. If $X=X^{(\alpha)}$, then every $X^s=(X_t^s, P_x)$ has the same law. In this case Theorem 3.1 is nothing but the corollary to Theorem 5.1 of [2].

4. The Sausage Problem for a Class of Lévy Processes on R^d

Let $S(x, \varepsilon)$ denote the sphere in R^d of radius $\varepsilon>0$ with center at $x \in R^d$. By the *sausage* of a symmetric Lévy process $X=(X_t, P_x: t \geqq 0, x \in R^d)$ we mean the random set $C_t^\varepsilon(X_\cdot) = \bigcup_{0 \leqq s < t} S(X_s, \varepsilon)$ (see [3]). Let $|A|$ denote the d-dimensional Lebesgue measure of any measurable subset A of R^d. Note that $|C_t^\varepsilon(X_\cdot)|$ is a functional of the path of X increasing with t.

Let $0<\alpha \leqq 2$ and let $X^{(\alpha)}=(X_t, P_x^{(\alpha)})$ be a symmetric stable process of order α with exponent $Q^{(\alpha)}(\xi)$ satisfying the nondegeneracy assumption $\inf_{|\xi|=1} Q^{(\alpha)}(\xi)>0$. Let $L^{(\alpha)}$ be the infinitesimal generator of $X^{(\alpha)}$ and let $E_x^{(\alpha)}$ denote the expectation with respect to $P_x^{(\alpha)}$. Donsker and Varadhan [3] have proved that, for each $x \in R^d$, $v>0$ and $\varepsilon>0$,

$$\lim_{t\to\infty} t^{-d/(d+\alpha)} \log E_x^{(\alpha)}[\exp\{-v|C_t^\varepsilon(X_\cdot)|\}] = -k(v, L^{(\alpha)}), \tag{4.1}$$

$$k(v, L^{(\alpha)}) = v^{\alpha/(d+\alpha)} \left(\frac{d+\alpha}{\alpha}\right) \left(\frac{\alpha\lambda_\alpha}{d}\right)^{d/(d+\alpha)}, \tag{4.2}$$

with $\lambda_\alpha = \inf_G \lambda(G)$, where the infimum is taken over all open sets G in R^d of unit volume and $\lambda(G)$ denotes the smallest eigenvalue of the eigenvalue problem $-L^{(\alpha)}u=\lambda u$ with the Dirichlet condition: $u(x)=0$, $x \in G^c$ (see [3] and [6; Sect. 4] for the precise definition of $\lambda(G)$).

The purpose of this section is to extend the above result to a class of Lévy processes which are close to $X^{(\alpha)}$. Let $X^{(\alpha)}$ be as above and $X=(X_t, P_x)$ another symmetric Lévy process. We assume that $X^{(\alpha)}$ and X satisfy conditions (Q_1) and (Q_2) in Lemma 3.1. In the theorem below we shall prove

$$\lim_{t\to\infty} t^{-d/(d+\alpha)} \log E_x[\exp\{-v|C_t^\varepsilon(X_\cdot)|\}] = -k(v, L^{(\alpha)}). \tag{4.3}$$

As in [3], however, we shall actually treat a more general functional $F(t, X.)$ defined below rather than $\exp\{-\nu|C_t^\varepsilon(X.)|\}$. Let $\varphi(x)$ be a $[0, \infty]$-valued Borel function on R^d. We define, for any $t>0$ and $\nu>0$,

$$F(t, X.)=\exp\left(-\nu\int_{R^d}\left(1-\exp\left\{-\int_0^t\varphi(X_s-y)\,ds\right\}\right)dy\right). \tag{4.4}$$

Note that if, in particular, $\varphi(x)=\infty$ for $|x|<\varepsilon$ and $\varphi(x)=0$ for $|x|\geqq\varepsilon$, then $F(t, X.)=\exp\{-\nu|C_t^\varepsilon(X.)|\}$.

Theorem 4.1. *Let $X^{(\alpha)}$ and X satisfy conditions* (Q_1) *and* (Q_2) *in Lemma* 3.1. *Suppose that $\int\varphi(x)\,dx>0$, then*

$$\limsup_{t\to\infty} t^{-d/(d+\alpha)}\log E_x[F(t, X.)]\leqq -k(\nu, L^{(\alpha)}). \tag{4.5}$$

Moreover, if $\varphi(x)=o(|x|^{-(d+\alpha)})\ (|x|\to\infty)$, then

$$\liminf_{t\to\infty} t^{-d/(d+\alpha)}\log E_x[F(t, X.)]\geqq -k(\nu, L^{(\alpha)}). \tag{4.6}$$

Here $k(\nu, L^{(\alpha)})$ is defined by (4.2).

