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Limit Theorems for Sums of Order Statistics*

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Let $\{X_n, n \geqq 1\}$ be a sequence of independent and identically distributed positive random variables and $S_n = \sum_{j=m}^{n} X_j^{(m)}$ and $\tilde{S}_n = \sum_{j=m}^{n} \tilde{X}_j^{(m)}$ where $X_j^{(m)}$ is the m-th order statistic of $(X_1, \ldots, X_j)$ and $\tilde{X}_j^{(m)}$ is the $(j-m+1)$-th order statistic of $(X_1, \ldots, X_j)$ for some fixed integer $m \geqq 1$. Asymptotic behaviour of S_n and $\tilde{S}_n$ are studied. If $m=1$, then $S_n = \sum_{j=1}^{n} \min(X_1, \ldots, X_j)$ and $\tilde{S}_n = \sum_{j=1}^{n} \max(X_1, \ldots, X_j)$. Results obtained here generalize those of Deheuvels, Grenander, Höglund, and Ghosh *et al.* for sums of minima and maxima of positive independent random variables.

1. Introduction

Let $\{X_n, n \geqq 1\}$ be a sequence of independent and identically distributed positive random variables. Let $\eta_k = \operatorname{Inf}(X_1, \ldots, X_k)$, $k \geqq 1$ and $S_n = \sum_{k=1}^{n} \eta_k$. Grenander (1965) proved that, if $F(t)$ is the c.d.f. of X_1, then

$$\frac{S_n}{\log n} \xrightarrow{p} F \tag{1.1}$$

where $F = \lim_{t \to 0} t/F(t)$. Höglund (1972) proved that S_n is asymptotically normal under some conditions. Ghosh *et al.* (1975) studied the almost sure behaviour of S_n. They proved that

$$\frac{S_n}{\log n} \to F \quad \text{a.s.} \tag{1.2}$$

where F is a defined above. Recently Deheuvels (1974) studied this problem in detail and he proved almost sure convergence and asymptotic normality of S_n after suitable normalization.

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Our aim in this paper is to generalize the above results for sums of order statistics. Let $X_j^{(m)}$ be the m-th order statistic of $(X_1, \ldots, X_j)$ and $S_n = \sum_{j=m}^{n} X_j^{(m)}$. We shall study the asymptotic behaviour of S_n. In case $m=1$, our results reduce to the ones obtained by Deheuvels (1974), Höglund (1972), Ghosh *et al.* (1975) and Grenander (1965) for sums of minima of positive random variables. The problem stated above was first considered by Feder (1967). He proved that

$$\frac{S_n}{m \log n} \xrightarrow{q \cdot m} F \tag{1.3}$$

where $F = \lim_{t \to 0} t/F(t)$. We shall show that S_n is asymptotically normal and for some function H related to $F(t)$,

$$\frac{S_n}{m H(\log n)} \to 1 \quad \text{a.s.}$$

under some conditions. The method used by us is that of Deheuvels (1974) and is of independent interest. Since proofs here are similar to those of Deheuvels (1974), details are given at those places where they are necessary in this general context.

Section 2 contains results for Uniform Distribution. Results in the general case are discussed in Section 3. Some examples are given in Section 4. Section 5 contains some results for sums of the form $\tilde{S}_n = \sum_{j=m}^{n} \tilde{X}_j^{(m)}$ where $\tilde{X}_j^{(m)}$ is the $(j-m+1)$-th order statistic of $(X_1, \ldots, X_j)$. When $m=1$, $\tilde{X}_j^{(m)} = \max(X_i, 1 \leqq i \leqq j)$.

2. Uniform Distribution

We shall first study the problem for uniform distribution. Let $\{X_n, n \geqq 1\}$ be a sequence of independent random variables uniformly distributed on $[0, 1]$ and $\{\varepsilon_n, n \geqq 1\}$ be a sequence of real numbers decreasing to zero. Define

$$\tau_\varepsilon = \operatorname{Inf}\{n \geqq m \mid X_n^{(m)} \leqq \varepsilon\} \tag{2.1}$$

and

$$S_n = \sum_{j=m}^{n} X_j^{(m)} \tag{2.2}$$

where $X_j^{(m)}$ is the m-th order statistic of $(X_1, \ldots, X_j)$. Assume WLOG that τ_ε is well defined for all $\varepsilon > 0$. It is easy to see that $\{\tau_{\varepsilon_n}, n \geqq 1\}$ is an increasing sequence of positive integer valued random variables. In fact

$$P[\tau_\varepsilon = r] = \binom{r-1}{m-1} \varepsilon^m (1-\varepsilon)^{r-m}, \qquad r \geqq m \tag{2.3}$$

and it can be shown, after some tedious calculations, that

$$\begin{aligned} P[\tau_{\varepsilon_2}-\tau_{\varepsilon_1}=r_1 \mid \tau_{\varepsilon_1}=r_0] &= \frac{\varepsilon_2^m}{\varepsilon_1^m} \quad \text{if } r_1=0 \\ &= \frac{\varepsilon_2^m}{\varepsilon_1^m}(\varepsilon_1-\varepsilon_2)^m(1-\varepsilon_2)^{r_1-m}\left\{\sum_{j=0}^{m-1}\binom{m}{j}\binom{r_1-1}{m-j-1}\left(\frac{1-\varepsilon_2}{\varepsilon_1-\varepsilon_2}\right)^j\right\} \\ &\qquad \text{if } r_1 \geqq 1. \end{aligned} \tag{2.4}$$

Since the expression for conditional probability is independent of the conditioning event, it follows that $\tau_{\varepsilon_2}-\tau_{\varepsilon_1}$ is independent of τ_{ε_1}. By similar arguments, it follows that

$$\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}}, \ldots, \tau_{\varepsilon_2}-\tau_{\varepsilon_1}, \tau_{\varepsilon_1} \tag{2.5}$$

are independent for every $n \geqq 1$ (define $\tau_0 \equiv 0$, $\varepsilon_0 \equiv 0$) and

$$\begin{aligned} P[\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}}=r] &= \frac{\varepsilon_n^m}{\varepsilon_{n-1}^m} \quad \text{if } r=0 \\ &= \frac{\varepsilon_n^m}{\varepsilon_{n-1}^m}(\varepsilon_{n-1}-\varepsilon_n)^m(1-\varepsilon_n)^{r-m}\left\{\sum_{j=0}^{m-1}\binom{m}{j}\binom{r-1}{m-j-1}\left(\frac{1-\varepsilon_n}{\varepsilon_{n-1}-\varepsilon_n}\right)^j\right\} \\ &\qquad \text{if } r \geqq 1 \end{aligned} \tag{2.6}$$

for every $n \geqq 2$. Let Z be a nonnegative integer valued random variable with the distribution

$$\begin{aligned} P(Z=r) &= \frac{\varepsilon_n}{\varepsilon_{n-1}} \quad \text{if } r=0 \\ &= (\varepsilon_{n-1}-\varepsilon_n)\frac{\varepsilon_n}{\varepsilon_{n-1}}(1-\varepsilon_n)^{r-1} \quad \text{if } r \geqq 1. \end{aligned} \tag{2.7}$$

The probability generating function $G_Z(z)$ of Z is given by

$$\begin{aligned} G_Z(z) &= \sum_{r=0}^{\infty} P(Z=r) z^r \\ &= \frac{\varepsilon_n}{\varepsilon_{n-1}}\left\{1+\frac{(\varepsilon_{n-1}-\varepsilon_n)z}{1-z(1-\varepsilon_n)}\right\}. \end{aligned} \tag{2.8}$$

