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Disintegration with Respect to L_p-density Functions and Singular Measures

P.H. Maserick* **

Pennsylvania State University, Dept. of Mathematics, 215 McAllister Bld.
University Park, University Park, PA 16802, USA

Summary. Let $\mathscr{A}$ be a real or complex commutative ordered algebra with identity and involution. Let Γ denote the set of positive multiplicative linear functionals ρ on $\mathscr{A}$. Equip Γ with the topology of simple convergence. For a fixed non-negative probability measure μ on Γ the set $\mathscr{L}_p$ of linear functionals f on $\mathscr{A}$ which admit an integral representation of the form $f(x)=\int_\Gamma \rho(x)\,F(\rho)\,d\mu(\rho)$ with $F\in L_p(\mu)$ $(1\leqq p\leqq\infty)$ is biuniquely identified with $L_p(\mu)$ via the map $f\to F$. The norm on $\mathscr{L}_p$ under which this map becomes an isometry is characterized and a formula for approximating F is derived. The linear functionals which admit representation of the form $\int_\Gamma \rho(x)\,d\nu(\rho)$ with $\nu\perp\mu$ are also characterized and appropriately normed. The theory is applied to solve abstract versions of trigonometric and n-dimensional moment problems as well as provide an alternate point of view to the theory of L_p-spaces. New proofs of classical theorems are offered.

0. Introduction

This work may be regarded as a sequel to [14]. Therein, those linear functionals f admitting an integral representation of the form $f(x)=\int_\Gamma \rho(x)\,d\mu_f(\rho)$ (with μ_f not necessarily non-negative) were described. Here we can further describe f when the natural restrictions on μ_f mentioned in the summary above are imposed. Not only do our results offer new proofs of classical theorems on trigonometric and n-dimensional moment problems, but also the abstract algebraic setting used enables us to unify the two here-to-fore only analogous theories, allowing new theorems from the one area to be carried over from

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** The results contained herein include the proofs of theorems announced in [15]

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known theorems in the other. This unification is made explicit in §4 where we solve what we call, the abstract moment problem, in broad generality. Motivating applications to Lebesgue spaces and to the field of classical moment problems are indicated by three examples given in §2. The set up of each of these differs from the subsumming set up of §4. The more recent results of Alo-Korvin [1] and Leader [9] are also easily attained by the theory presented here. In fact, these particular works were strong motivating influences on the present. For example, the proof of Theorem A, although more technical, is an upgrading of Leader's proof of Theorem 1 in [9]. To appreciate this the reader should refer to [16] where both the works of Leader and Alo-Korvin are generalized more in the flavor of the present. To maintain continuity, the main theorems are precisely stated in §1 but their proofs, which tend to be technical, are deferred until §3. The appropriate cross references are included parantheti-cally following the theorem numbers in the next section.

1. Notation and Statements of Main Theorems

Throughout the sequel, we will let $\mathscr{A}$ denote the algebra referred to in the summary above and assume the set up given in [14]. Explicitly, the identity and involution will be denote by 1 and * respectively, and we assume the existence of a subset τ of $\mathscr{A}$ satisfying

(i) $x^* = x$ for each $x \in \tau$.

(ii) $\{1\} - \tau \in \mathrm{Alg}^+ \operatorname{span}(\tau)$, i.e. $1 - x$ is a positive linear sum of products of members of τ for each $x \in \tau$.

(iii) $\mathscr{A} = \mathrm{Alg}\operatorname{span}(\tau)$, i.e. every $x \in \mathscr{A}$ is a linear sum of products of members of τ.

The cone P, defined by $\mathrm{Alg}^+ \operatorname{span}(\tau)$, orders $\mathscr{A}$ in the usual way. If the dual space $\mathscr{A}'$ of linear functionals on $\mathscr{A}$ is given the $\sigma(\mathscr{A}', \mathscr{A})$-topology, then the set Γ of non-negative, non-trivial, multiplicative linear functionals is compact. A finite, possibly repetitious, subset $(x_i)_i$ of P will be called a *partition of unity* provided $\Sigma_i x_i = 1$. If $(x_i)_i$ and $(y_j)_j$ are two partitions of unity then their product $(x_i y_j)_{i,j}$ is again a partition of unity. Let Ω be a subsemigroup of the semigroup of all partitions of unity such that $x \in \tau$ is a member of some $\Lambda \in \Omega$. Recall [14], $f \in \mathscr{A}'$ is said to be of *bounded variation* (BV) if $\sup_{\Lambda \in \Omega} \sum_{x \in \Lambda} |f(x)| = \|f\| < \infty$. Then $\|f\|$ is independent of the choice of Ω. Moreover, f admits *a*, necessarily unique, complex-valued, regular representing measure μ_f, if and only if f is BV; μ_f being non-negative if and only if f is non-negative. Let $g \in \mathscr{A}$ be fixed, non-negative, and normalized by the condition $g(1) = 1$. Then μ_g is a probability measure. Define $f \in \mathscr{A}'$ to be *g-continuous* if given $\varepsilon > 0$, there exist $\delta > 0$ such that $\sum_{x \in \Lambda_0} |f(x)| < \varepsilon$ whenever $\sum_{x \in \Lambda_0} g(x) < \delta$ and Λ_0 is a semipartition of Λ. The natural corresponding definition for a BV-functional to be g-singular will be defined later in §3.

Theorem A (3.1). *Every g-continuous functional is BV.*

The set $\mathscr{L}_1(g)$ of all g-continuous functionals is then a normed linear space with variation norm $\|\cdot\|_1=\|\cdot\|$. For $1<p<\infty$, $\Lambda\in\Omega$ and $f\in\mathscr{A}'$, define $\|f\|_{(p,\Lambda)} = \sum_{x\in\Lambda} (|f(x)|^p/[g(x)]^{p-1})^{1/p}$ where $0/0=0$, and set $\|f\|_p=\sup_{\Lambda}\|f\|_{(p,\Lambda)}$. Denote the set of all f with $\|f\|_p<\infty$ by $\mathscr{L}_p(g)$. A Lipschitz type norm is also available to define $\mathscr{L}_\infty(g)$. Moreover, $\mathscr{L}_p(g)\subset\mathscr{L}_1(g)$ for $1\leqq p\leqq\infty$.

Theorem B (3.12, 3.14, 3.16). *A linear functional f on $\mathscr{A}$ admits a disintegration of the form $f(x)=\int \rho(x)\,d\mu_f(\rho)$ where:*

(i) *$d\mu_f=f'(\rho)\,d\mu_g(\rho)$ with $f'\in L_p(\mu_g,\Gamma)$ if and only if $f\in\mathscr{L}_p(g)$ $(1\leqq p\leqq\infty)$,*
(ii) *μ_f is μ_g-singular if and only if f is g-singular.*

Moreover the spaces $\mathscr{L}_p(g)$ and $L_p(u_g,\Gamma)$ are linearly isometric via $f\to f'$ as also are the spaces of g-singular functionals and μ_g-singular measures via $f\to\mu_f$.

For each $f\in\mathscr{A}'$ and $\Lambda\in\Omega$, define a scalar-valued function f' on Γ by $f'_\Lambda(\rho) = \sum_{x\in\Lambda} (f(x)/g(x))\,\rho(x)$. If the natural ordering $\Lambda\Lambda'\geqq\Lambda$ on Ω is imposed, then the following approximation theorem holds.