Proof of the Upper Estimate. We shall prove (4.5) showing how Theorem 3.1 applies to the functional $F(t, X.)$ in (4.4). We can, without loss of generality, assume (see [3; p. 560]) that $\varphi(x)=a\tilde{k}(x)$, $x\in R^d$, where $\tilde{k}(x)$ is a probability density relative to the Lebesgue measure and $a>0$. For a given $M>0$, we define $g_t(\omega, y)$ for any path ω on T_M and $y\in T_M$ by (3.14), where $k_\varepsilon(x)$ is defined by (3.12) from the above $\tilde{k}(x)$ and $\varepsilon(t)$ by (3.13). Note that $g_t(\omega,\cdot)\in\gamma_M$.

By changes of variables and using the argument in [3; p. 562], we have

$$F(t, X.)\leqq\exp\{-\tau\Phi_\tau(g_\tau(\pi(X^s_\cdot),\cdot))\}, \tag{4.7}$$

where $\tau=\tau(t)=t^{d/(d+\alpha)}$, $s=s(\tau)=\tau^{1/d}=t^{1/(d+\alpha)}$ and

$$\Phi_\tau(f)=\nu\int_{T_M}(1-\exp\{-\tau^{\alpha/d}af(y)\})\,dy,\qquad f\in\gamma_M.$$

As was pointed out in [3; p. 563], the family of functionals $\Phi_\tau(f)$, $\tau>0$ on γ_M has the property that if $f_\tau\in\gamma_M$ converges to f in $L^1(\lambda)$, then $\liminf_{\tau\to\infty}\Phi_\tau(f_\tau)\geqq\Phi(f)$, where $\Phi(f)=\nu|\{x\in T_M; f(x)>0\}|$. Therefore, by Theorem 3.1, we have

$$\begin{aligned}&\limsup_{\tau\to\infty}\frac{1}{\tau}\log E_x[\exp\{-\tau\Phi_\tau(g_\tau(\pi(X^{s(\tau)}_\cdot),\cdot))\}]\\&\qquad\leqq-\inf_{f\in\gamma_M}[\Phi(f)+I_M^{(\alpha)}(f)],\end{aligned} \tag{4.8}$$

where $I_M^{(\alpha)}(f)$ is the I-functional corresponding to the projection $\pi(X^{(\alpha)})$ of $X^{(\alpha)}$ onto the torus T_M. By (4.7),

$$\limsup_{t\to\infty} t^{-d/(d+\alpha)}\log E_x[F(t, X.)]$$

is dominated by the left hand side of (4.8). Thus, taking infimum over $M>0$, we have

$$\limsup_{t\to\infty} t^{-d/(d+\alpha)} \log E_x[F(t,X.)] \leqq -\sup_{M>0} \inf_{f\in\gamma_M} [\nu|\{y\in T_M; f(y)>0\}| + I_M^{(\alpha)}(f)]. \tag{4.9}$$

It has already been shown in [3; Lemma 3.5, 3.6 and 3.9] that the right hand side of (4.9) is not greater than $-k(\nu, L^{(\alpha)})$. This completes the proof of (4.5).

Next we shall be concerned with the lower estimate (4.6). In the following we shall denote by $C_0^\infty(R^d)$ the space of all C^∞-functions on R^d with compact support and define, for any measurable function f on R^d, $\|f\|_\infty = \max_x |f(x)|$ and $\|f\|_p = (\int |f(x)|^p dx)^{1/p}$, $p=1,2$.

We shall prepare a generalized version of the lemma due to Pastur [7].

Lemma 4.1. *Let $\{q(x): x\in R^d\}$ be a stationary random field defined on a probability space with P and E denoting its probability measure and expectation, respectively. Let $\mathscr{E}(\cdot,\cdot)$ be the Dirichlet form (see [5]) of a symmetric Lévy process $X=(X_t, P_x)$ on R^d. Suppose that $E[e^{-tq(0)}]<\infty$ for each $t>0$. Then*

$$E[e^{-tq(0)}] \geqq E\times E_x\left[\exp\left\{-\int_0^t q(X_s)\,ds\right\}\right] \geqq (\|f\|_\infty \cdot \|f\|_1)^{-1} \exp\{-[t\mathscr{E}(f,f) + \Psi_t(f)]\} \tag{4.10}$$

for any $f\in C_0^\infty(R^d)$ such that $\|f\|_2=1$, where $E\times E_x$ denotes the expectation with respect to the product measure $P\times P_x$ and

$$\Psi_t(f) = -\log E[\exp\{-\int_{R^d} tq(x) f^2(x)\,dx\}]. \tag{4.11}$$

Pastur [7] has proved (4.10) in case of X being the Brownian motion. The proof of the general case is similar. Hence we omit the proof (see also [6; Theorem 7.1]).