It can be checked that $G_Z^m(z)$ is the probability generating function of $\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}}$ by expanding $G_Z^m(z)$ as a power series in z and comparing the coefficients of the power series with probabilities given by (2.6). Hence $\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}}$ has the same distribution as the sum of m independent random variables Z_i each distributed as Z with probability distribution given by (2.7). By simple calculations, it can be seen that

$$E(Z)=\frac{1}{\varepsilon_n}-\frac{1}{\varepsilon_{n-1}} \tag{2.9}$$

$$\operatorname{Var}(Z)=\left(\frac{1}{\varepsilon_n}-\frac{1}{\varepsilon_{n-1}}\right)\left(\frac{1}{\varepsilon_n}+\frac{1}{\varepsilon_{n-1}}-1\right) \tag{2.10}$$

and for any $p \geqq 1$

$$E(Z^p) \sim \left(\frac{\varepsilon_{n-1}-\varepsilon_n}{\varepsilon_{n-1}}\right)^p \cdot \frac{p!}{\varepsilon_n^p}. \tag{2.11}$$

Hence

$$E(\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}})=m\left(\frac{1}{\varepsilon_n}-\frac{1}{\varepsilon_{n-1}}\right), \tag{2.12}$$

$$\operatorname{Var}(\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}})=m\left(\frac{1}{\varepsilon_n}-\frac{1}{\varepsilon_{n-1}}\right)\left(\frac{1}{\varepsilon_n}+\frac{1}{\varepsilon_{n-1}}-1\right) \tag{2.13}$$

and for any $p \geqq 1$

$$\begin{aligned} E(\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}})^p &= E(Z_1+\cdots+Z_m)^p \\ &= \sum_{\substack{k_i \geqq 0 \\ \Sigma k_i = p}} \frac{p!}{k_1! \ldots k_m!} E(Z_1^{k_1}) \ldots E(Z_m^{k_m}) \\ &\sim \sum_{\substack{k_i \geqq 0 \\ \Sigma k_i = p}} \frac{p!}{k_1! \ldots k_m!} \left(\frac{\varepsilon_{n-1}-\varepsilon_n}{\varepsilon_{n-1}}\right)^{\Sigma k_i} \frac{k_1! \ldots k_m!}{\varepsilon_n^p} \\ &\sim m^p \left(\frac{\varepsilon_{n-1}-\varepsilon_n}{\varepsilon_{n-1}}\right)^p \cdot \frac{p!}{\varepsilon_n^p}. \end{aligned} \tag{2.14}$$

In particular, it follows that for any $p \geqq 1$

$$\begin{aligned} E|(\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}})-E\{(\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}})\}|^p &\leqq 2E(\tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}})^p \\ &\sim \frac{\varepsilon_{n-1}-\varepsilon_n}{\varepsilon_{n-1}} \cdot \frac{p!}{\varepsilon_n^p}. \end{aligned} \tag{2.15}$$

Lemma 2.1. *For every* $n \geqq 1$,

(i) $S_{\tau_{\varepsilon_{n+1}}}-S_{\tau_{\varepsilon_n}} \leqq (\tau_{\varepsilon_{n+1}}-\tau_{\varepsilon_n})\varepsilon_n$,

(ii) $S_{\tau_{\varepsilon_{n+1}}}-S_{\tau_{\varepsilon_n}}-1 \geqq (\tau_{\varepsilon_{n+1}}-\tau_{\varepsilon_n})\varepsilon_{n+1}$,

(iii) $-2+\sum_{j=1}^{n-1}(\tau_{\varepsilon_{j+1}}-\tau_{\varepsilon_j})\varepsilon_{j+1} \leqq S_{\tau_{\varepsilon_n}}-S_{\tau_{\varepsilon_1}} \leqq \sum_{j=1}^{n-1}(\tau_{\varepsilon_{j+1}}-\tau_{\varepsilon_j})\varepsilon_j$ *and*

(iv) *for all* $j \in [\tau_{\varepsilon_n}, \tau_{\varepsilon_{n+1}}]$,

$$-2+\sum_{j=1}^{n-1}(\tau_{\varepsilon_{j+1}}-\tau_{\varepsilon_j})\varepsilon_{j+1} \leqq S_j-S_{\tau_{\varepsilon_1}} \leqq \sum_{j=1}^{n}(\tau_{\varepsilon_{j+1}}-\tau_{\varepsilon_j})\varepsilon_j.$$

Proof. Follows from observing that the random variables X_i are non-negative,

$$X_{\tau_{\varepsilon_n}}^{(m)} \leqq \varepsilon_n \Rightarrow X_{\tau_{\varepsilon_n}+j}^{(m)} \leqq \varepsilon_n \quad \text{for any } j \geqq 0$$

and

$$X_j^{(m)} > \varepsilon_{n+1} \quad \text{for any } j \leqq \tau_{\varepsilon_{n+1}}-1.$$

Here after, we shall suppose that $\varepsilon_n = n^{-\alpha}$ for some $\alpha > 0$. Let

$$U_n = \sum_{j=1}^{n}(\tau_{\varepsilon_{j+1}}-\tau_{\varepsilon_j})\varepsilon_j \tag{2.16}$$

and

$$U_n' = \sum_{j=1}^{n} (\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, \varepsilon_{j+1}. \tag{2.17}$$

In view of (2.12), (2.13) and (2.14), one can prove the following lemma.

Lemma 2.2.

(i) $E(U_n) = m\alpha \log n + 0(1)$,
(ii) $E(U_n') = m\alpha \log n + 0(1)$,
(iii) $\operatorname{Var}(U_n) = 2m\alpha \log n + 0(1)$,
(iv) $\operatorname{Var}(U_n') = 2m\alpha \log n + 0(1)$, and
(v) for every $p \geqq 1$,

$$\sum_{j=1}^{n} E\{(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})^p \varepsilon_j^p\} \sim \sum_{j=1}^{n} E\{(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})^p \varepsilon_{j+1}^p\} \sim p! \log n.$$

Lemma 2.3. *$U_n - U_n'$ converges almost surely to a finite limit having moments of order 2.*

Proof. Observe that

$$U_n - U_n' = \sum_{j=1}^{n} (\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})(\varepsilon_j - \varepsilon_{j+1}). \tag{2.18}$$

It is sufficient to prove that the series

$$\sum_{j=1}^{\infty} (\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})(\varepsilon_j - \varepsilon_{j+1})$$

converges almost surely to a random variable with finite second moment. Since $\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j}$, $j \geqq 1$ are independent random variables with finite second moment, it is sufficient to show that

$$\sum_{j=1}^{\infty} E\{(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})(\varepsilon_j - \varepsilon_{j+1})\} < \infty \tag{2.19}$$

and

$$\sum_{j=1}^{\infty} \operatorname{Var}\{(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})(\varepsilon_j - \varepsilon_{j+1})\} < \infty. \tag{2.20}$$

Since $\varepsilon_j = j^{-\alpha}$, $\alpha > 0$, (2.19) and (2.20) hold in view of (2.12) and (2.13).

Theorem 2.1.

$$\frac{S_{\tau_{\varepsilon_n}}}{m \log n}$$

converges to α almost surely and in quadratic mean.

