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The Extreme Points of the Set of Decreasing Failure Rate Distributions

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Summary. Since the class of extended decreasing failure rate (EDFR) life distributions (i.e., distributions with support in $[0, \infty]$) is compact and convex, it follows from Choquet's Theorem that every EDFR life distribution can be represented as a mixture of extreme points of the EDFR class. We identify the extreme points of this class and of the standard class of decresing failure rate (DFR) life distributions. Further, we show that even though the convex class of DFR life distributions is not compact, every DFR life distribution can be represented as a mixture of extreme points of the DFR class.

1. Introduction and Summary

The class of decreasing failure rate (DFR) distributions plays an important role in the theory and application of reliability, biometry, and other fields of statistics (see, e.g., Barlow and Proschan, 1975, Chaps. 3 and 4, and Proschan, 1963).

Such distributions govern the lifelengths of systems which do not age adversely over time in the sense that the conditional survival probability given the age of the system is an increasing function of the age. For example, DFR distributions govern the lifelengths (i) of metals subject to "work-hardening", (ii) of many solid state components, (iii) of businesses, (iv) of mixtures of exponential distributions, etc.

The DFR class is convex, as is the class of extended decreasing failure rate distributions (EDFR) which contains distributions placing mass at ∞. This latter class is also compact in the topology of weak convergence of probability measures (see Sect. 2).

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From the Krein-Milman Theorem and Choquet's Theorem, we know that the basic building blocks of convex compact sets are their extreme points. In particular, it follows from Choquet's Theorem, stated in Sect. 2, that the set of extreme points is the smallest set of EDFR distributions with the property that every EDFR distribution may be represented as a mixture of its elements.

The main purpose of this paper is to identify the extreme points of the EDFR class (Sect. 3). They are those EDFR distributions having failure rate functions whose derivatives are close to zero in an appropriate sense. More specifically, an EDFR distribution F is *not* an extreme point of the EDFR class if and only if (i) $0<F(0)<1$, (ii) $0<F(\infty)<1$, or (iii) the derivative of the failure rate function is a.e. uniformly bounded away from 0 in some interval (see Theorem 3.1).

Since the DFR class is an extremal subset of the EDFR class, it follows that a distribution F is an extreme point of the DFR class if and only if F is an extreme point of the EDFR class and it places no mass at ∞. Thus the extreme points of the DFR class are also identified (see Corollary 4.2).

In Theorem 4.5 we show that even though the convex class of DFR distributions is not compact, the set of extreme points of this class is the smallest set of DFR distributions with the property that every DFR distribution may be represented as a mixture of its elements.

2. Preliminaries

Let F be an *extended life distribution*; i.e., F is a distribution function possibly placing mass at ∞ such that $F(0-)=0$. When an extended life distribution places no mass at ∞, it is simply called a *life distribution*. The function $\bar{F}\equiv 1-F$ is called the *survival probability*, and $R_f\equiv -\ln\bar{F}$ is called the *hazard function*. If R_F is absolutely continuous on every closed interval contained in $(0,\infty)$, then any measurable function r_F almost everywhere (a.e.) equal to R'_F, the derivative of R_F, is called a *failure rate function*. (Throughout, measurable means Borel measurable and all measures are Borel measures.) When F has a density f, the failure rate function $r_F=f/\bar{F}$ a.e. When the distribution F is clearly understood, the subscript f will be suppressed.

Definition 2.1. Let F be an extended life distribution. Then F is said to be an *extended decreasing failure rate* (EDFR) distribution if R is concave on $(0,\infty)$. If an EDFR distribution is a life distribution, it is simply called a *decreasing failure rate* (DFR) distribution.

The set of EDFR distributions will be denoted by $\mathscr{D}$. An EDFR distribution F can have no jump on $(0,\infty)$ since R is concave on $(0,\infty)$, but can have a jump at 0. Also, notice that we consider δ_0, the distribution degenerate at 0, to be an EDFR distribution. The hazard function of an EDFR distribution other than δ_0 is finite and concave on $(0,\infty)$. Consequently it is absolutely continuous on every closed interval of $(0,\infty)$ and has a right derivative which exists everywhere on $(0,\infty)$. In the remainder of this paper the failure rate function of an EDFR distribution other than δ_0 is always taken to be this right derivative and is, therefore, a decreasing function on $(0,\infty)$ with derivative a.e. less than or equal to zero. For convenience we shall define r to be identically zero when $F=\delta_0$.