Proof of the Lower Estimate. Let $\Pi(dy)$ denote a Poisson random measure on R^d with characteristic measure $\nu|\cdot|$, where $|\cdot|$ denotes the d-dimensional Lebesgue measure. Then

$$q(x) = \int_{R^d} \varphi(x-y)\,\Pi(dy) \tag{4.12}$$

defines a stationary random field $\{q(x): x\in R^d\}$, where $\varphi(x)$ is that appeared in (4.4). Then it is easy to see that

$$E\left[\exp\left\{-\int_0^t q(X_s)\,ds\right\}\right] = \exp\left(-\nu\int_{R^d}\left(1-\exp\left\{-\int_0^t \varphi(X_s-y)\,ds\right\}\right)dy\right),$$

$$E[\exp\{-\int_{R^d} tq(x) f^2(x)\,dx\}] = \exp(-\nu\int_{R^d}(1-\exp\{-\int_{R^d} t\varphi(x-y) f^2(x)\,dx\})\,dy).$$

Hence, by Lemma 4.1, we have

$$E_x[F(t,X.)]=E\times E_x\left[\exp\left\{-\int_0^t q(X_s)\,ds\right\}\right]$$
$$\geqq(\|f\|_\infty\cdot\|f\|_1)^{-1}\exp\{-[t\mathscr{E}(f,f)+\Psi_t(f)]\} \tag{4.13}$$

for any $f\in C_0^\infty(R^d)$ and, by Definitions (4.4) and (4.11), we have

$$F(t,X.)=E\left[\exp\left\{-\int_0^t q(X_s)\,ds\right\}\right],$$

$$\Psi_t(f)=\nu\int_{R^d}(1-\exp\{-\int_{R^d}t\varphi(x-y)f^2(x)\,dx\})\,dy.$$

For each $f\in C_0^\infty(R^d)$ with $\|f\|_2=1$ and $R>0$, define

$$f_R(x)=R^{-d/2}f(R^{-1}x),\qquad x\in R^d.$$

Let $R(t)=t^{1/(d+\alpha)}$, $t>0$. It has been proved [6; Lemma 8.2] that condition (Q_1) implies that

$$\mathscr{E}(f_{R(t)},f_{R(t)})=t^{-\alpha/(d+\alpha)}\mathscr{E}^{(\alpha)}(f,f)+o(t^{-\alpha/(d+\alpha)}) \tag{4.14}$$

as $t\to\infty$, where $\mathscr{E}^{(\alpha)}(\cdot,\cdot)$ denotes the Dirichlet form of $X^{(\alpha)}$. It has also been proved in [6; Lemma 8.3] that the condition

$$\varphi(x)=o(|x|^{-(d+\alpha)})\qquad(|x|\to\infty)$$

implies that

$$\Psi_t(f_{R(t)})\leqq t^{d/(d+\alpha)}\nu|E|+o(t^{d/(d+\alpha)}) \tag{4.15}$$

as $t\to\infty$, where E denotes the support of f. Thus, noting that

$$\|f_{R(t)}\|_\infty\cdot\|f_{R(t)}\|_1=\|f\|_\infty\cdot\|f\|_1,\qquad t>0,$$

we have, by (4.13), (4.14) and (4.15),

$$\liminf_{t\to\infty}t^{-d/(d+\alpha)}\log E_x[F(t,X.)]$$
$$\geqq-[\mathscr{E}^{(\alpha)}(f,f)+\nu|\{y;f^2(y)>0\}|]$$

for each $f\in C_0^\infty(R^d)$ with $\|f\|_2=1$, and hence

$$\liminf_{t\to\infty}t^{-d/(d+\alpha)}\log E_x[F(t,X.)]$$
$$\geqq-\inf_{f\in C_0^\infty(R^d),\,\|f\|_2=1}[\mathscr{E}^{(\alpha)}(f,f)+\nu|\{y;f^2(y)>0\}|]. \tag{4.16}$$