Proof. Lemma 2.1 shows that

$$-2 + U_{n-1}' \leqq S_{\tau_{\varepsilon_n}} - S_{\tau_{\varepsilon_1}} \leqq U_{n-1}. \tag{2.21}$$

(2.21) and Lemma 2.3 imply that theorem is true if

$$\frac{U_n}{m \log n}$$

converges to α almost surely and in quadratic mean. Since

$$E\left|\frac{U_n}{m \log n} - \alpha\right|^2 \leqq \frac{2}{m^2(\log n)^2}\left[\operatorname{Var}(U_n) + \{E(U_n) - m \log n\}^2\right]$$
$$\leqq \frac{2}{m^2(\log n)^2}\left[2m\alpha \log n + 0(1)\right]$$

by Lemma 2.2 and the last term tends to zero, it follows that

$$\frac{U_n}{m \log n} \xrightarrow{q \cdot m} \alpha. \tag{2.22}$$

Since

$$\frac{E(U_n)}{m \log n} \to \alpha \tag{2.23}$$

it is sufficient to prove that

$$\frac{U_n}{E(U_n)} \to 1 \quad \text{a.s.} \tag{2.24}$$

Let $\zeta_j = (\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, \varepsilon_j$. It is known from earlier remarks that $\zeta_j, j \geqq 1$ are independent with $E(\zeta_j) > 0$ and $\operatorname{Var}(\zeta_j) < \infty$. Further more

$$\sum_{j=1}^{\infty} E(\zeta_j) = m \sum_{j=1}^{\infty} [(j+1)^\alpha - j^\alpha]\, j^{-\alpha} \sim \sum_{j=1}^{\infty} \frac{1}{j} = +\infty$$

and

$$\sum_{j=1}^{\infty} \frac{\operatorname{Var}(\zeta_j)}{\left[\sum_{k=1}^{j} E(\zeta_k)\right]^2} \sim \sum_{j=1}^{\infty} \frac{1}{j(\log j)^2} < +\infty.$$

Hence by Rényi (1970), p. 435 (Exercise 17),

$$\frac{\sum_{j=1}^{n} \zeta_j}{\sum_{j=1}^{n} E(\zeta_j)} \to 1 \quad \text{a.s.} \tag{2.25}$$

But $U_n = \sum_{j=1}^{n} \zeta_j$. Hence (2.24) holds.

Theorem 2.2.

$$\frac{S_{\tau_{\varepsilon_n}} - m\alpha \log n}{(2m\alpha \log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0, 1) \tag{2.26}$$

where $N(0, 1)$ is the standard normal distribution.

Proof. Lemma 2.1 implies that

$$\begin{aligned} \varliminf_{n\to\infty} P\left(\frac{U_n' - m\alpha \log n}{(2m\alpha \log n)^{\frac{1}{2}}} \leqq u\right) &\leqq \varliminf_{n\to\infty} P\left(\frac{S_{\tau_{\varepsilon_n}} - m\alpha \log n}{(2m\alpha \log n)^{\frac{1}{2}}} \leqq u\right) \\ &\leqq \varlimsup_{n\to\infty} P\left(\frac{U_n - m\alpha \log n}{(2m\alpha \log n)^{\frac{1}{2}}} \leqq u\right) \end{aligned} \tag{2.27}$$

for every real u. But

$$\frac{U_n' - m\alpha \log n}{(2m\alpha \log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0, 1) \tag{2.28}$$

since $\{U_n'\}$ satisfies Liapunov's condition viz

$$\frac{\left[\sum_{j=1}^{n} E|(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, \varepsilon_{j+1} - E\{(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, \varepsilon_{j+1}\}|^3\right]^{\frac{1}{3}}}{\left[\sum_{j=1}^{n} \operatorname{Var}\{(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, \varepsilon_{j+1}\}\right]^{\frac{1}{2}}} \to 0$$

as $n \to \infty$ as the numerator is of the order $(\log n)^{\frac{1}{3}}$ and denominator is of the order $(\log n)^{\frac{1}{2}}$ by Lemma 2.2. Similarly one can show that

$$\frac{U_n - m\alpha \log n}{(2m\alpha \log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0, 1). \tag{2.29}$$

(2.28) and (2.29) prove (2.26) in the presence of (2.27).

Theorem 2.3.

$$\frac{S_n}{m \log n} \to 1 \quad \text{a.s.} \tag{2.30}$$

Proof. We shall first prove that

$$\alpha - 1 \leqq \varliminf \frac{\log \tau_{\varepsilon_n}}{\log n} \leqq \varlimsup \frac{\log \tau_{\varepsilon_n}}{\log n} \leqq \alpha. \tag{2.31}$$

Let $t > \alpha$. Then

$$\begin{aligned} P\left[\frac{\log \tau_{\varepsilon_n}}{\log n} > t\right] &= P[\tau_{\varepsilon_n} > n^t] \\ &= \sum_{j=0}^{m-1} \binom{n^t}{j} \left(\frac{1}{n^\alpha}\right)^j \left(1 - \frac{1}{n^\alpha}\right)^{n^t - j}. \end{aligned} \tag{2.32}$$

Since $\left(1-\frac{1}{n^{\alpha}}\right)^{n^{\alpha}} \to e^{-1}$, there exists an integer n_0 such that for $n \geqq n_0$,

$$\left(1-\frac{1}{n^{\alpha}}\right)^{n^{\alpha}} \leqq a < 1.$$

Hence, for $n \geqq n_0$,

$$P\left[\frac{\log \tau_{\varepsilon_n}}{\log n} > t\right] \leqq a^{n^{t-\alpha}} \sum_{j=0}^{m-1} \binom{n^t}{j} \frac{1}{n^{j\alpha}} \left(1-\frac{1}{n^{\alpha}}\right)^{-j}. \tag{2.33}$$

It is easy to see that

$$\sum_{j=0}^{m-1} \binom{n^t}{j} \frac{1}{n^{j\alpha}} \left(1-\frac{1}{n^{\alpha}}\right)^{-j} = 0\,(n^{(m-1)(t-\alpha)}). \tag{2.34}$$

Therefore there exists a constant C such that for $n \geqq n_1 > n_0$,

$$\sum_{n=n_1}^{\infty} P\left[\frac{\log \tau_{\varepsilon_n}}{\log n} > t\right] \leqq C \sum_{n=n_1}^{\infty} a^{n^{t-\alpha}}\, n^{(m-1)(t-\alpha)}. \tag{2.35}$$

Since $0 < a < 1$ and $t > \alpha$, the series on the right hand side converges. Borel-Cantelli lemma now proves that

$$\overline{\lim}\, \frac{\log \tau_{\varepsilon_n}}{\log n} \leqq \alpha \quad \text{a.s.} \tag{2.36}$$

On the other hand, for any $t < \alpha - 1$,

$$\begin{aligned} P\left[\frac{\log \tau_{\varepsilon_n}}{\log n} \leqq t\right] &= P[\tau_{\varepsilon_n} \leqq n^t] \\ &= 1 - \sum_{j=0}^{m-1} \binom{n^t}{j} \frac{1}{n^{j\alpha}} \left(1-\frac{1}{n^{\alpha}}\right)^{n^t - j} \\ &\leqq 1 - \left(1-\frac{1}{n^{\alpha}}\right)^{n^t} \end{aligned} \tag{2.37}$$

and the last term is of the order $n^{t-\alpha}$. But $\Sigma n^{t-\alpha} < \infty$ since $t - \alpha < -1$. Hence

$$\underline{\lim}\, \frac{\log \tau_{\varepsilon_n}}{\log n} \geqq \alpha - 1 \tag{2.38}$$

by Borel-Cantelli Lemma. (2.36) and (2.38) prove (2.31) which implies that

$$\frac{\alpha-1}{\alpha} \leqq \underline{\lim}\, \frac{\log \tau_{\varepsilon_n}}{\log \tau_{\varepsilon_{n+1}}} \leqq \overline{\lim}\, \frac{\log \tau_{\varepsilon_n}}{\log \tau_{\varepsilon_{n+1}}} \leqq \frac{\alpha}{\alpha-1} \tag{2.39}$$