Theorem C (3.13). *If $f\in\mathscr{L}_p(g)$ $(1\leqq p<\infty)$ then the representing measure μ_f for f is of the form $f'\,d\mu_g$ where $\|f'-f'_\Lambda\|_p \xrightarrow[\Lambda]{} 0$.*

We conclude the theory by describing the dual $\mathscr{L}_p(g)^*$, of $\mathscr{L}_p(g)$ intrinsically.

Theorem D (3.15). *If $1\leqq p<\infty$, $1<q\leqq\infty$ and $(1/p)+(1/q)=1$, then $\mathscr{L}_p(g)^*$ is linearly isometric to $\mathscr{L}_q(g)$ via the pairing $\langle f,h\rangle=\operatorname{Lim}_{\Lambda}\sum_{x\in\Lambda}\dfrac{f(x)\,h(x)}{g(x)}$ $(f\in\mathscr{L}_p(g)$, $h\in\mathscr{L}_q(g))$.*

Remarks. Theorem A, which is used to prove Theorem B, also implies a decomposition of BV-functionals into singular and non-singular parts. This specializes to a theorem of Darst [5], c.f. also [16]. Theorem B implies solvability conditions for both moment problems on convex bodies as well a trigonometric moment problems; both with respect to measures which are either of L_p-density or singular. Hausdorff's solution to the "little" moment problem with measures of L_p-density, c.f. [20], is an example of the former. More generally, Theorem B extends to all of the examples of disintegration of functions on semigroups discussed in [14], so that necessary and sufficient conditions for the representing measures either to be of L_p-density or singular can be formulated. The algebra $\mathscr{A}$ to which Theorem B is applied is the linear span of the translation operators on the semigroup. Theorem C can be viewed as an inversion theorem for recovery of μ_f since f'_Λ and $d\mu_g$ are known. For the case where the semigroup operation is idempotent, these considerations recover and generalize the theory of V^p-spaces introduced by Bochner [3]. We refer the reader to [16] where a direct approach, devoid of the above algebraic setting is offered.

2. Applications to Classical Settings

2.1. Moment Problems and Convex Bodies in n-space with L_p-density Functions

For notational convenience set $n=2$ and let K be a closed bounded convex set with non-void interior in $\mathbb{R}^2$. The classical problem of classifying those doubly indexed sequences $F(m_1, m_2)$ which admit disintegrations of the form $F(m_1, m_2) = \int_K t_1^{m_1} t_2^{m_2}\, d\mu_F$, where μ_F satisfies select properties, can be solved by appealing to Theorem B. Let $\mathscr{A}$ be the real algebra of all polynomials in two variables. There exists a subset τ of appropriately scaled polynomials in two variables of degree less than 2 such that $K = \bigcap_{p\in\tau} \{t\colon p(t) \geqq 0\}$, [14, § 3]. Let E_1 and E_2 be the two unit coordinate shift operators defined by $E_1(F(m_1, m_2)) = F(m_1+1, m_2)$ and $E_2(F(m_1, m_2)) = F(m_1, m_2+1)$. The dual $\mathscr{A}'$ admits a natural identification with the doubly indexed sequences; P' being identified with those F satisfying $(p_1^{i_1} \dots p_k^{i_k}(E_1, E_2))\, F(0,0) \geqq 0$, with $p_1, \dots, p_k \in \tau$. Let G be such an F normalized by the condition $G(0,0)=1$. For definiteness assume K is the simplex $\{t\colon t_1 \geqq 0,\ t_2 \geqq 0,\ 1-t_1-t_2 \geqq 0\}$. Then τ can be taken to be $\{p_1(t) = t_1, p_2(t) = t_2,\ p_3(t) = 1-t_1-t_2\}$. For $1<p<\infty$, the p-norm of F takes the form:

$$\|F\|_p = \operatorname*{Lim}_n \left(\sum \begin{pmatrix} & & n \\ m_1 & m_2 & n-m_1-m_2 \end{pmatrix} \frac{|(I-E_1-E_2)\, F(m_1, m_2)|^p}{[(I-E_1-E_2)\, G(m_1, m_2)]^{p-1}} \right)^{1/p}.$$

Part (i) of Theorem B asserts that $\|F\|_p < \infty$ if and only if $d\mu_F = F'\, d\mu_G$ with $F' \in L_p(K, \mu_G)$.

The case where $n=1$, $K=[0,1]$, $1<p\leqq\infty$ and $G=1/(k+1)$ is the well known "little" moment problem discussed by Hausdorff, cf. [21, p. 109]. Here μ_G is Lebesgue measure. Theorem B can be applied to yield new proofs in more generality. It is interesting to note that P' is identified with the completely monotonic sequences, and the classical Lebesgue space $L_p[0,1]$ with the subset $\mathscr{L}_p(G)$ of BV-sequences (BV-in the sense of Hausdorff).

2.2. Lebesgue Spaces as Subspaces of Absolutely Continuous Functions

Let $\mathscr{A}$ be the real algebra spanned by the characteristic functions $1_{[0,b]}$ of closed subintervals of the form $[0,b]$ of $[0,1]$. Then $\mathscr{A}$ contains the characteristic functions of the left-open and right-closed intervals. Let $\tau = \{1_{(t,1]}, 1_{[0,t]}\colon 0 \leqq t \leqq 1\}$. Each $f \in \mathscr{A}'$ can be biuniquely identified with functions F on $[0,1]$ by $F(t) = f(1_{[0,t]})$ and the map $f \to F$ is an isomorphism. The positive linear functionals P' correspond to the non-decreasing functions on $[0,1]$; $[0,1]$ itself is the continuous image of Γ. Moreover the BV-norm carries over to the classical real-variable BV-norm for functions on $[0,1]$. If we choose the linear functional g defined by $g(1_{[0,t]}) = t$, as the control function, then it corresponds to the monotonic function $G(t) = t$ so that $\mathscr{L}_1(g)$ corresponds to the space $\mathscr{L}_1(G)$ of all absolutely continuous functions F on $[0,1]$, such that $F(0)=0$. For $1<p<\infty$, the p-norm of $F\in\mathscr{L}_p(g)$ takes the form

$$\operatorname{Lim}\left(\sum_i \frac{|F(t_i)-F(t_{i-1})|^p}{(t_i-t_{i-1})^{p-1}}\right)^{1/p}$$

where the limit is taken over all refinements of subpartitions of the form $0\leqq t_0<\ldots<t_n\leqq 1$. The space $\mathscr{L}_\infty(G)$ is just those functions satisfying the Lipschitz condition with the Lipschitz norm. Each $F\in\mathscr{L}_p(G)$ is differentiable almost everywhere, and even though the map $F\to F'$ is not ordinary differentiation, it follows that the map $F\to(dF/dt)$ is a linear isometry of $\mathscr{L}_p(G)$ onto $L_p[0,1]$ for all p, [16].