It is known [3; Theorem 3.2 and Lemma 3.9] that

$$\inf_{E:\text{compact}}[\nu|E|+\inf_{\mu:\mu(E)=1}I(\mu)]=k(\nu,L^{(\alpha)}),$$

where $I(\mu)$ denotes the I-functional corresponding to the symmetric stable process $X^{(\alpha)}$ and μ denotes any probability measure on R^d. Thus we have only to show that

$$\inf_{f\in C_0^\infty(R^d),\, \|f\|_2=1} [\nu|\{y; f^2(y)>0\}| + \mathscr{E}^{(\alpha)}(f,f)] \\ \leqq \inf_{E:\text{compact}} [\nu|E| + \inf_{\mu:\mu(E)=1} I(\mu)]. \tag{4.17}$$

To this end, we note the relation between $I(\mu)$ and $\mathscr{E}^{(\alpha)}(f,f)$. It is known [3; p. 533] that if $I(\mu)<\infty$, then μ has the density g relative to the Lebesgue measure on R^d such that $I(\mu)=\mathscr{E}^{(\alpha)}(\sqrt{g},\sqrt{g})$. Let E be any compact subset of R^d and μ a probability measure supported on E with the density g such that $I(\mu)=\mathscr{E}^{(\alpha)}(\sqrt{g},\sqrt{g})<\infty$. Then by a standard mollification argument one can find a family f_δ, $\delta>0$ of functions in $C_0^\infty(R^d)$ with $\|f_\delta\|_2=1$ such that

$$\lim_{\delta\to 0} \mathscr{E}^{(\alpha)}(f_\delta, f_\delta) = \mathscr{E}^{(\alpha)}(\sqrt{g},\sqrt{g}) \quad \text{and} \quad \limsup_{\delta\to 0} |\{y; f_\delta^2(y)>0\}| \leqq |E|,$$

which proves (4.17). This completes the proof of (4.6).

5. Necessary and Sufficient Conditions for (Q_1) and a Sufficient Condition for (Q_2)

Let $Q(\xi)$ be the exponent of a symmetric Lévy process $X=(X_t, P_x)$. Then, by the Lévy-Hintčin formula, we have

$$Q(\xi)=\tfrac{1}{2}\langle \xi, a\xi\rangle + \int_{R^d-\{0\}} (1-\cos\langle \xi, y\rangle)\, n(dy), \qquad \xi\in R^d \tag{5.1}$$

where a is a d-dimensional symmetric non-negative definite matrix and $n(dy)$ a symmetric Radon measure on $R^d-\{0\}$ satisfying $\int(|y|^2\wedge 1)\, n(dy)<\infty$. The measure $n(dy)$ is called the Lévy measure. Let $Q^{(\alpha)}(\xi)$ be the exponent of a symmetric stable process $X^{(\alpha)}=(X_t, P_x^{(\alpha)})$ of order α. It is known that $Q^{(\alpha)}(\xi)$ has the form

$$\begin{aligned} Q^{(\alpha)}(\xi) &= \int_0^\infty \int_{S^{d-1}} (1-\cos\langle \xi, r\sigma\rangle) \frac{\tilde{n}(d\sigma)}{r^{\alpha+1}} dr && \text{if } 0<\alpha<2, \\ &= \tfrac{1}{2}\langle \xi, \tilde{a}\xi\rangle && \text{if } \alpha=2, \end{aligned} \tag{5.2}$$

where $\tilde{n}(d\sigma)$ is a symmetric finite measure on the unit sphere S^{d-1} and $\tilde{a}$ is a symmetric non-negative definite matrix. In case of $0<\alpha<2$ the Lévy measure $n^{(\alpha)}(dy)$ of $X^{(\alpha)}$ is determined by the relation $n^{(\alpha)}(\Sigma(r))=\alpha^{-1} r^{-\alpha}\tilde{n}(\Sigma)$, where

$$\Sigma(r)=\{y\in R^d; |y|>r, |y|^{-1}y\in\Sigma\}, \qquad r>0$$

and Σ is any Borel subset of S^{d-1}. We assume that $\inf_{|\xi|=1} Q^{(\alpha)}(\xi)>0$ (nondegeneracy assumption); this is satisfied if and only if, for $0<\alpha<2$, the support $S_0\subset S^{d-1}$ of $\tilde{n}(d\sigma)$ spans R^d as a vector space; for $\alpha=2$, $\tilde{a}$ is positive definite.

Recall that $X_t^s = X_0 + s^{-1}(X_{s^\alpha t} - X_0)$, $t \geqq 0$, $s \in (0, \infty)$. We define the infinitely divisible distributions μ_s, $s > 0$ by $\mu_s(dx) = P_0(X_1^s \in dx)$ and the stable distribution $\mu^{(\alpha)}$ by $\mu^{(\alpha)}(dx) = P_0^{(\alpha)}(X_1 \in dx)$.