for any $\alpha > 1$. Since

$$\frac{S_{\tau_{\varepsilon_n}}}{m \log n} \to \alpha \quad \text{a.s.}$$

by Theorem 2.1, it follows that

$$1 \leqq \underline{\lim}\, \frac{S_{\tau_{\varepsilon_n}}}{m \log \tau_{\varepsilon_n}} \leqq \overline{\lim}\, \frac{S_{\tau_{\varepsilon_{n+1}}}}{m \log \tau_{\varepsilon_{n+1}}} \leqq \frac{\alpha}{\alpha-1} \tag{2.40}$$

for every $\alpha > 1$. But, for any j such that

$$\tau_{\varepsilon_n} \leqq j \leqq \tau_{\varepsilon_{n+1}}$$

one has

$$\frac{S_{\tau_{\varepsilon_n}}}{m \log \tau_{\varepsilon_{n+1}}} \leqq \frac{S_j}{m \log j} \leqq \frac{S_{\tau_{\varepsilon_{n+1}}}}{m \log \tau_{\varepsilon_n}}.$$

Hence

$$\frac{\alpha-1}{\alpha} \leqq \underline{\lim} \frac{S_n}{m \log n} \leqq \overline{\lim} \frac{S_n}{m \log n} \leqq \left(\frac{\alpha}{\alpha-1}\right)^2$$

for every $\alpha > 1$. Taking limit as $\alpha \to \infty$, we obtain (2.30).

Remark. It can be seen from the previous proof that

$$\frac{\log \tau_{\varepsilon_n}}{\log n} \to \alpha \quad \text{a.s.} \tag{2.41}$$

and

$$\frac{S_n}{E(S_n)} \to 1 \quad \text{a.s.} \tag{2.42}$$

since $E(S_n) = m \log n + 0(1)$.

Let $\log_p n$ denote $\log(\log_{p-1} n)$ for any $p \geqq 2$ where $\log_2 n = \log \log n$.

Lemma 2.4. *For any* $p \geqq 2$,

$$\overline{\lim_n} \frac{(\tau_{\varepsilon_{n+1}} - \tau_{\varepsilon_n}) - m n^\alpha (\log_2 n + \cdots + \log_{p-1} n)}{m n^\alpha \log_p n} = 1 \quad \text{a.s.} \tag{2.43}$$

Proof. From our earlier remarks, it is known that

$$\tau_{\varepsilon_{n+1}} - \tau_{\varepsilon_n} = Z_{n1} + \cdots + Z_{nm}$$

where Z_{nj}, $1 \leqq j \leqq m$ are independent and identically distributed with

$$\begin{aligned} P(Z_{n1} = r) &= \frac{\varepsilon_{n+1}}{\varepsilon_n} \quad \text{if } r = 0 \\ &= (\varepsilon_n - \varepsilon_{n+1}) \frac{\varepsilon_{n+1}}{\varepsilon_n} (1 - \varepsilon_{n+1})^{r-1} \quad \text{if } r \geqq 1. \end{aligned}$$

Hence

$$\begin{aligned} &\overline{\lim_n} \frac{(\tau_{\varepsilon_{n+1}} - \tau_{\varepsilon_n}) - m n^\alpha (\log_2 n + \cdots + \log_{p-1} n)}{m n^\alpha \log_p n} \\ &= \overline{\lim_n} \frac{\sum_{j=1}^m \{Z_{nj} - n^\alpha (\log_2 n + \cdots + \log_{p-1} n)\}}{m n^\alpha \log_p n} \\ &\leqq \frac{1}{m} \sum_{j=1}^m \overline{\lim_n} \frac{Z_{nj} - n\alpha (\log_2 n + \cdots + \log_{p-1} n)}{n^\alpha \log_p n} \\ &\leqq \frac{1}{m} \cdot m = 1 \quad \text{a.s.} \end{aligned} \tag{2.44}$$

by Proposition 11 of Deheuvels (1974). On the other hand, by the same result of Deheuvels (1974),

$$\frac{Z_{nj}-n^{\alpha}(\log_2 n+\cdots+\log_{p-1} n)}{n^{\alpha}\log_p n}>1-\varepsilon \quad \text{a.s.}$$

for infinitely many n and for every $1\leqq j\leqq m$. Hence

$$\frac{\sum_{j=1}^{m}\{Z_{nj}-n^{\alpha}(\log_2 n+\log_3 n+\cdots+\log_{p-1} n)\}}{mn^{\alpha}\log_p n}>\frac{m(1-\varepsilon)}{m}=1-\varepsilon \quad \text{a.s.}$$

for infinitely many n which implies that

$$\overline{\lim_{n}}\, \frac{(\tau_{\varepsilon_{n+1}}-\tau_{\varepsilon_n})-mn^{\alpha}(\log_2 n+\cdots+\log_{p-1} n)}{mn^{\alpha}\log_p n}\geqq 1-\varepsilon \quad \text{a.s.}$$

for every $\varepsilon>0$. This fact, together with (2.44), prove (2.43).

Lemma 2.5. *For all* $p>3$,

$$\overline{\lim_{u\to 0}}\, \frac{\tau_u-\frac{m}{u}\left\{\log\left(\frac{1}{u}\right)+\cdots+\log_{p-1}\left(\frac{1}{u}\right)\right\}}{\frac{m}{u}\log_p\left(\frac{1}{u}\right)}\leqq 1 \quad \text{a.s.} \tag{2.45}$$

Proof. Since $\tau_u=Z_{1u}+\cdots+Z_{mu}$ where Z_{iu}, $1\leqq i\leqq m$ are independent and identically distributed with

$$P(Z_{1u}=r)=u(1-u)^{r-1}, \quad r\geqq 1$$

$$\begin{aligned}
&\overline{\lim_{u\to 0}}\, \frac{\tau_u-\frac{m}{u}\left\{\log_2\left(\frac{1}{u}\right)+\cdots+\log_{p-1}\left(\frac{1}{u}\right)\right\}}{\frac{m}{u}\log_p\left(\frac{1}{u}\right)} \\
&\quad\leqq \overline{\lim_{u\to 0}}\, \frac{\sum_{j=1}^{m}\left[Z_{ju}-\frac{1}{u}\left\{\log_2\left(\frac{1}{u}\right)+\cdots+\log_{p-1}\left(\frac{1}{u}\right)\right\}\right]}{\frac{m}{u}\log_p\left(\frac{1}{u}\right)} \\
&\quad\leqq \frac{1}{m}\sum_{j=1}^{m}\overline{\lim_{u\to 0}}\, \frac{Z_{ju}-\frac{1}{u}\left\{\log_2\left(\frac{1}{u}\right)+\cdots+\log_{p-1}\left(\frac{1}{u}\right)\right\}}{\frac{1}{u}\log_p\left(\frac{1}{u}\right)} \\
&\quad\leqq \frac{1}{m}\cdot m=1 \quad \text{a.s.}
\end{aligned} \tag{2.46}$$

by Theorem 3 of Deheuvels (1974).

Remark. Choosing $u=\varepsilon_n=n^{-\alpha}$, $\alpha>0$, it follows that for all $A>0$,

$$\tau_{\varepsilon_n} \leqq m n^{\alpha}(\log_2 n+\log_3 n+\cdots+\log_{p-1} n+(1+A)\log_p n) \quad \text{a.s.} \tag{2.46}$$

for large n.