Let $\mathscr{L}$ denote the class of extended life distributions, and $\sigma(\mathscr{L})$ denote the smallest σ-field of subsets of $\mathscr{L}$ such that the map $F \to F(t)$ from $\mathscr{L}$ into $[0,1]$ is $\sigma(\mathscr{L})$-measurable for all $t \in [0,\infty]$.

Definition 2.2. An extended life distribution F is a *mixture* of elements in a set $S \in \mathscr{L}$ if $S \in \sigma(\mathscr{L})$ and there is a probability measure μ_F defined on $\sigma(\mathscr{L})$ such that

$$\mu_F(S)=1 \tag{2.1}$$

and

$$F(t)=\int G(t)\,\mu_F(dG) \qquad \text{for all } t \in [0,\infty].$$

When (3.1) holds, μ is said to have *support* S.

Remark 2.3. It can be shown that $\sigma(\mathscr{L})$ is the Borel σ-field of $\mathscr{L}$, when $\mathscr{L}$ is given the topology of weak convergence of probability measures. In particular, μ_F is a Borel measure.

We recall that an element x of a convex set K is an *extreme point* of K if $x=py+(1-p)z$ with $y, z, \in K$ and $p \in (0,1)$ implies that $y=z=x$.

We need Choquet's Theorem, stated below, to show that every EDFR life distribution can be represented as a mixture of extreme points of the EDFR class.

Choquet's Theorem (Phelps, 1966, pp. 19–20). *Let K be a metrizable, compact, convex subset of a locally convex space X. Let $x_0 \in K$. Then there is a probability measure μ_{x_0} supported by the extreme points of K such that*

$$L(x_0)=\int L(x)\,\mu_{x_0}(dx)$$

for all continuous linear functionals L defined on X.

In this paper we take X to be $M[0,\infty]$, the space of finite signed measures on $[0,\infty]$, with the topology of weak convergence, and K to be $\mathscr{D}$.

It is well known that with the above topology, $M[0,\infty]$ is locally convex and $\mathscr{D}$ is metrizable, a convenient metric being the Lévy metric. An argument using Helly's Compactness Theorem and the fact that pointwise limits of concave functions are concave shows that $\mathscr{D}$ is compact. The convexity of $\mathscr{D}$ follows since the family of positive log-convex functions is closed under addition and multiplication (see Roberts and Varberg, 1973, Sect. 13).

To see that every EDFR can be represented as a mixture of extreme points of $\mathscr{D}$, we notice that $L\colon M[0,\infty] \to R$ defined by $L(v) \equiv \int f(x)\,v(dx)$ is a continuous linear functional on $M[0,\infty]$ for each $f \in C[0,\infty]$. Hence the conclusion of Choquet's Theorem implies that for every EDFR distribution F there exists a probability measure, μ_F, supported by the extreme points of $\mathscr{D}$ such that $\int f(x)\,F(dx)=\int [\int f(x)\,G(dx)]\,\mu_F(dG)$ for all $f \in C[0,\infty]$. Since $C[0,\infty]$ is a separating class of functions (see Breiman, 1968, p. 165) we have that $F(t)=\int G(t)\,\mu_F(dG)$ for all $t \in [0,\infty]$. Equivalently, by Remark 2.3, every EDFR distribution is a mixture of extreme points of $\mathscr{D}$. Further, by the definition of extreme point, we see that the set of extreme points of $\mathscr{D}$ is the smallest set of EDFR distributions with the property that every EDFR may be represented as a mixture of its elements.

We identify the extreme points of $\mathscr{D}$ in the next section.

3. The Extreme Points of the EDFR Class

In this section we identify $\mathscr{E}$, the set of extreme points of $\mathscr{D}$. In essence, we show that an EDFR distribution F is not an extreme point of $\mathscr{D}$ if and only if either (i) $0<F(0)<1$, (ii) $0<F(\infty)<1$, or (iii) r', the derivative of the failure rate function, is a.e. uniformly bounded away from 0 in some interval. More precisely, we show the following theorem.