Proposition 5.1. *The following conditions are equivalent.*

(i) *Condition* (Q_1) *holds for* $Q(\xi)$ *and* $Q^{(\alpha)}(\xi)$.

(ii) μ_s *converges weakly to* $\mu^{(\alpha)}$ *as* $s \to \infty$.

(iii) *The distribution* μ_1 *belongs to the domain of normal attraction of* $\mu^{(\alpha)}$, i.e., ν_n *converges weakly to* $\mu^{(\alpha)}$ *as* $n \to \infty$, *where* $\nu_n(dx) = \mu_{n^{1/\alpha}}(dx) = P_0(n^{-1/\alpha} X_n \in dx)$.

(iv) *In case of* $0 < \alpha < 2$, $\alpha r^\alpha n(\Sigma(r)) \to \tilde{n}(\Sigma)$ *as* $r \to \infty$ *for each continuity set* $\Sigma \subset S^{d-1}$ *of* $\tilde{n}(d\sigma)$. *In case of* $\alpha = 2$,

$$\langle \xi, \tilde{a}\xi \rangle = \langle \xi, a\xi \rangle + \int \langle \xi, y \rangle^2 n(dy).$$

Proof. First we assume that condition (i) holds. Then one can easily see that $\exp\{-Q^s(\xi)\}$ tends to $\exp\{-Q^{(\alpha)}(\xi)\}$ uniformly on any compact subset of R^d as $s \to \infty$, which is equivalent to condition (ii). The implication (ii) $\Rightarrow$ (iii) is obvious. We next show (iii) $\Rightarrow$ (i). Suppose that condition (iii) holds, then $\exp\{-nQ(n^{-1/\alpha}\xi)\}$ tends to $\exp\{-Q^{(\alpha)}(\xi)\}$ uniformly on any compact subset of R^d as $n \to \infty$. Thus, using the inequality

$$|e^{-x} - e^{-y}| \geqq \min\{e^{-x}, e^{-y}\}\, |x - y| \qquad \text{for } x \geqq 0 \text{ and } y \geqq 0,$$

we have $nQ(n^{-1/\alpha}\xi) \to Q^{(\alpha)}(\xi)$ uniformly on any compact subset of R^d as $n \to \infty$. Let K_n, $n \geqq 1$ be a sequence of compact subsets of R^d of the form $K_n = \{\xi \in R^d;\ (2n)^{-1/\alpha} \leqq |\xi| \leqq n^{-1/\alpha}\}$. Then, for $k \geqq 1$, $\{\xi \in R^d; 0 < |\xi| \leqq k^{-1/\alpha}\} = \bigcup_{n \geqq k} K_n$ and $|\xi|^{-\alpha} \leqq 2n$ for $\xi \in K_n$. Hence we obtain

$$\begin{aligned}
&\sup_{0 < |\xi| \leqq k^{-1/\alpha}} |\xi|^{-\alpha} |Q(\xi) - Q^{(\alpha)}(\xi)| \\
&\quad = \sup_{n \geqq k} \sup_{\xi \in K_n} |\xi|^{-\alpha} |Q(\xi) - Q^{(\alpha)}(\xi)| \\
&\quad \leqq 2 \sup_{n \geqq k} \sup_{\xi \in K_n} n\, |Q(\xi) - Q^{(\alpha)}(\xi)| \\
&\quad = 2 \sup_{n \geqq k} \sup_{\xi \in K_1} |nQ(n^{-1/\alpha}\xi) - Q^{(\alpha)}(\xi)| \to 0
\end{aligned}$$

as $k \to \infty$, which is condition (i). Finally we show (ii) $\Leftrightarrow$ (iv). Let a^s and $n^s(dy)$ be the matrix and Lévy measure, respectively, in the representation (5.1) for the exponent $Q^s(\xi)$. Then it follows from Theorem 1.2 of [8] that condition (ii) holds if and only if the following two conditions hold.

(a) $n^s(\Sigma(r)) \to n^{(\alpha)}(\Sigma(r))$ as $s \to \infty$ for each $\Sigma \subset S^{d-1}$ and $r > 0$ such that $\Sigma(r)$ is a continuity set of $n^{(\alpha)}(dy)$.

(b) $$\begin{aligned}
&\lim_{\varepsilon \to 0} \limsup_{s \to \infty} \Big[\langle \xi, a^s \xi \rangle + \int_{0 < |y| < \varepsilon} \langle \xi, y \rangle^2 n^s(dy)\Big] \\
&= \lim_{\varepsilon \to 0} \liminf_{s \to \infty} \Big[\langle \xi, a^s \xi \rangle + \int_{0 < |y| < \varepsilon} \langle \xi, y \rangle^2 n^s(dy)\Big] \\
&= \langle \xi, \tilde{a}\xi \rangle.
\end{aligned}$$

Here we use the convention that in case of $\alpha=2$, $n^{(2)}(dy)=0$ and in case of $0<\alpha<2$, $\tilde{a}=0$.