Lemma 2.6.

$$\overline{\lim_{u\to 0}} \frac{u\tau_u}{m\log_2\left(\frac{1}{u}\right)}=1 \quad \text{a.s.} \tag{2.47}$$

Proof. Since $\tau_{\varepsilon_n} \geqq \tau_{\varepsilon_n}-\tau_{\varepsilon_{n-1}}$ for every n, the result follows from Lemmas 2.4 and 2.5.

Lemma 2.7. *For every $A>0$ and for every $p\geqq 2$*

$$\underline{\lim_{u\to 0}}\, u\tau_u \log\left(\frac{1}{u}\right)\log_2\left(\frac{1}{u}\right)\ldots\log_{p-1}\left(\frac{1}{u}\right)\left\{\log_p\left(\frac{1}{u}\right)\right\}^{1+A}=\infty \quad \text{a.s.} \tag{2.48}$$

Proof. Since $\tau_u=Z_{1u}+\cdots+Z_{mu}$ where Z_{iu} are as defined in Lemma 2.5 and $\tau_u \geqq Z_{1u}$, it follows that

$$\underline{\lim_{u\to 0}}\, u\tau_u \log\left(\frac{1}{u}\right)\log_2\left(\frac{1}{u}\right)\ldots\log_{p-1}\left(\frac{1}{u}\right)\left(\log_p\left(\frac{1}{u}\right)\right)^{1+A}$$
$$\geqq \underline{\lim_{u\to 0}}\, uZ_{1u} \log\left(\frac{1}{u}\right)\log_2\left(\frac{1}{u}\right)\ldots\log_{p-1}\left(\frac{1}{u}\right)\left(\log_p\left(\frac{1}{u}\right)\right)^{1+A}=+\infty \quad \text{a.s.}$$

by Theorem 4 of Deheuvels (1974).

Theorem 2.4. *For all $A>0$,*

$$\begin{aligned}\log\left(\frac{1}{u}\right)-(1+A)\log_2\left(\frac{1}{u}\right) &\leqq \log\left(\frac{\tau_u}{u}\right)\\ &\leqq \log\left(\frac{1}{u}\right)+(1+A)\log_3\left(\frac{1}{u}\right) \quad \text{a.s.}\end{aligned} \tag{2.49}$$

for sufficiently small u.

Proof. This result follows from Lemmas 2.6 and 2.7.

Theorem 2.5. *For any $v>0$,*

$$P(u\tau_u>v)\to\frac{1}{r(m)}\int_v^{\infty} x^{m-1}e^{-x}\,dx \quad \textit{as } u\to 0.$$

Proof. Since $u\tau_u=\sum_{j=1}^{m} uZ_{ju}$ where Z_{ju}, $1\leqq j\leqq m$ are independent and identically distributed with

$$P(Z_{1u}=r)=u(1-u)^{r-1}, \quad r\geqq 1$$

and

$$P(uZ_{1u}>v)\to e^{-v} \quad \text{as } u\to 0$$

it follows that $u\tau_u$ converges in law to the m-fold convolution of exponential with mean one which is the Gamma Distribution as given in the theorem.

Theorem 2.6.

$$\frac{S_n - m\log n}{(2m\log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0,1). \tag{2.50}$$

Proof. By choosing $u = \frac{1}{n}$ and $\varepsilon_n = \frac{1}{n}$, it can be shown that

$$\tau_{\varepsilon_{[nm^{-1}(\log_2 n)^{-(1+A)}]}} \leqq n \leqq \tau_{\varepsilon_{[nm^{-1}(\log n)^{1+A}]}} \quad \text{a.s.}$$

for large n by Theorem 2.4 and the fact that

$$\frac{\log \tau_{\varepsilon_n}}{\log n} \to 1 \quad \text{a.s.}$$

It is sufficient to prove that

$$\frac{S_{\tau_{\varepsilon[nm^{-1}(\log_2 n)^{-(1+A)}]}} - m\log n}{(2m\log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0,1) \tag{2.51}$$

and

$$\frac{S_{\tau_{\varepsilon[nm^{-1}(\log n)^{1+A}]}} - m\log n}{(2m\log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0,1). \tag{2.52}$$

But (2.51) and (2.52) follow from Theorem 2.2 since $\alpha = 1$.

Remark. By some tedious computations, it can be shown that

$$\operatorname{Var}(S_n) = 2m\log n + 0(1). \tag{2.53}$$

3. General Case

Let $\{Y_n, n \geqq 1\}$ be independent, positive random variables with the same distribution function F. Further suppose that for all $\varepsilon > 0$, $P(Y_1 < \varepsilon) > 0$. Let $G(t) = \operatorname{Inf}\{x \geqq 0 \mid F(x) \geqq t\}$. G is a monotone non decreasing function. Let $\{X_n, n \geqq 1\}$ be a sequence of independent random variables uniformly distributed on $[0, 1]$. Then $\{G(X_n), n \geqq 1\}$ is identical in law with $\{Y_n, n \geqq 1\}$. The study of the behaviour of

$$S_n = \sum_{j=m}^{n} Y_j^{(m)} \tag{3.1}$$

is equivalent to that of

$$S_n = \sum_{j=m}^{n} [m\text{-th order statistic of } \{G(X_1), \dots, G(X_j)\}]$$

which is again equivalent to that of

$$S_n = \sum_{j=m}^{n} G(X_j^{(m)}) \tag{3.2}$$

since G is monotone non-decreasing. Let $\varepsilon_n = n^{-1}$. Define

$$V_n = \sum_{j=1}^{n} (\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, G(\varepsilon_{j+1}), \tag{3.3}$$

$$V_n' = \sum_{j=1}^{n} (\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, G(\varepsilon_j) \tag{3.4}$$

where τ_ε is as defined in the previous section. The following lemmas can be proved by methods similar to those in Section 2.

Lemma 3.1. *For all* $n \geqq 1$

$$-2 + V_{n-1} \leqq S_{\tau_{\varepsilon_n}} - S_{\tau_{\varepsilon_1}} \leqq V_{n-1}' \tag{3.5}$$

and for all $j \in [\tau_{\varepsilon_n}, \tau_{\varepsilon_{n+1}}]$,

$$-2 + V_{n-1} \leqq S_j - S_{\tau_{\varepsilon_1}} \leqq V_n'. \tag{3.6}$$

Lemma 3.2.

$$M_n = E(V_n) = m \sum_{j=1}^{n} G\left(\frac{1}{j+1}\right);\; M_n' = E(V_n') = m \sum_{j=1}^{n} G\left(\frac{1}{j}\right)$$

$$D_n^2 = \operatorname{Var}(V_n) = 2m \sum_{j=1}^{n} j G^2\left(\frac{1}{j+1}\right), \qquad D_n'^2 = \operatorname{Var}(V_n') = 2m \sum_{j=1}^{n} j G^2\left(\frac{1}{j}\right).$$

Lemma 3.3. $V_n' - V_n$ *converges almost surely to a finite limit.*

Let

$$h(u) = G(e^{-u}) \cdot e^u$$

and

$$H(u) = \int_0^u h(u)\, du. \tag{3.7}$$

Lemma 3.2 shows that

$$M_n = mH(\log n) + 0(1) \tag{3.8}$$

and

$$M_n' = mH(\log n) + 0(1). \tag{3.9}$$

Lemma 3.4. *Suppose* $\lim_{n\to\infty} H(n) = \infty$. *In order that*

$$\frac{S_{\tau_{\varepsilon_n}}}{E(S_{\tau_{\varepsilon_n}})} \to 1 \quad \text{a.s. } (\textit{in probability})$$

it is necessary and sufficient that

$$\frac{V_n}{E(V_n)} \to 1 \quad \text{a.s. } (\textit{in probability})$$

or

$$\frac{V_n'}{E(V_n')} \to 1 \quad \text{a.s. } (\textit{in probability}).$$

More over

$$E(S_{\tau_{\varepsilon_n}}) - mH(\log n) = 0(1). \tag{3.10}$$

Proof. The above lemma follows from Lemmas 3.1 and 3.3 and relations (3.8) and (3.9).