Theorem 3.1. *The EDFR distribution F is an extreme point of the set of EDFR distributions if and only if either* (i) *F is degenerate at 0, or at ∞, or* (ii) $F(0)=0$, $F(\infty-)=1$, *and $\{t\colon r'(t)$ exists, $a\leqq t\leqq b$, and $r'(t)>-\delta\}$ has nonzero Lebesgue measure for all $0<a<b$ and $\delta>0$.*

Let δ_∞ denote the distribution degenerate at ∞.

Remark 3.2. The degree of smoothness of r for $F\in\mathscr{E}$ characterizes certain types of extreme points. For example, let $F\in\mathscr{E}\setminus\{\delta_0,\delta_\infty\}$. Then (i) r' is continuous if and only if F is an exponential distribution, and (ii) r' has a countable number of isolated discontinuities if and only if F is piecewise exponential, that is, R is piecewise linear. We remark that the EDFR piecewise exponentials are dense in $\mathscr{D}$. This example therefore shows that $\mathscr{E}$ is a dense subset of $\mathscr{D}$.

Two other types of interesting life distributions which are extreme points of $\mathscr{D}$ are given below. These life distributions are extreme points of $\mathscr{D}$ since their failure rates are decreasing functions with derivatives almost everywhere equal to 0. Notice that the failure rate function in (iii) below is discontinuous in a countable dense subset of $[0,\infty)$ while the failure rate function in (iv) below is continuous everywhere on $[0,\infty)$.

(iii) Let $\{a_j\}_{j=1}^\infty$ be a countable dense subset of $[0,\infty)$ and $\{b_j\}_{j=1}^\infty$ be a sequence of positive numbers such that $\Sigma b_j<\infty$ and $\Sigma a_j b_j=\infty$. For $t\geqq 0$, let $r(t)\equiv\Sigma b_j I_{\{a_j\geqq t\}}$. Then $F(t)\equiv 1-e^{-\int_0^t r(u)\,du}$ is an extreme point of $\mathscr{D}$ since $r'=0$ a.e. on $[0,\infty)$, $F(0)=0$, and $F(\infty-)=1$.

(iv) Let $r(t)$ be positive, decreasing, continuous, and singular (with respect to Lebesgue measure). Further assume that $\int_0^\infty r(u)\,du=\infty$. Then $r'=0$ a.e. on $[0,\infty)$, $F(0)=0$, and $F(\infty-)=1$. Consequently, $F(t)=1-e^{-\int_0^t r(u)\,du}$ is an extreme point of $\mathscr{D}$.

In proving Theorem 3.1, we use the following notation. For each $F\in\mathscr{D}$ let $E_F\equiv\{t\colon r_F'$ exists at $t\}$. Denote the Lebesgue measure by m. Let $\mathscr{O}\equiv\{F\in\mathscr{D}\setminus\{\delta_0,\delta_\infty\}\colon m(t\in E_F\colon a\leqq t\leqq b$ and $r'(t)>-\delta)>0$ for all $0<a<b$ and all $\delta>0\}$. Let $\mathscr{C}=\{F\in\mathscr{L}\colon F(0)=0$ and $F(\infty-)=1\}$, i.e., $\mathscr{C}$ is the class of life distributions which place no mass at 0.

In this notation, Theorem 3.1 is equivalent to

$$\mathscr{E}=(\mathscr{O}\cap\mathscr{C})\cup\{\delta_0,\delta_\infty\}. \tag{3.1}$$

We prove (3.1) by showing (i) $\mathscr{E}\subset\mathscr{O}\cup\{\delta_0,\delta_\infty\}$ (Lemma 3.3), (ii) $\mathscr{E}\subset\mathscr{C}\cup\{\delta_0,\delta_\infty\}$ (Lemma 3.4), and (iii) $\mathscr{E}\supset(\mathscr{O}\cap\mathscr{C})\cup\{\delta_0,\delta_\infty\}$ (Lemma 3.5).