2.3. The Trigonometric Moment Problem: Functions on Discrete Groups Whose Fourier Transforms are L_p-functions

Let S be an additive discrete abelian group and $\mathscr{A}$ the complex convolution algebra of all functions on S with finite support. Set $\tau=\left\{\frac{1}{2}\left(1_{\{0\}}+\frac{\sigma}{2}1_{\{s\}}+\frac{\bar\sigma}{2}1_{\{-s\}}\right): s\in S,\ \sigma^4=1\right\}$. Then the τ-positive linear functionals can be identified with the positive definite functions on S and Γ with the character group $\hat S$ [13]. If we choose the positive definite function $1_{\{0\}}$ as the control function, then Theorem B gives an explicit description, in terms of iterated differences, of those functions whose Fourier transforms are L_p-functions with respect to Haar measure on $\hat S$. From duality, we have therefore described the transforms of those measures on $\hat S$ which have L_p-density. The theme of characterizing measures on locally compact abelian groups by their Fourier Transforms was initiated by Schoenberg [19] and extended by both Eberlein [cf. 18] and Stewart [20].

3. Proofs of Main Theorems

Theorem 3.1 (A). *Every g-continuous linear functional is BV.*

Proof. For each $x\in\mathscr{A}$, let E_x denote the shift operator on $\mathscr{A}'$ defined by $E_x f(y)=f(xy)$, $(y\in\mathscr{A},\ f\in\mathscr{A}')$. Assume f is a g-continuous linear functional which is real-valued on P. If $\Lambda\in\Omega$ then the triangle inequality for $\|\cdot\|$, implies $\sum_{x\in\Lambda}\|E_x f\|\geqq\|f\|$. Suppose $\|f\|=\infty$. Let $M\geqq 2\pm|f(1)|$ and $a^\pm=\operatorname{Max}(\pm a,0)$. There exist Λ_1 such that

$$\sum_{x\in\Lambda_1}((f(x))^+ +(f(x))^-)\geqq M. \qquad (3.1.1)$$

Moreover,

$$\sum_{x\in\Lambda_1}((f(x))^+ -(f(x))^-)=f(1). \qquad (3.1.2)$$

Then x_1 can be found in Λ_1 such that $\|E_{x_1} f\| = \infty$. Without loss of generality, assume $f(x_1) \leqq 0$. Averaging (3.1.1) and (3.1.2) gives $\sum_{y_1} |f(y_1)| \geqq (M + f(1))/2 \geqq 1$, $(y_1 \in \Lambda_1 \setminus \{x_1\})$. This argument can be repeated with f replaced by $E_{x_1} f$ to produce Λ_2 with $x_2 \in \Lambda_2$ satisfying both $\|E_{x_2 x_1} f\| = \infty$ and $\Sigma_{y_2} |f(y_2 x_1)| \geqq 1$, $(y_2 \in \Lambda_2 \setminus \{x_2\})$. Continuing by induction we obtain a sequence $\{\Lambda_k\}_k$ of members of Ω, such that each Λ_k contains a select x_k satisfying

$$\|E_{\Pi_i x_i} f\| = \infty \qquad (i = 1, 2, \ldots, k) \tag{3.1.3}$$

$$\Sigma_{y_k} |f(y_k \Pi_j x_j)| \geqq 1 \qquad (j = 1, \ldots, k-1;\ y_k \in \Lambda_k \setminus \{x_k\}). \tag{3.1.4}$$

If $j = 1, \ldots, k-1$ and $y_k \in \Lambda_k \setminus \{x_k\}$ then $\Sigma_{y_k} g(y_k \Pi_j x_j) = g((\Sigma_{y_k} y_k) \Pi_j x_j) = g((1 - x_k) \Pi_j x_j) = g(\Pi_j x_j) - g(x_k \Pi_j x_j)$. Thus we can "telescope" the series $\sum_{k=1}^{n} \Sigma_{y_k} g(y_k \Pi_j x_j)$ to obtain, $(g(1) - g(x_1)) + (g(x_1) - g(x_1 x_2)) + \ldots + (g(x_1 \ldots x_{n-1}) - g(x_1 \ldots x_n)) \leqq g(1)$. Therefore,

$$\sum_{k=1}^{\infty} (\Sigma_{y_k} g(y_k \Pi_j x_j)) < \infty. \tag{3.1.5}$$

Let $\Lambda_k' = \Lambda_1 \ldots \Lambda_k$ and $\Lambda_k'' = \{y_k \Pi_j x_j\}_j$. Then $\Lambda_k' \supset \Lambda_k''$ and (3.1.5) implies $\lim_k \sum_{x \in \Lambda_k''} g(x) = 0$, so that g-continuity of f contradicts (3.1.4). Hence f is BV whenever f is real on P. If f is not real on P then we can write $f = f_1 + i f_2$, where $f_*(x)$ is the conjugate of $f(x^*)$, $f_1 = (f + f_*)/2$ and $f_2 = (f - f_*)/2i$. It follows that f_1 and f_2 are each g-continuous, BV-functionals which are real on P. Thus f is BV.

The existence of a representing μ_f for each g-continuous functional f follows from [14], and we are now in a position to prove

Theorem 3.2. *A functional f is g-continuous if and only if μ_f is absolutely continuous with respect to μ_g.*

Proof. Suppose μ_f is absolutely continuous with respect to μ_g and let f' denote the Radon-Nikodym derivative of μ_f. Let $\varepsilon > 0$ be given, and $\hat{x}(\rho)$ denote the evaluation function $\rho \to \rho(x)$. Observe that $\sum_{x \in \Lambda} \hat{x}(\rho) = 1$ for each $\Lambda \in \Omega$. Since the continuous functions, $C(\Gamma)$, are dense in $L_1(\Gamma, \mu_g)$, there exists $F \in C(\Gamma)$ such that $\|F - f'\|_1 < \varepsilon/2$. For any semipartition Λ_0 of a partition $\Lambda \in \Omega$, we have

$$\begin{aligned}\sum_{x \in \Lambda_0} |f(x)| &\leqq \sum_{x \in \Lambda_0} \int_\Gamma \hat{x} |f'|\, d\mu_g \leqq \sum_{x \in \Lambda} \int \hat{x} |f' - F|\, d\mu_g + \sum_{x \in \Lambda_0} \int \hat{x} |F|\, d\mu_g \\ &= \|f' - F\|_1 + \|F\|_\infty \sum_{x \in \Lambda_0} g(x), \qquad (\|F\|_\infty = \sup_\Gamma |F(\rho)|).\end{aligned}$$

If $\sum_{x \in \Lambda_0} g(x) < \varepsilon/(2\|F\|)_\infty$ then $\sum_{x \in \Lambda_0} |f(x)| < \varepsilon$; proving that f is g-continuous.

Conversely, assume f is g-continuous. Then, as in the proof of the previous theorem, we may assume that f is real-valued on P. Since μ_f is regular, we need only prove $\mu_f(D) = 0$ whenever D is compact and $\mu_g(D) = 0$. Let $\varepsilon > 0$ be given and δ be as in the definition of g-continuity. By regularity, there exists an open set G such that $D \subset G$, $\mu_g(G) < \delta$ and

$$|\mu_f|(G \setminus D) < \varepsilon. \tag{3.2.1}$$

By Lemma 2.2.1 of [14] (note the misprint in the statement therein), for each natural number n, there exists a semipartition Λ_n of a partition in Ω such that

$$\sum_{x \in \Lambda_n} \rho(x) \begin{cases} \geqq 1 - 1/n & \text{for } \rho \in D \\ < 1/n & \text{for } \rho \in \Gamma \setminus G. \end{cases}$$

Then,

$$\sum_{x \in \Lambda_n} g(x) = \int \sum_x \hat{x} \, d\mu_g \leqq \mu_g(G) + \frac{1}{n} \mu_g(\Gamma \setminus G) \leqq \delta + \frac{1}{n}.$$

Since f is g-continuous, we have

$$\begin{aligned} \varepsilon > \sum_{x \in \Lambda_n} |f(x)| &\geqq \Big| \int_\Gamma \sum_x \hat{x} \, d\mu_f \Big| \\ &\geqq \Big| \int_D \ldots \Big| - \Big| \int_{G \setminus D} \ldots \Big| - \Big| \int_{\Gamma \setminus G} \ldots \Big| \quad \text{for sufficiently large } n. \end{aligned}$$

Applying Lemma 2.2.2 of [14] and (3.2.1) above gives, $2\varepsilon > |(1 - 4/n)|\mu_f(D)| - (4/n)|\mu_f(\Gamma \setminus G)|$. Letting $n \to \infty$, shows that $2\varepsilon > |\mu_f(D)|$ and the assertion follows.

