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A Note on PM Spaces Determined by Measure Preserving Transformations

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In [1] Schweizer and Sklar introduced a very interesting class of probabilistic metric spaces. The most important results in that paper are found in the next paragraph.

Let (S, d) be a separable metric space. Let S be the carrier of a probability measure P defined on a σ-field that contains the Borel sets of S. Let M be a mapping from S into S which is measure preserving with respect to P. Let P^2 denote the product measure on $S \times S$ induced by P. For each real number x, let $D(x) = \{(p, q)\colon d(p, q) < x\}$. For any subset A of $S \times S$ let χ_A denote the indicator function of A. Then there exists a subset W_0 of $S \times S$, with $P^2(W_0) = 1$, such that for every pair (p, q) in W_0 there exists a left-continuous distribution function F_{pq} such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} \chi_{D(x)}(M^m p, M^m q) = F_{pq}(x),$$

at every continuity point x of F_{pq}. Furthermore, if $\mathscr{F}$ is the function defined by $\mathscr{F}(p, q) = F_{pq}$ for each (p, q) in W_0, then $(W_0, \mathscr{F})$ is a probabilistic pseudo-metric space, more specifically, a Menger Space under the t-norm T_m. Finally, if $(p, q) \in W_0$ then $\mathscr{F}(Mp, Mq) = \mathscr{F}(p, q)$.

The domain W_0 of $\mathscr{F}$ need not be a Cartesian product. Thus $\mathscr{F}$ is not a probabilistic pseudo-metric in the usual sense. Notice that (p, r) could fail to belong to W_0 even though both (p, q) and (q, r) do belong. This deficiency could significantly reduce the usefulness of the triangle inequality. The following theorem removes this difficulty.

Theorem. *The mapping $\mathscr{F}$ has an extention $\mathscr{F}^*$ whose domain is all of $S \times S$ such that $(S, \mathscr{F}^*)$ is a probabilistic pseudo-metric space satisfying Menger's triangle inequality under the t-norm T_m. Moreover, for every p, q in S, $\mathscr{F}^*(Mp, Mq) = \mathscr{F}^*(p, q)$.*

Proof. To construct the mapping $\mathscr{F}^*$ we first let

$$G_{pq}(x) = \liminf_{n \to \infty} F_{pq}^{(n)}(x)$$

where

$$F_{pq}^{(n)}(x)=\frac{1}{n}\sum_{m=0}^{n-1}\chi_{D(x)}(M^m p, M^m q).$$

We now define $\mathscr{F}^*$ on $S\times S$ via $\mathscr{F}^*(p,q)=F_{pq}^*$ where F_{pq}^* is the left-continuous function which agrees with G_{pq} at each continuity point of G_{pq}.

It is obvious that $\mathscr{F}^*$ extends $\mathscr{F}$ to all of $S\times S$. In showing that $(S,\mathscr{F}^*)$ is a Menger space, it is only the triangle inequality which is not immediate. To verify the triangle inequality we use the fact that T_m is nondecreasing in each place and the inequality

$$F_{pr}^{(n)}(x+y)\geqq T_m(F_{pq}^{(n)}(x), F_{qr}^{(n)}(y))$$

which is proved in [1]. It follows that for each $j\geqq n$,

$$\begin{aligned}F_{pr}^{(j)}(x+y)&\geqq T_m(F_{pq}^{(j)}(x), F_{qr}^{(j)}(y))\\&\geqq T_m(\inf\{F_{pq}^{(k)}(x)\colon k\geqq n\}, \inf\{F_{qr}^{(k)}(y)\colon k\geqq n\}).\end{aligned}$$

Thus

$$\inf\{F_{pr}^{(k)}(x+y)\colon k\geqq n\}\geqq T_m(\inf\{F_{pq}^{(k)}(x)\colon k\geqq n\}, \inf\{F_{qr}^{(k)}(y)\colon k\geqq n\}).$$

Taking the limit and utilizing the continuity of T_m we obtain

$$G_{pr}(x+y)\geqq T_m(G_{pq}(x), G_{qr}(y)).$$

Since G_{pr} is a nondecreasing function there exists a nondecreasing sequence x_n converging to x with the property that x_n+y is a continuity point of G_{pr} for each n. Thus

$$F_{pr}^*(x_n+y)=G_{pr}(x_n+y)\geqq T_m(G_{pq}(x_n), G_{qr}(y))\geqq T_m(F_{pq}^*(x_n), F_{qr}^*(y)).$$

Using the left-continuity of F_{pr}^* and F_{pq}^* and the continuity of T_m and taking the limit as $n\to\infty$ we obtain

$$F_{pr}^*(x+y)\geqq T_m(F_{pq}^*(x), F_{qr}^*(y)).$$

Finally it will be shown that for each p,q in S $\mathscr{F}^*(Mp,Mq)=\mathscr{F}^*(p,q)$. To this end, let p,q be in S and let x be real. Then,

$$\begin{aligned}G_{MpMq}(x)&=\liminf_{n\to\infty} F_{MpMq}^{(n)}=\liminf_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}\chi_{D(x)}(M^m Mp, M^m Mq)\\&=\liminf\frac{1}{n}\left[\sum_{k=0}^{n-1}\chi_{D(x)}(M^k p, M^k q)+\chi_{D(x)}(M^n p, M^n q)-\chi_{D(x)}(p,q)\right].\\&=G_{pq}(x).\end{aligned}$$

This implies that $\mathscr{F}^*(Mp,Mq)=\mathscr{F}^*(p,q)$.

Reference

1. Schweizer, B., Sklar, A.: Probabilistic Metric Spaces Determined by Measure Preserving Transformations. Z. Wahrscheinlichkeitstheorie verw. Gebiete **26**, 235–239 (1973)

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