On the other hand it is easy to see that $n^s(\Sigma(r))=s^\alpha n(\Sigma(sr))$, $n^{(\alpha)}(\Sigma(r)) =\alpha^{-1}r^{-\alpha}\tilde{n}(\Sigma)$, $a^s=s^{\alpha-2}a$ and

$$\int_{0<|y|<\varepsilon}\langle\xi,y\rangle^2 n^s(dy)=s^{\alpha-2}\int_{0<|y|<s\varepsilon}\langle\xi,y\rangle^2 n(dy).$$

Hence conditions (a) and (b) can be written as follows.

In case of $0<\alpha<2$:

(a)′ $\alpha r^\alpha n(\Sigma(r))\to\tilde{n}(\Sigma)$ as $r\to\infty$ for each continuity set $\Sigma\subset S^{d-1}$ of $\tilde{n}$.

(b)′ $\lim_{\varepsilon\to 0}\limsup_{s\to\infty} s^{\alpha-2}\int_{0<|y|<s\varepsilon}|y|^2 n(dy)=0.$

In case of $\alpha=2$:

(a)″ $n(y;|y|>r)=o(r^{-2})\,(r\to\infty)$.

(b)″ $\langle\xi,a\xi\rangle+\int_{|y|>0}\langle\xi,y\rangle^2 n(dy)=\langle\xi,\tilde{a}\xi\rangle$.

Thus we have only to prove the implications (a)′ ⇒ (b)′ and (b)″ ⇒ (a)″. Let $0<\alpha<2$. Then condition (a)′ implies that $N(r)\equiv n(y;|y|>r)=O(r^{-\alpha})\,(r\to\infty)$. Thus we get

$$\begin{aligned}\int_{0<|y|<r}|y|^2 n(dy)&=-\int_0^r\rho^2\,dN(\rho)\\&=2\int_0^r\rho(N(\rho)-N(r))\,d\rho\\&=O(r^{2-\alpha})\qquad(r\to\infty).\end{aligned}$$

Hence we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{s\to\infty} s^{\alpha-2}\int_{0<|y|<s\varepsilon}|y|^2 n(dy)\\&\quad=\lim_{\varepsilon\to 0}\varepsilon^{2-\alpha}\limsup_{r\to\infty} r^{\alpha-2}\int_{0<|y|<r}|y|^2 n(dy)=0,\end{aligned}$$

which is (b)′. Let $\alpha=2$. Then (b)″ implies that $\int|y|^2 n(dy)<\infty$. Hence we have

$$r^2 n(y;|y|>r)\leqq\int_{|y|>r}|y|^2 n(dy)\to 0\quad\text{as}\quad r\to\infty,$$

which is (a)″. This completes the proof.

Next we shall be converned with condition (Q_2).

Proposition 5.2. *Suppose that condition* (Q_1) *holds for* $Q(\xi)$ *and* $Q^{(\alpha)}(\xi)$. *Then condition* (Q_2) *holds if*

$$\lim_{|\xi|\to\infty}(\log|\xi|)^{-1}Q(\xi)=\infty. \tag{5.3}$$

Proof. By (5.3) there exists a constant $R>0$ and a positive function $c(x)$ on $[R,\infty)$ increasing to infinity with $x\geqq R$ such that

$$Q(\xi)\geqq c(|\xi|)\log|\xi|\qquad\text{for }|\xi|\geqq R. \tag{5.4}$$

Let β be fixed such as $0<\beta<\alpha$ and let $\varepsilon=\alpha-\beta>0$. We can assume that $R\geqq e^{1/\varepsilon}$ and the function $c(x)$, $x\geqq R$ has the property that $c(x)/c(y)\leqq(x/y)^{\beta}$ for $x\geqq y\geqq R$. In fact, if we define a minorant $c_*(x)$ of $c(x)$ by $c_*(x) = \inf_{x\geqq z\geqq R} c(z)(x/z)^{\beta}$, $x\geqq R$, then $c_*(x)$ has the above property and it also increases to infinity with x.