Corollary 3.3. *$\mathscr{L}_1(g)$ is linearly isometric to $L_1(\mu_g)$ via Radon-Nikodym differentiation $(f \to f')$.*

Proof. Theorem 2.2 of [14] implies $\mathrm{BV}(\mathscr{A})$ is linearly isometric to the space $M(\Gamma)$ of Radon measures on Γ. The assertion follows from the above Theorem along with standard measure theory since we have imposed the BV-norm on $\mathscr{L}_1(g)$.

Later, we will replace $\mathscr{L}_1(g)$ by $\mathscr{L}_p(g)$ in the above to establish (i) of Theorem B. The inequalities established in (3.4) through (3.8) are instrumental in setting up this replacement.

Proposition 3.4. *If $\Lambda, \Lambda' \in \Omega$ and $1 < p < \infty$ then $\|f\|_{(p, \Lambda\Lambda')} \geqq \|f\|_{(p, \Lambda)}$.*

Proof. Let $\{\alpha_i\}_{i=1,2,\ldots,k}$, $\{\beta_i\}_{i=1,2,\ldots,k}$ be finite sets of real numbers such that $\Sigma_i \alpha_i = 1$, $\Sigma_i \beta_i = 1$ with $\beta_i > 0$ and $q = p/(p-1)$. Keeping in mind our convention that $0/0 = 0$, Holder's inequality implies:

$$1 \leqq \Sigma_i \left(\frac{|\alpha_i| \cdot 1}{\beta_i} \right) \beta_i \leqq \left(\Sigma_i \frac{|\alpha_i|^p}{\beta_i^p} \beta_i \right)^{1/p} \cdot (\Sigma_i 1^q \cdot \beta_i)^{1/q},$$

hence

$$1 \leqq \Sigma_i |\alpha_i|^p / \beta_i^{p-1}. \tag{3.4.1}$$

Since

$$\sum_{xx' \in \Lambda\Lambda'} |f(xx')|^p / [g(xx')]^{p-1} = \sum_{x \in \Lambda} \sum_{x' \in \Lambda'} |f(xx')|^p / [g(xx')]^{p-1},$$

we need only show

$$\sum_{x'\in\Lambda'} |f(xx')|^p/[g(xx')]^{p-1} \geqq |f(x)|^p/[g(x)]^{p-1}.$$

However, this follows from (3.4.1) upon setting $\beta_{x'}=g(xx')/g(x)$ and $\alpha_{x'}=f(xx')/f(x)$.

Remark 3.5. $\|f\|_p=\operatorname{Lim}_{\Lambda} \|f\|_{(p,\Lambda)}$, where Ω is directed by the natural semigroup ordering of divisibility as explained in § 1.

Lemma 3.6. (i) $\mathscr{L}_p(g)\subset\mathscr{L}_1(g)$ $(1<p<\infty)$
(ii) $\|f\|_p\leqq\|f'\|_p$ *whenever* $f\in\mathscr{L}_1(g)$.

Proof. Let $q=p/(p-1)$ and $\Lambda_0\subset\Lambda\in\Omega$. Then Holder's inequality implies,

$$\begin{aligned}\sum_{x\in\Lambda_0} |f(x)| &\leqq \sum_{x\in\Lambda_0} \left(\frac{|f(x)|}{(g(x))^{1/q}}\cdot(g(x))^{1/q}\right)\\ &\leqq \Big(\sum_{x\in\Lambda_0} |f(x)|^p/[g(x)]^{p-1}\Big)^{1/p}\cdot\Big(\sum_{x\in\Lambda_0} g(x)\Big)^{1/q}\end{aligned}$$

which establishes (i). To prove (ii) we again apply Holder's inequality to obtain,

$$|f(x)|=\Big|\int_\Gamma \hat{x} f'\,d\mu_g\Big|\leqq\Big(\int_\Gamma |f'|^p\,\hat{x}\,d\mu_g\Big)^{1/p}\Big(\int_\Gamma 1^q\,\hat{x}\,d\mu_g\Big)^{1/q}.$$

Therefore

$$|f(x)|^p/[g(x)]^{p-1}\leqq\int_\Gamma |f'|^p\,\hat{x}\,d\mu_g$$

so that

$$\sum_{x\in\Lambda} |f(x)|^p/[g(x)]^{p-1}\leqq\sum_{x\in\Lambda}\int_\Gamma |f'|^p\,\hat{x}\,d\mu_g=\int_\Gamma |f'|^p\sum_{x\in\Lambda}\hat{x}\,d\mu_g=\int_\Gamma |f'|^p\,d\mu_g.$$

Hence, $\|f\|_p\leqq\|f'\|_p$.

In the proof of Theorem 3.1 we defined the shift operator $E_x|\mathscr{A}'\to\mathscr{A}'$ by $(E_x f)(y)=f(xy)$ for each $x\in\mathscr{A}$. For each $\Lambda\in\Omega$, and $f\in\mathscr{A}'$, we now introduce the notation f_Λ to denote $\sum_{x\in\Lambda}\frac{f(x)}{g(x)}E_x g$. Then the map $f\to f_\Lambda$ is linear and $f'_\Lambda=\sum_{x\in\Lambda}\frac{f(x)}{g(x)}\hat{x}$.

Lemma 3.7. *If* $f\in\mathrm{BV}(\mathscr{A})$ *then* $\|f_\Lambda\|_{(p,\Lambda')}\leqq\|f\|_{(p,\Lambda)}$ $(1\leqq p<\infty)$ *for all* $\Lambda, \Lambda'\in\Omega$.

Proof. The proof is again a consequence of Holder's inequality. Indeed,

$$\begin{aligned}\|f_\Lambda\|^p_{(p,\Lambda')} &= \sum_{y\in\Lambda'} |f_\Lambda(y)|^p/[g(y)]^{p-1}\\ &= \sum_{y\in\Lambda'}\left|\sum_{x\in\Lambda}\frac{f(x)}{g(x)}g(xy)\right|^p\Big/[g(y)]^{p-1}\\ &\leqq \sum_y\left(\Big(\sum_x\left|\frac{f(x)}{g(x)}\right|^p g(xy)\Big)\Big(\sum_x g(xy)\Big)^{p/q}\right)\Big/[g(y)]^{p-1}\\ &= \sum_y\sum_x\left|\frac{f(x)}{g(x)}\right|^p g(xy)=\sum_x\left|\frac{f(x)}{g(x)}\right|^p\sum_y g(xy)=\|f\|^p_{(p,\Lambda)}.\end{aligned}$$

Remark 3.8. $\|f_\Lambda\|_p \leqq \|f\|_{(p,\Lambda)} \leqq \|f\|_p$, by the above 3.5. In particular, the linear map $f \to f_\Lambda$ is norm decreasing and hence continuous.