Lemma 3.3. $\mathscr{E}\subset\mathscr{O}\cup\{\delta_0,\delta_\infty\}$

Proof. It is enough to show that if $\mathscr{E}\in\mathscr{D}$ and $F\notin\mathscr{O}\cup\{\delta_0,\delta_\infty\}$, then $F\notin\mathscr{E}$. To prove $F\notin\mathscr{E}$, it suffices to show that there exist F_1 and $F_2\in\mathscr{D}$ such that $(F_1+F_2)/2=F$.

Let $F\in\mathscr{D}$ and $F\notin\mathscr{O}\cup\{\delta_0,\delta_\infty\}$. Then there exists an interval $[a,b]$, $(0<a<b)$, and a value $\delta>0$ such that $r'(t)<-\delta$ a.e. on $[a,b]$. Without loss of generality, we may assume that r is continuous at a and b. For each positive integer n, let $l_n(x)$ be $=\frac{1}{n}(x-a)^3(b-x)^3$ for $a\leqq x\leqq b$ and 0 otherwise.

Choose n so that $\sup l_n<\ln 2$. Let ${}_nR_1\equiv R-l_n$ and let ${}_nR_2\equiv R+c(l_n)$, where $c(x)=-\ln(2-e^x)$ for $x<\ln 2$. We show that for n sufficiently large, say $n=n_0$, $F_i\equiv 1-\exp\{{}_{n_0}R_i\}$ $(l=1,2)$ are two different well-defined EDFR distributions such that $(F_1+F_2)/2=F$.

Since R is concave on $(0,\infty)$, it follows that r and r' exist a.e. on $[0,\infty)$. Further a.e. on $[0,\infty)$, we have that

$$ {}_nR_1'=r-l_n', $$

$$ {}_nR_2'=r+c'(l_n)\cdot l_n', $$

$$ {}_nR_1''=r'-l_n'', $$

and

$$ {}_nR_2''=r'+c'(l_n)\cdot l_n''+c''(l_n)^2. $$

Since $r'(t)<\delta$ a.e. on $[a,b]$, from the continuity of c' and c'', it follows that for n sufficiently large, say $n=n_0$, ${}_{n_0}R_i(t)<-\delta/2$ a.e. on $[a,b]\,(i=1,2)$. Thus, writing R_i for ${}_{n_0}R_i(i=1,2)$, we have that R_i is strictly concave on $[a,b]$. We notice that R and R_i and also their derivatives agree off (a,b). Thus R_i is an increasing function on $[0,\infty)$, is concave on $(0,\infty)$, and satisfies $R_i(\infty-)\equiv\infty$. It follows that $F_i\equiv 1-e^{-R_i}(i=1,2)$ are two different well-defined EDFR distributions.

We show that $(F_1+F_2)/2=F$. By the definitions of c, R_1, and R_2, we see that

$$ R_2=R-\ln(2-e^{(R-R_1)}). $$

Hence

$$ e^{-R_2}=e^{-R}(2-e^{(R-R_1)})=2e^{-R}-e^{-R_1}. $$

Thus

$$ \bar{F}=e^{-R}=(e^{-R_1}+e^{-R_2})/2=(\bar{F}_1+\bar{F}_2)/2. \quad \square $$

Lemma 3.4. $\mathscr{E}\subset\mathscr{C}\cup\{\delta_0,\delta_\infty\}$.

Proof. Let $F\in\mathscr{C}\cup\{\delta_0,\delta_\infty\}$. Then either (i) $F(0)=\alpha$, where $0<\alpha<1$, or (ii) $F(\infty)=\beta$, where $0<\beta<1$. Assume (i) holds. Then $F=\alpha\,\delta_0+(1-\alpha)\left(\frac{F-F(0)}{1-F(0)}\right)$. Since $F_1\equiv\delta_0$ and $F_2\equiv\frac{F-F(0)}{1-F(0)}$ are both DFR distributions, $F\notin\mathscr{E}$. A similar argument shows $F\notin\mathscr{E}$ when case (ii) holds. $\square$

Lemma 3.5. $(\mathcal{O}\cap\mathcal{C})\cup\{\delta_0,\delta_\infty\}\subset\mathcal{E}$.