First we shall show that, for each $s\geqq 1$,

$$Q^s(\xi)\equiv s^{\alpha}Q(s^{-1}\xi)\geqq c(|\xi|)\log|\xi| \quad \text{if } |\xi|\geqq sR. \tag{5.5}$$

To this end we note that $Q^s(\xi)\geqq s^{\alpha}c(|\xi|/s)\log(|\xi|/s)$ if $|\xi|\geqq sR$ by (5.4). Let $s\geqq 1$ and $|\xi|\geqq sR$. Then we have $c(|\xi|)\leqq s^{\beta}c(|\xi|/s)$, and hence

$$s^{\alpha}c(|\xi|/s)\log(|\xi|/s)-c(|\xi|)\log|\xi|\geqq c(|\xi|)\{s^{\varepsilon}\log(|\xi|/s)-\log|\xi|\}.$$

One can prove that the right hand side is non-negative by differentiating with respect to s and noting the relation $|\xi|/s\geqq R\geqq e^{1/\varepsilon}$. This proves (5.5).

Next we shall prove that there exists a constant $b>0$ such that,

$$Q_*(\xi)\equiv \inf_{s\geqq 1}Q^s(\xi)\geqq bc(|\xi|)\log|\xi| \quad \text{if } |\xi|\geqq R. \tag{5.6}$$

To this end we note that there exists a constant $c_0>0$ such that

$$Q(\xi)\geqq c_0|\xi|^{\alpha} \quad \text{if } |\xi|\leqq R, \tag{5.7}$$

which follows from (Q_1) and the nondegeneracy assumption. Since (5.7) implies that $Q^s(\xi)\geqq c_0|\xi|^{\alpha}$ if $|\xi|\leqq sR$, we have

$$Q^s(\xi)\geqq \min\{c_0|\xi|^{\alpha}, c(|\xi|)\log|\xi|\}, \quad |\xi|\geqq R,\ s\geqq 1.$$

Thus we have only to show that there exists *a* constant $b>0$ such that

$$c_0|\xi|^{\alpha}\geqq bc(|\xi|)\log|\xi| \quad \text{if } |\xi|\geqq R. \tag{5.8}$$

For this it suffices to show that $\sup_{x\geqq R}x^{-\alpha}c(x)\log x<\infty$. Let $x\geqq R$. Then we have $c(x)\leqq c(R)(x/R)^{\beta}$ and $x^{-\varepsilon}\log x\leqq R^{-\varepsilon}\log R$ since $R\geqq e^{1/\varepsilon}$. Thus we obtain

$$x^{-\alpha}c(x)\log x\leqq c(R)R^{-\beta}x^{-\varepsilon}\log x\leqq c(R)R^{-\alpha}\log R.$$

This proves (5.6).

Finally we shall check condition (Q_2). For given $t>0$ and $r>0$ we choose $R'\geqq R$ so that $tbc(R')\geqq d+1$ and let $\tilde{G}=(rZ)^d$. Then, by (5.6), we have

$$\begin{aligned}\sum_{\xi\in\tilde{G}}\exp\{-tQ_*(\xi)\}&\leqq C_{R'}+\sum_{\xi\in\tilde{G},|\xi|\geqq R'}\exp\{-tbc(|\xi|)\log|\xi|\}\\&\leqq C_{R'}+\sum_{\xi\in\tilde{G},|\xi|\geqq R'}|\xi|^{-d-1}<\infty,\end{aligned}$$

where $C_{R'}$ denotes the cardinality of the set $\{\xi\in\tilde{G};|\xi|\leqq R'\}$. This completes the proof.

Remark. One can easily see that if $\limsup_{|\xi|\to\infty}(\log|\xi|)^{-1}Q(\xi)<\infty$, then (Q_2) does not hold.

Example 1. Let $0<\alpha=\alpha_0<\alpha_1<\ldots<\alpha_n\leqq 2$ and let $Q(\xi)=\sum_{i=0}^{n} Q^{(\alpha_i)}(\xi)$, where $Q^{(\alpha_i)}(\xi)$ is the exponent of a symmetric stable process of order α_i for each i. Then condition (Q_1) holds for $Q^{(\alpha)}(\xi)$ and $Q(\xi)$. Further condition (Q_2) holds if and only if $Q^{(\alpha)}(\xi)$ is non-degenerate.