Let $x, y \in \mathscr{A}$ and α be any scalar. Then $E_x E_y = E_{xy}$, $E_x + E_y = E_{x+y}$ and $\alpha E_x = E_{\alpha x}$. Thus if $p(t_1, \ldots, t_k)$ is a scalar valued polynomial in k variables and $x_1, \ldots, x_k \in \tau$, then $p(E_{x_1}, \ldots, E_{x_k}) g$ is a well defined member of $\mathscr{A}'$. Such a linear functional will be called a *polygonal functional* and the set of all such linear functionals (in any number of variables) will be denoted by $\mathscr{P}$.

Proposition 3.9. *If* $f = p(E_{x_1}, \ldots, E_{x_k}) g$ *then*

(i) $f = E_{p(x_1, \ldots, x_k)} g$

(ii) $f' = \hat{p}$ *where* $\hat{p} = p(\hat{x}_1, \ldots, \hat{x}_k)$ $(\hat{x}_1(\rho) = \rho(x_1))$

(iii) $\bar{f}' = \bar{\hat{p}}$.

Moreover, $\mathscr{P}' = \{f' : f \in \mathscr{P}\}$ *is uniformly dense in* $C(\Gamma)$.

Proof. (i) Let $y \in \mathscr{A}$ and $p(t_1, \ldots, t_k) = \sum \alpha_{j_1, \ldots, j_k} t_1^{j_1} \ldots t_k^{j_k}$. Then

$$f(y) = \sum \alpha_{j_1, \ldots, j_k} E_{x_1}^{j_1} \ldots E_{x_k}^{j_k} g(y) = g(y \sum \alpha_{j_1, \ldots, j_k} x_1^{j_1} \ldots x_k^{j_k})$$
$$= g(p(x_1, \ldots, x_k)\, y) = E_{p(x_1, \ldots, x_k)}\, g(y).$$

(ii) $$f'(\rho) = (E_{p(x_1, \ldots, x_k)}\, g)'\,(\rho) = \rho(p(x_1, \ldots, x_k)) = p(\hat{x}_1, \ldots, \hat{x}_k)\,(\rho).$$

(iii) Since $\rho(x) = \bar{\rho}(x)$ for all $x \in \tau$ we have

$$\bar{f}'(\rho) = \sum \bar{\alpha}_{j_1, \ldots, j_k}\, \rho(x_1^{j_1} \ldots x_k^{j_k})$$
$$= \rho(\sum \bar{\alpha}_{j_1, \ldots, j_k} (x_1^{j_1} \ldots x_k^{j_k})) = \rho((p(x_1, \ldots, x_k))^*) = \bar{\hat{p}}(\rho).$$

For the last assertion, suppose $\rho_1, \rho_2 \in \Gamma$ with $\rho_1 \neq \rho_2$. There exists $x \in \mathscr{A}$ with $\rho_1(x) \neq \rho_2(x)$. Since τ generates $\mathscr{A}$, we have, $x = p(x_1, \ldots, x_k)$ for some $x_1, \ldots, x_k \in \tau$ and polynomial p. Thus $\mathscr{P}'$ is a point separating subalgebra of $C(\Gamma)$ by (ii), which is closed under complex conjugation by (iii), so that the Stone-Weierstrass Theorem applies to complete the proof.

The following technical lemma is crucial to the proof of the $\mathscr{L}_p$-isometry (Theorem B(i)) and $\mathscr{L}_p$-inversion (Theorem C) Theorems. The proof appeals to the theory of Bernstein polynomials. The reader will recall that if $p(s, t, \ldots)$ is a function of several variables $s, t, \ldots$ then the Bernstein polynomial $B_{m,n,\ldots}$ is defined by

$$B_{m,n,\ldots}\, p(s, t, \ldots) = \sum_{i,j=1}^{m,n} \binom{m}{i} \binom{n}{j} \ldots p\left(\frac{i}{m}, \frac{j}{n}, \ldots\right) s^i t^j \ldots (1-s)^{m-i} (1-t)^{n-j} \ldots \tag{3.9.1}$$

c.f. [11].

Lemma 3.10. *Let p be a polynomial in several variables; $x, y, \ldots$ belong to τ; f be the polygonal functional, $p(E_x, E_y, \ldots) g$ and $\Lambda_x, \Lambda_y, \ldots$ be partitions in Ω which contain $x, y, \ldots$ respectively. Then,*

(i) $$\operatorname*{Lim}_{m,n,\ldots} \int_\Gamma |f'_{\Lambda \Lambda_x^m \Lambda_y^n \ldots} - B_{m,n\ldots}\, p(\hat{x}, \hat{y}, \ldots)|\, d\mu_g = 0$$

for each $\Lambda \in \Omega$.

and

(ii) $$\|f'\|_q = \|f\|_q.$$

Proof. For notational convenience assume the number of variables is two. Let $\Lambda_x = \{x_1, x_2, \ldots\}$; $\Lambda_y = \{y_1, y_2, \ldots\}$ with $x = x_1$ and $y = y_1$. The Multinominal Theorem and the remark preceding Lemma 3.7 imply

$$f'_{\Lambda \Lambda_x^m \Lambda_y^n} = \sum \binom{m}{i_1 \ldots} \binom{n}{j_1 \ldots} \frac{f(w(z, i, j))}{g(w(z, i, j))} \cdot \hat{w}(z, i, j), \tag{3.10.1}$$

where $\binom{m}{i_1 \ldots}$ and $\binom{n}{j_1 \ldots}$ are multinominal coefficients;

$$w \equiv w(z, i, j) = z x_1^{i_1} x_2^{i_2} \ldots y_1^{j_1} y_2^{j_2} \ldots$$

and the summation is over all choices of $z \in \Lambda$, $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$.

Moreover an application of the multinominal theorem to (3.9.1) gives:

$$B_{m,n} p(\hat{x}, \hat{y}) = \sum \binom{m}{i_1 \ldots} \binom{n}{j_1 \ldots} p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \hat{w}. \tag{3.10.2}$$

But since, $f(w) = \int p(\hat{x}, \hat{y}) \, \hat{w} \, d\mu_g$ we have:

$$\left| (f(w)/g(w)) - p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right| \leq \frac{1}{g(w)} \int_\Gamma \left| p(\hat{x}, \hat{y}) - p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right| \hat{w} \, d\mu_g. \tag{3.10.3}$$