Proof. Let $F\in(\mathcal{O}\cap\mathcal{C})\cup\{\delta_0,\delta_\infty\}$. We show that $F\in\mathcal{E}$. If $F=\delta_0$ or δ_∞, then clearly $F\in\mathcal{E}$. Assume then that $F\in\mathcal{O}\cap\mathcal{C}$. Since $\mathcal{D}$ is convex, it suffices to prove that

$$F=(F_1+F_2)/2 \quad \text{with } F_1, F_2\in\mathcal{D} \tag{3.2}$$

implies that

$$F_1=F_2=F. \tag{3.3}$$

Thus assume (3.2) holds. We show (3.3) follows.

For the distribution $F_i(i=1,2)$, let r_i be the failure rate function and R_i be the hazard function. Let $t\in M\equiv\{t\colon r_1'$ and r_2' exist at $t\}$. Since $m(M^c)=0$ and $F\in\mathcal{O}$, it follows from the definition of $\mathcal{O}$ that for each positive integer n,

$$m([t,t+n^{-1}]\cap M\cap\{t\colon r'>-n^{-1}\})>0.$$

Hence we may choose a sequence $\{t_n\}\subset M$ such that $t_n\to t$ and $r'(t_n)\to 0$ as $n\to\infty$. By a subsequence argument we may also assume that the sequences $\{r_1'(t_n)\}$ and $\{r_2'(t_n)\}$ both have limits (possibly infinite). Now reversing the steps in the last paragraph of the proof of Lemma 3.3 we can show that $(R_2-R)(t_n)=c((R-R_1)(t_n))$, where $c(x)\equiv-\ln(2-e^x)$ for $x<\ln 2$. Differentiating twice and taking limits, we get

$$\begin{aligned}\lim r_2'(t_n)= & -c'((R-R_1)(t))\cdot(\lim r_1'(t_n)\\ & +c''((R-R_1)(t))\cdot(r(t)-r_1(t))^2,\end{aligned} \tag{3.4}$$

since $r'(t_n)\to 0$ and $t_n\to t$ as $n\to\infty$ and R, R_1, r, and r_1 are continuous at t.

Now for $x<\ln 2$, $c'(x)=e^x/(2-e^x)>0$ and $c''(x)=2e^x/(2-e^x)^2>0$. Hence, using the fact that $\lim r_1'(t_n)\leqq 0$, we see that the expression on the right of (3.4) is nonnegative. Since the term on the left of (3.4) is nonpositive, it follows that the expression on the right of (3.4) is equal to 0. We conclude that $r(t)=r_1(t)$ for $t\in M$. Reversing the roles of R_1 and R_2 above, we get $r(t)=r_2(t)$ for $t\in M$. Thus

$$r=r_1=r_2 \qquad \text{a.e. on } [0,\infty). \tag{3.5}$$

Now $F(0)=0$ implies that $F_1(0)=F_2(0)=F(0)$ and, consequently, that $R(0)=R_1(0)=R_2(0)$. It follows by (3.5) that $R=R_1=R_2$ or, equivalently, that $F=F_1=F_2$. $\square$

4. Convex Extremal Subsets of $\mathcal{D}$

In this section we identify the extreme points of two convex subclasses of $\mathcal{D}$, namely, (i) $\mathcal{D}_P$, the class of (proper) DFR distributions (Corollary 4.2), and (ii) $\mathcal{D}_C\equiv\{F\in\mathcal{D}\colon r_F$ is continuous on $(0,\infty)\}$ (Corollary 4.3). In addition in Theorem 4.5, we show that every DFR life distribution can be represented as a mixture of the extreme points of $\mathcal{D}_P$.

To identify the extreme points of $\mathscr{D}_P$ and $\mathscr{D}_C$ we need Lemma 3.1 below concerning extremal subsets. Recall that a subset K' of a convex set K is an *extremal subset* of K if $x=py+(1-p)\,z$ with y, $z\in K$ and $p\in(0,1)$ implies that y and z are in K'.

Let ext K denote the set of extreme points of K.