Theorem 3.1. (i) *If* $\lim_{n\to\infty} H(n) < \infty$, *then* S_n *tends almost surely to a finite limit with finite expectation.*

(ii) *If* $\lim_{n\to\infty} H(n) = \infty$, *then*

(a) *if*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} jG^2\left(\frac{1}{j}\right)}{\left[\sum_{j=1}^{n} G\left(\frac{1}{j}\right)\right]^2} = 0 \tag{3.11}$$

then

$$\frac{S_{\tau_{\varepsilon_n}}}{mH(\log n)} \xrightarrow{p} 1; \tag{3.12}$$

(b) *if*

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{jG^2\left(\frac{1}{j}\right)}{\left[\sum_{i=1}^{j} G\left(\frac{1}{i}\right)\right]^2} < \infty \tag{3.13}$$

then

$$\frac{S_{\tau_{\varepsilon_n}}}{mH(\log n)} \to 1 \quad \text{a.s.} \tag{3.14}$$

Proof. (i) If $H(n)$ is bounded, then $E(S_n)$ is bounded in n. Hence $\sup_n S_n(\omega) < \infty$ a.s. But $S_n(\omega)$ is monotone non-decreasing for each ω. Hence S_n tends to a finite limit almost surely. It is easily seen that this limit has a finite expectation.

(ii) (a) follows from Lemmas 3.2, 3.4 and Čebyšev's inequality.

(ii) (b) follows from Lemmas 3.2, 3.4, Rényi (1970), p. 435 (Exercise 17) and the fact that $\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j}$, $j \geqq 1$ are independent.

Remark. Let

$$H_p(u) = \int_0^u h^p(u)\, du$$

for any $p \geqq 2$. It can be shown that

$$(3.11) \Leftrightarrow H_2(u)/H^2(u) \to 0 \quad \text{as } u \to \infty \tag{3.15}$$

and

$$(3.13) \Leftrightarrow \int_0^\infty \frac{h^2(u)}{H^2(u)}\, du < \infty. \tag{3.16}$$

Theorem 3.2. *If there exists* $p>2$ *such that*

$$\lim_{n\to\infty} \frac{\left[\sum_{j=1}^{n} j^{p-1} G^p\left(\frac{1}{j}\right)\right]^{1/p}}{\left[\sum_{j=1}^{n} jG^2\left(\frac{1}{j}\right)\right]^{\frac{1}{2}}} = 0 \tag{3.17}$$

then

$$\frac{S_{\tau_{\varepsilon_n}} - mH(\log n)}{\{2mH_2(\log n)\}^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0,1). \tag{3.18}$$

Proof. It is easy to see from Lemma 3.2 and the definition of $H_2(u)$ that

$$\operatorname{Var}(V_n) = 2mH_2(\log n) + 0(1) \tag{3.19}$$

and

$$\operatorname{Var}(V_n') = 2mH_2(\log n) + 0(1). \tag{3.20}$$

The result now follows from Lemmas 3.1 and 3.3 and relations (3.8), (3.9) by using methods similar to those used in Theorem 2.2.

Remark.

$$(3.17) \Leftrightarrow \text{for some } p>2, \quad \lim_{u\to\infty} \frac{H_p(u)}{H^p(u)} = 0. \tag{3.21}$$

Theorem 3.3.

$$E(S_{\tau_{\varepsilon_n}}) = mH(\log n) + 0(1) \tag{3.22}$$

and

$$\operatorname{Var}(S_{\tau_{\varepsilon_n}}) = 2mH_2(\log n) + 0\left(\sum_{j=1}^{n} G^2\left(\frac{1}{j}\right)\right). \tag{3.23}$$

Proof. (3.22) and (3.23) follow from Lemmas 3.1 and 3.2.

We shall now state a result regarding the asymptotic behaviour of $S_{\tau_{\varepsilon_n}}/E(S_{\tau_{\varepsilon_n}})$ or equivalently that of

$$\frac{S_{\tau_{\varepsilon_n}}}{mH(\log n)}.$$

Suppose that

$$\lim_{u\to\infty} \frac{k!\, H_k(u)}{H^k(u)} = A_k < \infty, \qquad k=1,2,\ldots \tag{3.24}$$

and $A_k > 0$ for all k. Let

$$B_p = \sum_{r_1+\cdots+r_i=p} \frac{p!}{r_1!\ldots r_i!\, S(r_1,\ldots,r_i)} A_{r_1}\ldots A_{r_i} \tag{3.25}$$

for every $p \geqq 1$, where $S(r_1, \ldots, r_i)$ denotes the number of permutations leaving $r_1, \ldots, r_i$ invariant.

Theorem 3.4. *If* (3.24) *holds, then there exists a distribution L with moments* B_p *defined by* (3.25) *and*

$$\frac{S_{\tau_{\varepsilon_n}}}{mH(\log n)} \xrightarrow{\mathscr{L}} L. \tag{3.26}$$

Proof of this theorem is the same as that of Theorem 9 of Deheuvels (1974) in view of the estimate (2.14).

Theorem 3.5. *If* $\lim_{n\to\infty} H(n) = \infty$, *then*

$$\varliminf_{n} \frac{\operatorname{Var}(S_{\tau_{\varepsilon_n}})}{[E(S_{\tau_{\varepsilon_n}})]^2} \cdot \log n \geqq \frac{2}{m}. \tag{3.27}$$

Proof. This result follows from Theorem 3.3 since

$$\frac{H_2(\log n)}{H^2(\log n)} \cdot \log n \geqq 1$$

by Schwartz inequality.

Theorem 3.6. *If* $jG(1/j)$ *is an increasing sequence, then, for all* $A > 0$,

$$\begin{aligned} &\{(\log n)(\log_2 n) \ldots (\log_{p-1} n)(\log_p n)^{1+A}\}^{-1} \\ &\qquad \leqq \frac{S_{\tau_{\varepsilon_n}}}{mnG\left(\frac{1}{n}\right)} \\ &\qquad \leqq (\log n)(\log_2 n) \ldots (\log_{p-1} n)(\log_p n)^{1+A} \quad \text{a.s.} \end{aligned} \tag{3.28}$$

for large n.

Proof. Let $A > 0$ and define

$$a_n = nG\left(\frac{1}{n}\right)(\log n)(\log_2 n) \ldots (\log_{p-1} n)(\log_p n)^{1+A}. \tag{3.29}$$

Then a_n is a positive increasing sequence and

$$\begin{aligned} \frac{V_n'}{a_n} &= \frac{1}{a_n} \sum_{j=1}^{n} (\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, G(\varepsilon_j) \\ &\leqq \sum_{j=1}^{n} \frac{(\tau_{\varepsilon_{j+1}} - \tau_{\varepsilon_j})\, G(\varepsilon_j)}{a_j}. \end{aligned}$$

Therefore

$$\begin{aligned} E\left(\frac{V_n'}{a_n}\right) &\leqq m \sum_{j=1}^{n} \frac{1}{a_j}\, G\left(\frac{1}{j}\right) \\ &= m \sum_{j=1}^{n} \frac{1}{j \log j (\log_2 j) \ldots (\log_{p-1} j)(\log_p j)^{1+A}}. \end{aligned} \tag{3.30}$$

Since the series

$$\sum_{j=1}^{\infty} \frac{1}{j \log j (\log_2 j) \dots (\log_{p-1} j)^{1+A}} < \infty$$

if follows that V_n'/a_n is bounded above almost surely. On the other hand

$$S_{\tau_{\varepsilon_n}} \geqq [\tau_{\varepsilon_n} - (m-1)]\, G\left(\frac{1}{n}\right).$$

Lemma 3.1 and Lemma 2.7 give the desired result.