Applying (3.10.3) to the integral of the absolute value of the difference between (3.10.1) and (3.10.2) gives:

$$\begin{aligned}
&\int_\Gamma |f'_{\Lambda \Lambda_x^m \Lambda_y^n}(\rho_1) - B_{m,n} p(\hat{x}, \hat{y})(\rho_1)| \, d\mu_g(\rho_1) \\
&\quad \leq \int_\Gamma \sum \binom{m}{i_1 \ldots} \binom{n}{j_1 \ldots} \left| (f(w))/g(w) - p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right| \hat{w}(\rho_1) \, d\mu_g(\rho_1) \\
&\quad \leq \int_\Gamma \sum \binom{m}{i_1 \ldots} \binom{n}{j_1 \ldots} \left(\frac{\hat{w}(\rho_1)}{g(w)}\right) \int_\Gamma \left| p(\hat{x}, \hat{y})(\rho_2) \right. \\
&\qquad\qquad \left. - p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right| \hat{w}(\rho_2) \, d\mu_g(\rho_2) \, d\mu_g(\rho_1) \\
&\quad = \sum \binom{m}{i_1 \ldots} \binom{n}{j_1 \ldots} \int_\Gamma |\ldots| \, d\mu_g(\rho_2) \int_\Gamma \frac{\hat{w}(\rho_1)}{g(w)} \, d\mu_g(\rho_1) \\
&\quad = \int_\Gamma \sum \binom{m}{i_1 \ldots} \binom{n}{j_1 \ldots} \left| p(\hat{x}, \hat{y}) - p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right| \hat{w} \, d\mu_g \\
&\quad = \int_\Gamma \sum \binom{m}{i} \binom{n}{j} \left| p(\hat{x}, \hat{y}) - p\left(\frac{i}{m}, \frac{j}{n}\right) \right| \hat{x}^i \hat{y}^j (1 - \hat{x})^{m-i} (1 - \hat{y})^{n-j} \, d\mu_g.
\end{aligned} \tag{3.10.4}$$

The last term tends to 0 as $m \to \infty$ and $n \to \infty$, since the integrand converges to zero by the theory of Bernstein polynomials. This completes the proof of (i). To prove (ii), we first take the limit of the second expression in (3.10.4) as $m \to \infty$ and $n \to \infty$ to see

$$\sum \binom{m}{i_1 \dots} \binom{n}{j_1 \dots} \left| \frac{f(w)}{g(w)} - p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right| g(w) \to 0. \tag{3.10.5}$$

Set $f(w)/g(w) = a$ and $p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) = b$. Since $a = (\int_\Gamma \hat{w} f' \, d\mu_g)/(\int_\Gamma \hat{w} \, d\mu_g) \leqq \sup_\Gamma |f'(\rho)|$ and $b \leqq \sup\{p(s,t) \mid s, t \in [0,1]\}$, then there exists a constant $M \geqq 1$ independent of $i_1, \dots, j_1 \dots$ and z such that $a, b \leqq M$. But if q is a positive integer then replacing α and β by a/M and b/M in the identity,

$$||\alpha|^q - |\beta|^q| = ||\alpha| - |\beta|| \, (|\alpha|^{q-1} + |\alpha|^{q-2}|\beta| + \dots + |\beta|^{q-1}),$$

shows:

$$||a|^q - |b|^q| \leqq q M^{q-1} ||a| - |b||.$$

Differentiation shows the function $q \to ||a|^q - |b|^q|$ to be non-decreasing, so that

$$\left| \left| \frac{f(w)}{g(w)} \right|^q - \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right|^q \right| \leqq (q+1) M^q \left| \left| \frac{f(w)}{g(w)} \right| - \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right| \right|. \tag{3.10.6}$$

Using (3.10.6) in conjunction with (3.10.5) and the triangle inequality gives:

$$\sum \binom{m}{i_1 \dots} \binom{n}{j_1 \dots} \left| \left| \frac{f(w)}{g(w)} \right|^q - \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right|^q \right| g(w) \xrightarrow[m,n]{} 0. \tag{3.10.7}$$

In conclusion, we return to the theory of Bernstein polynomials to observe that

$$\|f'\|_q^q = \int_\Gamma |f'|^q \, d\mu_g \leftarrow \int_\Gamma B_{m,n} |f'|^q \, d\mu_g = \sum \int_\Gamma \binom{m}{i_1 \dots} \binom{n}{j_1 \dots} \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right|^q \hat{w} \, d\mu_g$$
$$= \sum \binom{m}{i_1 \dots} \binom{n}{j_1 \dots} \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right|^q g(w)$$

or

$$\|f'\|_q^q \leftarrow \sum \binom{m}{i_1 \dots} \binom{n}{j_1 \dots} \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right|^q g(w). \tag{3.10.8}$$

However,

$$|\|f'\|_q^q - \|f\|_q^q| \leqq \left| \|f'\|_q^q - \sum \binom{m}{i_1 \dots} \binom{n}{j_1 \dots} \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right|^q g(w) \right|$$
$$+ \sum \binom{m}{i_1 \dots} \binom{n}{j_1 \dots} \left| \left| p\left(\frac{i_1}{m}, \frac{j_1}{n}\right) \right|^q - \left| \frac{f(w)}{g(w)} \right|^q \right| g(w) + \left| \|f\|_{(q, A\Lambda_x^m \Lambda_y^n)}^q - \|f\|_q^q \right|.$$

But (3.10.8), (3.10.7) and Remark 3.5 imply each of three terms on the right of this inequality can be made arbitrarily small. Assertion (ii) follows.

The following, which is needed to establish the $\mathscr{L}_p$-isometry, will also imply the $\mathscr{L}_p$-approximation (Theorem C) once the isometry has been proven in Theorem 3.12 below.

Lemma 3.11. *If* $f' \in L_p$, *then* $\|f' - f'_\Lambda\|_p \xrightarrow[\Lambda]{} 0$.

Proof. We first prove the assertion for f polygonal. Thus assume $\varepsilon > 0$ is given and $f = p(E_x, E_y, \ldots)\, g$, where $x, y, \ldots \in \tau$ and $p(s, t, \ldots)$ is a polynomial in several variables. By the theory of Bernstein polynomials and Lemma 3.10(i), there exists a Bernstein polynomial $B_{m,n,\ldots}$ such that both

$$\int_\Gamma |p(\hat{x}, \hat{y}, \ldots) - B_{m,n\ldots}\, p(\hat{x}, \hat{y}, \ldots)|\, d\mu_g < \varepsilon/2$$

$$\text{and} \quad \int_\Gamma |B_{m,n,\ldots} f' - f'_\Lambda|\, d\mu_g < \varepsilon/2 \ \text{(for sufficiently large } \Lambda \in \Omega).$$

Since $f' = p(\hat{x}, \hat{y}, \ldots)$, the triangle inequality implies, $\|f' - f'_\Lambda\|_p < \varepsilon$ for the case, $p = 1$. For p: $1 < p < \infty$,

$$|f'_\Lambda| \leqq \sum_{x \in \Lambda} \frac{|f(x)|}{g(x)}\, \hat{x} \leqq \sum_\Lambda \frac{\int_\Gamma \hat{x}\, |f'|\, d\mu_g}{\int_\Gamma \hat{x}\, d\mu_g} \leqq \sup_\Gamma |f'(\rho)|,$$

so that $\{f'_\Lambda\}_\Lambda$ is uniformly bounded. But, $\|f' - f'_\Lambda\|_1 \xrightarrow[\Lambda]{} 0$ implies $f'_\Lambda \to f'$ in μ_g-measure and the Dominated Convergence Theorem [6, p. 125] implies the assertion for the case where f is polygonal. To complete the proof for general $f' \in L_p$, let $\varepsilon > 0$ be given. Since $C(\Gamma)$ is norm dense in L_p, Proposition 3.9 implies the existence of $p \in \mathscr{P}$ such that $\|f' - p'\|_p < \varepsilon/3$. Applying the first part of the above proof, we obtain $\|p' - p'_\Lambda\|_p < \varepsilon/3$. Remark 3.8 gives, $\|(p' - f')_\Lambda\|_p \leqq \|p' - f'\|_p < \varepsilon/3$ so that the triangle inequality implies the assertion.