Lemma 4.1. *Let K' be a convex extremal subset of a convex set K. Then*

$$\operatorname{ext} K' = K' \cap \operatorname{ext} K.$$

We omit the elementary proof of this lemma.

From Lemma 4.1 it follows that to identify the extreme points of $\mathscr{D}_P$ and $\mathscr{D}_C$ is is only necessary to show that these classes are convex extremal subsets of $\mathscr{D}$.

Since it is immediate that $\mathscr{D}_P$ is a convex extremal subset of $\mathscr{D}$, we have the following corollary of Lemma 4.1.

Corollary 4.2. *The DFR distribution F is an extreme point of the DFR class if and only if either* (i) *F is degenerate at* 0, *or* (ii) *$F(0)=0$ and $\{t\colon r'(t)$ exists, $a\leqq t\leqq b$, and $r'(t)>-\delta\}$ has nonzero Lebesgue measure for all $0<a<b$ and $\delta>0$.*

To show that $\mathscr{D}_C$ is an extremal subset of $\mathscr{D}$ we need the following lemma. Let r, r_1, and r_2 be the failure rates of F, F_1, and F_2.

Lemma 4.3. *Let $F=pF_1+(1-p)\,F_2$ for some $p\in(0,1)$ and F_1, $F_2\in\mathscr{D}$. Let r be continuous at t. Then r_1 and r_2 are continuous.*

Proof. Assume that either r_1 or r_2 is not continuous at t. We show that this contradicts the continuity of r at t.

Since r_1 and r_2 are decreasing and F_1 and F_2 are continuous on $(0,\infty)$, then

$$\begin{aligned} r(t+) &= [\bar F(t)]^{-1}\,[p r_1(t+)\,\bar F_1(t)+(1-p)\,r_2(t+)\,\bar F_2(t)] \\ &< [\bar F(t)]^{-1}\,[p r_1(t-)\,\bar F_1(t)+(1-p)\,r_2(t-)\,\bar F_2(t)] = r(t-), \end{aligned}$$

which contradicts the continuity of r at t. ▯

It is clear from Lemma 4.3 that $\mathscr{D}_C$ is an extremal subset of $\mathscr{D}$. Also a straightforward argument similar to that in the proof of Lemma 4.3 shows that $\mathscr{D}_C$ is convex. Hence we obtain the following corollary of Lemma 4.1.

Corollary 4.4. *The set of extreme points of $\mathscr{D}_S$ is $\mathscr{E}\cap\mathscr{D}_S$.*

Remark 4.5. Let $\mathscr{D}_A\equiv\{F\in\mathscr{D}\colon r_F$ is absolutely continuous on every closed interval contained in $(0,\infty)\}$. It can be shown that $\mathscr{D}_A$ is also a convex extremal subset of $\mathscr{D}$, and consequently that $\operatorname{ext}\mathscr{D}_A=\mathscr{E}\cap\mathscr{D}_A$.

We conclude this section with the following representation theorem for the DFR class.

Theorem 4.5. *The life distribution F is DFR if and only if F may be represented as a mixture of distributions in* $\operatorname{ext}\mathscr{D}_P$. *Further* $\operatorname{ext}\mathscr{D}_P$ *is the smallest set with the property that every DFR may be represented as a mixture of its elements.*

Proof. Let $F \in \mathscr{D}_P$. Then $F \in \mathscr{D}$ and, therefore, by the result stated in the next to the last paragraph of Sect. 2, we have that $F(t) = \int G(t)\, \mu_F(dG)$ for all $t \in [0, \infty]$, where μ_F is a probability measure such that $\mu_F(\mathscr{E}) = 1$. Since $F(\infty -) = 1$ by the Bounded Convergence Theorem, $1 = \int G(\infty -)\, \mu_F(dG)$. Hence $\mu_F(\mathscr{E} \cap \mathscr{D}_P) = \mu_F\{G \in \mathscr{E} \colon G(\infty -) = 1\} = 1$. Since $\operatorname{ext} \mathscr{D}_P = \mathscr{E} \cap \mathscr{D}_P$ by Corollary 4.2, the conclusion of the first part of Theorem 4.5 follows. The second part of Theorem 4.5 is an immediate consequence of the definition of an extreme point. □

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