Remark. It can be shown that (3.28) holds if there exists a constant $K > 0$ such that for every t_0 and $t \geqq t_0$,

$$\frac{F(t)}{t} \geqq K \frac{F(t_0)}{t_0}. \tag{3.31}$$

Theorem 3.7. (i) *If*

$$\lim_{u \to \infty} \frac{H(u + \log u)}{H(u)} = 1 \quad \text{and} \quad \int_0^{\infty} \frac{h^2(u)}{H^2(u)}\, du < \infty \tag{3.32}$$

then

$$\frac{S_n}{mH(\log n)} \to 1 \qquad \text{a.s.;} \tag{3.33}$$

(ii) *if there exists a sequence $u_n \uparrow \infty$ such that*

$$\lim_{n \to \infty} \frac{H(\log n + u_n)}{H(\log n)} = 1 \tag{3.34}$$

and if

$$\lim_{u \to \infty} \frac{H_2(u)}{H^2(u)} = 0 \tag{3.35}$$

then

$$\frac{S_n}{mH(\log n)} \xrightarrow{p} 1. \tag{3.36}$$

Proof. (3.33) follows from Theorems 3.1 and 2.4 since S_n lies between

$$S_{\tau_{\varepsilon_n}/m(\log_2 n)^{1+A}} \quad \text{and} \quad S_{\tau_{\varepsilon_n} m^{-1} (\log n)^{1+A}}$$

for any $A > 0$ almost surely for large n. Similar arguments give (3.36) since $\tau_{\varepsilon_n} u_n \to \infty$ and $\tau_{\varepsilon_n} u_n^{-1} \xrightarrow{p} 0$.

Theorem 3.8. *If $u_n \uparrow \infty$ such that*

$$\lim_{n \to \infty} \frac{H_2(\log n + u_n)}{H_2(\log n)} = 1, \tag{3.37}$$

$$\lim_{n \to \infty} \frac{H(u_n + \log n) - H(\log n)}{(H_2(\log n))^{\frac{1}{2}}} = 0, \tag{3.38}$$

and if for some $p>2$

$$\lim_{u\to\infty} \frac{H_p(u)}{H^p(u)} = 0 \tag{3.39}$$

then

$$\frac{S_n - m\,H(\log n)}{\{2m\,H_2(\log n)\}^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0,1). \tag{3.40}$$

Proof. Proof is similar to that of Theorem 2.6 in view of Theorems 3.2 and 2.4.

Finally, we have the following theorem which gives bounds on S_n.

Theorem 3.9. *If there exists a constant* $K>0$ *such that for all* t_0 *and* $t \geqq t_0$,

$$\frac{F(t)}{t} \geqq K\,\frac{F(t_0)}{t_0} \tag{3.41}$$

then, for all $A>0$,

$$\begin{aligned} &\frac{mnG\left(\frac{1}{n}\right)}{\log n \ldots \log_{p-1} n (\log_p n)^{1+A}} \\ &\quad \leqq S_n \\ &\quad \leqq mnG\left(\frac{1}{n}\right) \log n \ldots \log_{p-1} n (\log_p n)^{1+A} \quad \text{a.s.} \end{aligned} \tag{3.42}$$

for large n.

Proof. This result follows from Theorem 3.6 and Lemmas 2.5 and 2.7.

4. Examples

Example 1. Suppose $F(t) \sim f_0\, t^a$, $a>0$. Then

(i) if $a<1$, S_n tends a.s. to a finite limit;

(ii) if $a=1$, $\dfrac{S_n}{m \log n} \to \dfrac{1}{f_0}$ a.s. and

$$\frac{S_n - \frac{m}{f_0}\log n}{\left(\frac{2m}{f_0^2}\log n\right)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0,1);$$

(iii) if $a>1$, there is a distribution L such that

$$\frac{S_{\tau_{\varepsilon n}}}{m\left\{\frac{n^{1-1/a}}{1-1/a}\right\}} \xrightarrow{\mathscr{L}} L$$

and for sufficiently large n, for every $A>0$,

$$\begin{aligned} & m n^{1-1/a}(\log n)^{-1} \ldots (\log_p n)^{-(1+A)} f_0^{-1/a} \\ & \quad \leqq S_n \\ & \quad \leqq m n^{1-1/a}(\log n)(\log_2 n) \ldots (\log_p n)^{1+A} f_0^{-1/a} \quad \text{a.s.} \end{aligned}$$

Example 2. If

$$0<A\leqq \underline{\lim}\,\frac{F(t)}{t}\leqq \overline{\lim}\,\frac{F(t)}{t}\leqq B<\infty$$

then

$$\frac{S_n}{mH(\log n)}\to 1 \quad \text{a.s.}$$

and

$$\frac{S_n-m\,H(\log n)}{\{2m\,H_2(\log n)\}^{\frac{1}{2}}}\xrightarrow{\mathscr{L}} N(0,1).$$

Example 3. Suppose $F(t)\sim f t\left\{\log\left(\frac{1}{t}\right)\right\}^a$. Then,

(i) if $a<-1$, S_n tends a.s. to a finite limit;
(ii) if $a\geqq -1$,

$$\frac{S_n}{mH(\log n)}\to 1 \quad \text{a.s.;}$$

(iii) if $a\geqq -\frac{1}{2}$

$$\frac{S_n-m\,H(\log n)}{\{2m\,H_2(\log n)\}^{\frac{1}{2}}}\xrightarrow{\mathscr{L}} N(0,1).$$

These examples were discussed in Deheuvels (1974) for the case $m=1$.

5. Some Further Results

Suppose $\{X_n, n\geqq 1\}$ are independent uniformly distributed random variables on $[0,1]$. Let $\tilde{X}_j^{(m)}$ be the $(j-m+1)$-th order statistic of $(X_1,\ldots,X_j)$. Define

$$\tilde{S}_n=\sum_{j=m}^{n}\tilde{X}_j^{(m)}. \tag{5.1}$$

One can obtain the limiting behaviour of $\tilde{S}_n$ from that of S_n derived in Section 2.

Theorem 5.1.

$$\frac{\tilde{S}_n}{n}\to 1 \quad \text{a.s.} \tag{5.2}$$

Proof. Observe that

$$\tilde{S}_n=(n-m+1)-S_n$$

where S_n is the sum of the m-th order statistics of $(1-X_1, \ldots, 1-X_j)$. But

$$\{1-X_n, n \geqq 1\}$$

are independent uniformly distributed on [0, 1]. Hence

$$\frac{S_n}{m \log n} \to 1 \quad \text{a.s.}$$

by Theorem 2.6. Hence

$$\frac{\tilde{S}_n}{n} = \frac{(n-m+1)-S_n}{n} = \frac{n-m+1}{n} - \frac{S_n}{m \log n} \cdot \frac{m \log n}{n}$$

converges to 1 almost surely as $n \to \infty$.