The theorem below establishes Theorem B(i) for $1 \leqq p < \infty$.

Theorem 3.12. *If* $f \in \mathscr{L}_p(g)$ *then* $\|f\|_p = \|f'\|_p$.

Proof. From Lemma 3.11 as well as Corollary 3.3 it follows that $\|f - f_\Lambda\|_1 = \|f' - f'_\Lambda\|_1 \to 0$. Thus $f'_\Lambda \to f'$ in μ_g-measure. Consequently, there exists a subsequence f'_{Λ_n} such that $f'_{\Lambda_n} \to f'$ (μ_g-a.e.). Hence,

$$\begin{aligned}
\|f'\|_p^p &= \int_\Gamma |f'|^p\, d\mu_g \\
&\leqq \underline{\mathrm{Lim}}_n \int_\Gamma |f'_{\Lambda_n}|^p\, d\mu_g && \text{(Fatou's Lemma)} \\
&= \underline{\mathrm{Lim}}_n \|f_{\Lambda_n}\|_p^p && \text{(Lemma 3.10(ii))} \\
&\leqq \|f\|_p^p && \text{(Remark 3.8)} \\
&\leqq \|f'\|_p^p, && \text{(Lemma 3.6)}
\end{aligned}$$

and the proof is complete.

The following $\mathscr{L}_p$-approximation formula is a direct consequence of Theorem 3.12 and Lemma 3.11. Moreover, it and the Theorem imply Theorem C.

Corollary 3.13. *If* $f \in \mathscr{L}_p$ *then* $\|f - f_\Lambda\|_p \xrightarrow[\Lambda]{} 0$.

The only unproved assertion of Theorem B(i) concerns the space $\mathscr{L}_\infty$. For each $f \in \mathscr{A}'$ we define $\|f\|_\infty = \operatorname{Sup}_{x \in \tau} \frac{|f(x)|}{g(x)}$ (0/0=0) and set $\mathscr{L}_\infty(g) = \{f \in \mathscr{A}' \mid \|f\|_\infty < \infty\}$. The following proposition completes the proof of Theorem B(i).

Corollary 3.14. $\mathscr{L}_\infty$ *is linearly isometric to* L_∞ *via the differentiation map* $f \to f'$.

Proof. Let $f \in \mathscr{A}'$ and $1 < p < \infty$, then $\sum_{x \in \Lambda} |f(x)|^p/(g(x))^{p-1} \leqq \|f\|_\infty^p \sum_x g(x)$ so that $\|f\|_p \leqq \|f\|_\infty$. Moreover if $x \in \tau$ and $f \in \mathscr{L}_1(g)$ then, $|f(x)| \leqq \int_\Gamma |f'| \hat{x}\, d\mu_g \leqq \|f'\|_\infty \int_\Gamma \hat{x}\, d\mu_g \leqq \|f'\|_\infty g(x)$, so that $\|f\|_\infty \leqq \|f'\|_\infty$. Applying the familiar relation $\|f'\|_p \xrightarrow[p]{} \|f'\|_\infty$ (even if $\|f'\|_\infty = \infty$), we see that for $f \in \mathscr{L}_1(g)$,

$$\|f\|_\infty \geqq \|f\|_p = \|f'\|_p \to \|f'\|_\infty \geqq \|f\|_\infty.$$

Thus the map $f \to f'$ is an isometry, as required.

Standard measure theory now implies that the dual $\mathscr{L}_p^*(g)$ of $\mathscr{L}_p(g)$ is linearly isometric to $\mathscr{L}_q(g)$ for $1 \leqq p < \infty$, $1 < q \leqq \infty$ and $(1/p) + (1/q) = 1$. To give an explicit description of the pairing, recall from §1,

$$\langle f, h \rangle = \operatorname{Lim}_{\Lambda} \sum_{x \in \Lambda} \frac{f(x)\, h(x)}{g(x)} \qquad (0/0=0).$$

Theorem 3.15. *If* $1 \leqq p < \infty$, $1 < q \leqq \infty$ *and* $(1/p) + (1/q) = 1$, *then* $\mathscr{L}_p^*(g)$ *is linearly isometric to* $\mathscr{L}_q(g)$ *via the pairing* $\langle f, h \rangle$ *with* $f \in \mathscr{L}_p(g)$ *and* $h \in \mathscr{L}_q(g)$.

Proof. The assertion follows from Corollary 3.13, since:

$$\int_\Gamma f' h'\, d\mu_g \xleftarrow[\Lambda]{} \int_\Gamma f'_\Lambda h'\, d\mu_g = \int_\Gamma \left(\sum_{x \in \Lambda} \frac{f(x)}{g(x)} \hat{x} \right) h'\, d\mu_g = \sum_{x \in \Lambda} \frac{f(x)\, h(x)}{g(x)} \xrightarrow[\Lambda]{} \langle f, h \rangle.$$

In order to finish the proof of Theorem B we define a BV-functional f to be *g-singular* if given $\varepsilon > 0$ there exists $\Lambda \in \Omega$ and $\Lambda_0 \subset \Lambda$ such that both $\sum_{x \in \Lambda_0} |f(x)| < \varepsilon$ and $\sum_{x \in \Lambda \setminus \Lambda_0} g(x) < \varepsilon$.

Theorem 3.16. *A* BV-*functional* f *is g-singular if and only if its representing measure* μ_f *is* μ_g*-singular.*

Proof. Suppose μ_f is singular with respect to μ_g and $\varepsilon > 0$ is given. Without loss of generality, we will assume $\|\mu_f\| = 1$. There exists a Borel set $A \subset \Gamma$ such that $|\mu_f|(A) = 0$ and $\mu_g(\Gamma \setminus A) = 0$. By regularity, one finds compact subsets K_1 and K_2 with $K_1 \subset A$ and $K_2 \subset \Gamma \setminus A$ such that $|\mu_f|(K_2) \geqq 1 - \varepsilon/2$ and $\mu_g(K_1) \geqq 1 - \varepsilon/2$. Then,

$$|\mu_f|(\Gamma \setminus K_2) < \varepsilon/2 \tag{3.16.1}$$

and

$$\mu_g(\Gamma\setminus K_1) < \varepsilon/2. \tag{3.16.2}$$

By Lemma 2.2.1 of [14], there exists $\Lambda \in \Omega$ with $\Lambda_0 \subset \Lambda$ such that

$$\sum_{x\in\Lambda_0} \hat{x}(\rho) \begin{cases} \geqq 1-\varepsilon/2 & \text{for } \rho\in K_1 \\ <\varepsilon/2 & \text{for } \rho\in K_2. \end{cases} \tag{3.16.3}$$

Thus,

$$\sum_{x\in\Lambda\setminus\Lambda_0} \hat{x}(\rho) \begin{cases} <\varepsilon/2 & \text{for } \rho\in K_1 \\ \geqq 1-\varepsilon/2 & \text{for } \rho\in K_2. \end{cases} \tag{3.16.4}$$

Therefore,

$$\begin{aligned}\sum_{x\in\Lambda_0} |f(x)| &\leqq \sum_{x\in\Lambda_0} \int_\Gamma \hat{x}\, d|\mu_f| \\ &= \int_{\Gamma\setminus K_2} \sum_{\Lambda_0} \hat{x}\, d|\mu_f| + \int_{K_2} \sum_{\Lambda_0} \hat{x}\, d|\mu_f| \leqq |\mu_f|(\Gamma\setminus K_2) + (\varepsilon/2)\,|\mu_f|(K_2) < \varepsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{x\in\Lambda\setminus\Lambda_0} g(x) &= \int_\Gamma \sum_{\Lambda\setminus\Lambda_0} \hat{x}\, d\mu_g \\ &= \int_{\Gamma\setminus K_1} \sum_{\Lambda\setminus\Lambda_0} \hat{x}\, d\mu_g + \int_{K_1} \sum_{\Lambda\setminus\Lambda_0} \hat{x}\, d\mu_g = \mu_g(\Gamma\setminus K_1) + (\varepsilon/2)\,\mu_g(K_1) < \varepsilon\end{aligned}$$

thus f is g-singular.