Example 2. Let $\hat{n}(d\sigma)$ be a symmetric finite measure on S^{d-1} and let $f(\sigma,r)$ be a non-negative measurable function on $S^{d-1}\times(0,\infty)$ satisfying

$$\int_0^\infty \int_{S^{d-1}} (r^2\wedge 1) f(\sigma,r)\hat{n}(d\sigma)\,dr<\infty \quad \text{and} \quad f(-\sigma,r)=f(\sigma,r).$$

Define the exponent $Q(\xi)$ of a symmetric Lévy process by

$$Q(\xi)=\int_0^\infty \int_{S^{d-1}} (1-\cos\langle\xi,r\sigma\rangle) f(\sigma,r)\hat{n}(d\sigma)\,dr, \qquad \xi\in R^d;$$

the corresponding Lévy measure $n(dy)$ is determined by the relation

$$n(\Sigma(r))=\int_r^\infty \int_\Sigma f(\sigma,\rho)\hat{n}(d\sigma)\,d\rho$$

for any Borel subset Σ of S^{d-1} and $r>0$. Let $0<\alpha<2$. Suppose that there exists a non-negative measurable function $c(\sigma)$ on S^{d-1} such that

$$\int_{|\langle\xi,\sigma\rangle|>0} c(\sigma)\hat{n}(d\sigma)>0 \qquad \text{for any } \xi\neq 0 \tag{5.9}$$

and

$$\int_{S^{d-1}} \left| f(\sigma,r)-\frac{c(\sigma)}{r^{\alpha+1}}\right| \hat{n}(d\sigma)=o\left(\frac{1}{r^{\alpha+1}}\right) \qquad (r\to\infty). \tag{5.10}$$

Let $\tilde{n}(d\sigma)$ be the symmetric finite measure on S^{d-1} defined by $\tilde{n}(d\sigma)=c(\sigma)\hat{n}(d\sigma)$ and let $Q^{(\alpha)}(\xi)$ be the exponent of a symmetric stable process defined by the first half of (5.2) with the above $\tilde{n}(d\sigma)$. Then, by (5.9), $Q^{(\alpha)}(\xi)$ satisfies the nondegeneracy assumption and condition (Q_1) holds for $Q^{(\alpha)}(\xi)$ and $Q(\xi)$. In fact by (5.10) we have, for any continuity set $\Sigma\subset S^{d-1}$ of $\tilde{n}(d\sigma)$,

$$\begin{aligned}|\alpha r^\alpha n(\Sigma(r))-\tilde{n}(\Sigma)| &= \alpha r^\alpha \left| \int_r^\infty \int_\Sigma f(\sigma,\rho)\hat{n}(d\sigma)\,d\rho - \int_r^\infty \int_\Sigma \frac{\tilde{n}(d\sigma)}{\rho^{\alpha+1}}\,d\rho \right| \\ &\leqq \alpha r^\alpha \int_r^\infty d\rho \int_{S^{d-1}} \left| f(\sigma,\rho)-\frac{c(\sigma)}{\rho^{\alpha+1}}\right| \hat{n}(d\sigma)\to 0 \qquad (r\to\infty).\end{aligned}$$

Thus, by Proposition 5.1, condition (Q_1) holds.

Moreover, suppose that

$$\lim_{r\to 0}\frac{g(r)}{(1/r)\log(1/r)}=\infty, \tag{5.11}$$

where $g(r)=\inf_{|\xi|=1}\int_{S^{d-1}}\langle\xi,\sigma\rangle^2 f(\sigma,r)\hat{n}(d\sigma)$. Then one can show that condition (5.3) in Proposition 5.2 holds by the following observation, and hence (Q_2) holds. Noting that

$$1-\cos\langle\xi,r\sigma\rangle \geqq \frac{1}{\pi}r^2\langle\xi,\sigma\rangle^2 \quad \text{if } r\leqq|\xi|^{-1},$$

we get

$$\begin{aligned} Q(\xi) &\geqq \frac{1}{\pi}\int_0^{|\xi|^{-1}} r^2\,dr\int_{S^{d-1}}\langle\xi,\sigma\rangle^2 f(\sigma,r)\hat{n}(d\sigma) \\ &\geqq \frac{1}{\pi}\int_0^{|\xi|^{-1}} r^2|\xi|^2 g(r)\,dr \\ &= \frac{1}{\pi}\int_0^1 r^2|\xi|^{-1} g(r/|\xi|)\,dr. \end{aligned}$$

Thus, by Fatou's lemma, we have

$$\begin{aligned} \liminf_{|\xi|\to\infty}\frac{Q(\xi)}{\log|\xi|} &\geqq \frac{1}{\pi}\int_0^1 r^2\left(\liminf_{|\xi|\to\infty}\frac{g(r/|\xi|)}{|\xi|\log|\xi|}\right)dr \\ &= \frac{1}{\pi}\int_0^1 r\left(\liminf_{|\xi|\to\infty}\frac{g(r/|\xi|)}{(|\xi|/r)\log(|\xi|/r)}\right)dr=\infty. \end{aligned}$$

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