Remark. In particular, if $m=1$, one obtains that

$$\frac{1}{n} \sum_{j=1}^{n} \max(X_1, \ldots, X_j) \to 1 \quad \text{a.s.} \tag{5.3}$$

Theorem 5.2.

$$\frac{\tilde{S}_n - n - m \log n}{(2m \log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0, 1). \tag{5.4}$$

Proof. This theorem follows from Theorem 2.6 by noting that

$$\tilde{S}_n = (n-m+1) - S_n,$$

$$E(\tilde{S}_n) = (n-m+1) - E(S_n),$$

and

$$\operatorname{Var}(\tilde{S}_n) = \operatorname{Var}(S_n).$$

Remark. In particular, if $m=1$, we obtain that

$$\frac{\sum_{j=1}^{n} \max(X_1, \ldots, X_j) - n - \log n}{(2 \log n)^{\frac{1}{2}}} \xrightarrow{\mathscr{L}} N(0, 1). \tag{5.5}$$

Theorem 5.3. *Let $\{Y_n, n \geqq 1\}$ be a sequence of positive, independent and identically distributed random variables bounded by a constant $b > 0$ i.e.,*

$$b = \operatorname{Inf}\{a: P(0 \leqq Y_1 \leqq a) = 1\}.$$

Then

$$\frac{\tilde{S}_n}{n} \to b \quad \text{a.s.} \tag{5.6}$$

where $\tilde{S}_n = \sum_{j=m}^{n} \tilde{Y}_j^{(m)}$ and $\tilde{Y}_j^{(m)}$ *is the* $(j-m+1)$*-th order statistic of* $(Y_1, \ldots, Y_j)$.

Proof. Let $\{X_n, n \geqq 1\}$ be independent uniformly distributed on [0, 1] and $\tilde{X}_j^{(m)}$ be the $(j-m+1)$-th order statistic of $(X_1, \ldots, X_j)$. Since $Y_i \leqq b$ a.s., it can be shown that

$$\tilde{Y}_j^{(m)} \to b \qquad \text{a.s. as } j \to \infty \tag{5.7}$$

and

$$\tilde{X}_j^{(m)} \to 1 \qquad \text{a.s. as } j \to \infty. \tag{5.8}$$

Hence

$$\eta_j \equiv \frac{\tilde{Y}_j^{(m)}}{\tilde{X}_j^{(m)}} \to b \qquad \text{a.s. as } j \to \infty. \tag{5.9}$$

But

$$\frac{\tilde{S}_n}{n} = \frac{1}{n} \sum_{j=1}^{n} \tilde{Y}_j^{(m)} = \frac{1}{n} \sum_{j=1}^{n} \eta_j \tilde{X}_j^{(m)}.$$

Hence

$$\frac{\tilde{S}_n}{n} \to b \cdot 1 = b \qquad \text{a.s. as } n \to \infty$$

by (5.9) and Theorem 5.1.

Remark. Theorem 5.3 reduces to Theorem 1 of Ghosh *et al.* (1975) when $m=1$ and the proof uses the same technique as in their paper.

Let $\{Y_n, n \geqq 1\}$ be a sequence of positive, independent random variables with a common distribution function F and $G(t) = \operatorname{Inf}\{x \geqq 0 \mid F(x) \geqq t\}$. Let $\{X_n, n \geqq 1\}$ be a sequence of independent exponential random variables with mean one. It is easy to see that $\{Y_n, n \geqq 1\}$ is identical in law with $\{G(1-e^{-X_n}), n \geqq 1\}$. Hence the asymptotic behaviour of

$$\tilde{S}_n \equiv \sum_{j=m}^{n} \tilde{Y}_j^{(m)}$$

is the same as the asymptotic behaviour of

$$\tilde{S}_n = \sum_{j=m}^{n} G(1 - e^{-\tilde{X}_j^{(m)}}) \tag{5.10}$$

where $\tilde{X}_j^{(m)}$ is the $(j-m+1)$-th order statistic of $(X_1, \ldots, X_j)$. Suppose F is absolutely continuous with density f which is continuously differentiable and there exists $0 < b < \infty$ such that

(i) $f(x) \geqq \alpha > 0$ for all $0 \leqq x \leqq b$, (5.11)

(ii) $|f'(x)| \leqq \beta < \infty$ for all $0 \leqq x \leqq b$, and (5.12)

(iii) $b = \operatorname{Inf}\{a\colon P(0 \leqq Y \leqq a) = 1\}$.

(i) and (ii) imply that

$$G(1-x) = G(1) - x\,G'(1) + \frac{x^2}{2}\,0(1)$$

in any neighbourhood of zero. But $G(1)=b$. Hence

$$\begin{aligned}\tilde{S}_n&=\sum_{j=m}^{n} G(1-e^{-\tilde{X}_j^{(m)}})\\&=(n-m+1)b-G'(1)\left\{\sum_{j=m}^{n} e^{-\tilde{X}_j^{(m)}}\right\}+\sum_{j=m}^{n}(e^{-\tilde{X}_j^{(m)}})^2\cdot 0(1) \quad \text{a.s.} \qquad (5.13)\\&=(n-m+1)b-G'(1)\left\{\sum_{j=m}^{n}(e^{-X})_j^{(m)}\right\}+\sum_{j=m}^{n}[(e^{-X})_j^{(m)}]^2\cdot 0(1) \quad \text{a.s.}\end{aligned}$$

Since e^{-X} has a uniform distribution on [0, 1],

$$E[(e^{-X})_j^{(m)}]^2=\frac{m(m+1)}{(j+1)(j+2)}.$$

Since the series $\sum_{j=m}^{\infty}\frac{1}{j^2}<\infty$, it follows that

$$\sum_{j=m}^{n}[(e^{-X})_j^{(m)}]^2=0_p(1). \qquad (5.14)$$

Hence

$$\tilde{S}_n=(n-m+1)b-\frac{1}{f(1)}\left\{\sum_{j=m}^{n}(e^{-X})_j^{(m)}\right\}+0_p(1). \qquad (5.15)$$

Let

$$S_n=\sum_{j=m}^{n}(e^{-X})_j^{(m)}. \qquad (5.16)$$

Since e^{-X} is uniformly distributed on [0, 1], Theorem 2.6 implies that

$$\frac{S_n-m\log n}{(2m\log n)^{\frac{1}{2}}}\xrightarrow{\mathscr{L}}N(0,1). \qquad (5.17)$$

But

$$-f(1)\{\tilde{S}_n-(n-m+1)b\}-m\log n+0_p(1)=S_n$$

and hence

$$\frac{-f(1)\{\tilde{S}_n-(n-m+1)b\}-m\log n+0_p(1)}{(2m\log n)^{\frac{1}{2}}}\xrightarrow{\mathscr{L}}N(0,1)$$

i.e.,

$$\frac{f(1)\{\tilde{S}_n-nb\}-m\log n}{(2m\log n)^{\frac{1}{2}}}\xrightarrow{\mathscr{L}}N(0,1).$$

Hence we have the following theorem.

Theorem 5.4. *Under the conditions of Theorem* 5.3 *and relations* (5.11) *and* (5.12),

$$\frac{\tilde{S}_n-nb-\frac{m}{f(1)}\log n}{\{2m\log n/f^2(1)\}^{\frac{1}{2}}}\xrightarrow{\mathscr{L}}N(0,1). \qquad (5.18)$$

Remark. We were unable to study the behaviour of $\tilde{S}_n$ for unbounded random variables by the methods described above. We conjecture that

$$\frac{\tilde{S}_n}{n \log n} \to c = \lim_{t \uparrow \infty} \frac{t}{-\log(1-F(t))} \tag{5.19}$$

almost surely. Ghosh *et al.* (1975) have proved that (5.19) holds for $m=1$.

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