To prove the converse, we first show that the variation, $|f|$, of f is g-singular. Recall that $|\nu|(E) \leqq 4|\nu(E)|$, for all bounded measures ν and all Borel sets E. Since the variation of $\hat{x}\, d\mu_f$ is $\hat{x}\, d|\mu_f|$ for each $x\in\tau$, we have $|f|(x) = \int_\Gamma \hat{x}\, d|\mu_f| \leqq 4|\int_\Gamma \hat{x}\, d\mu_f| = 4|f(x)|$. But given $\varepsilon>0$, there exists $\Lambda\in\Omega$ and $\Lambda_0 \subset \Lambda$ such that $\sum_{x\in\Lambda_0} |f(x)| < \varepsilon/4$ and $\sum_{x\in\Lambda\setminus\Lambda_0} g(x) < \varepsilon/4$ from which it follows that $|f|$ is also singular. But since $|\mu_f|$-singularity implies μ_f-singularity, we need only show singularity of $|\mu_f|$. By the Lebesgue decomposition theorem, there exist τ-positive linear functionals f_1 and f_2 such that $|\mu_f| = \mu_{f_1} + \mu_{f_2}$, with μ_{f_1} singular and μ_{f_2} continuous. Then f_2 is g-continuous by Theorem 3.2. If $f_2(1)$ were zero then $0 \leqq f_2(x) = \int_\Gamma \hat{x}\, d\mu_{f_2} \leqq f_2(1)$ would imply $f_2 = 0$ or $|\mu_f| = \mu_{f_1}$, so that $|\mu_f|$ would be singular. Thus we will assume $f_2(1) > 0$. Choose $\delta(<\varepsilon)$ corresponding to $\varepsilon = f_2(1)/2$ in the definition of g-continuity of f_2. By singularity of $|f|$, there exists $\Lambda\in\Omega$ with $\Lambda_0 \in \Lambda$ such that both

$$\sum_{x\in\Lambda_0} f_2(x) \leqq \sum_{x\in\Lambda_0} |f|(x) < \delta < f_2(1)/2 \tag{3.16.5}$$

and

$$\sum_{x\in\Lambda\setminus\Lambda_0} g(x) < \delta. \tag{3.16.6}$$

But (3.16.6) and the choice of δ, imply

$$\sum_{x\in\Lambda\setminus\Lambda_0} f_2(x) < f_2(1)/2. \tag{3.16.7}$$

Using (3.16.5) and (3.16.7), we get:

$$f_2(1)=\sum_{x\in\Lambda_0} f_2(x)+\sum_{x\in\Lambda\setminus\Lambda_0} f_2(x)<f_2(1).$$

This contradiction shows that $f_2=0$ and there by completes the proof.

4. Applications to the Abstract Moment Problem

The three examples included in § 2, can be subsumed under what we call the *abstract moment problem.* That is, given a locally compact subset Γ' of semicharacters on a commutative semigroup S with identity and given a subcollection $M(\Gamma')$ of regular Borel measures on Γ', what conditions on a function $F\colon S\to\mathbb{C}$ will insure the existence of a $\mu\in M(\Gamma')$ such that $F(s)=\int_{\Gamma'}\rho(s)\,d\mu(\rho)$? To accomplish this, let $\mathscr{A}$ be the linear span of the shift operators, $\{E_s\colon s\in S\}$; where E_s acts on the complex-valued functions F on S by $E_s F(t)=F(s+t)$. When S admits an involution $*$, the involution on $\mathscr{A}$ is defined by $(E_s)^*=E_{s^*}$. The dual $\mathscr{A}'$ is identified with the functions F on S by $F(E_s)=F(s)$. The set of multiplicative linear functionals is homeomorphic to the semicharacters when the topology of simple convergence is imposed on the latter. Various choices of τ are available. For example, if S is taken to be the additive semigroup of pairs of non-negative integers then the algebra of polynomials in two variables is isomorphic to the algebraic span of the shift operators and the simplex example of § 2 is recovered upon setting $\tau=\{E_{(1,0)}, E_{(0,1)}, I-E_{(1,0)}-E_{(0,1)}\}$. When S is the additive semigroup of non-negative integers and $\tau=\{E_1, I-E_1\}$ then the abstract moment problem reduces to the "little" moment problem discussed in § 2.1. More generally if S is an arbitrary commutative semigroup with identity, X is a subset which spans S and $\tau=\{E_x, I-E_x\colon x\in X\}$; then it is proved in [14] that the positive linear functionals on $\mathscr{A}$ can be identified with the completely monotonic functions discussed in [2, 4, 7, 10 and 12] and the BV-functionals with the BV-functions on S which were induced in [12]. For $1<p<\infty$ and completely monotonic control function G, the p-norm takes the form

$$\|F\|_p=\operatorname*{Lim}_{(n,X)}\sum\binom{n}{i_1}\cdots\binom{n}{i_k}\frac{|\Pi_j E_{x_j}^{i_j}(I-E_{x_j})^{n-i_j}F(0)|^p}{[\Pi_j E_{x_j}^{i_j}(I-E_{x_j})^{n-i_j}G(0)]^{p-1}}.$$

The special case where the semigroup operation of S is idempotent (i.e. S is a semilattice) has been considered in [1, 8, 16 and 17]. For the special case where S is the semilattice $[0,1]$ with operation $s\wedge t=\min[s,t]$, the theory reduces to the classical theory of BV-functions of a real variable mentioned above in § 2.2. Note that the algebra used in § 2.2 is isomorphic to the algebra of shift operators on $([0,1],\wedge)$.

The trigonometric moment problem referred to in § 2.3 can be generalized to arbitrary semigroups with involution. For this purpose we set

$$\tau=\left\{\tfrac{1}{2}I+\frac{\sigma}{4}E_s+\frac{\bar\sigma}{4}E_{s^*}\colon s\in S,\ \sigma^4=1\right\}.$$

The functions on S which correspond to the BV-functionals on $\mathscr{A}$ are discussed in [13 and 14]. The p-norms can easily be computed as in the above examples. The positive linear functionals on $\mathscr{A}$ can be realized as those $F: S \to \mathbb{C}$ such that all finite products $\prod_{T \in \tau} [T(F)](0)$ are non-negative [14]. It is shown in [13] that the latter agrees with those functions which are bounded and positive definite in the sense of [10], i.e. all quadratic forms $\sum_{i,j} c_i \bar{c}_j F(s_i + s_j^*)$ are nonnegative.

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