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On Glivenko-Cantelli Convergence*

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0. In this paper ideas which were first developed by the author in [6] are applied to the investigation of the almost sure convergence behavior of empirical distribution functions (the proofs, however, do not depend on results of the previous article). Generally speaking, the problems in question suggest themselves once the concept of almost sure convergence is reformulated in terms of the weak* topology of measures: let $(\xi_1, \xi_2, \dots)$ be a sequence of random variables taking values in a separable metric vector space E; then

$$P\big[\lim_{n\to\infty} \xi_n = 0\big] = 1 \quad \text{if and only if} \quad \lim_{n\to\infty} \mathscr{L}(\xi_n, \xi_{n+1}, \dots) = \mathscr{L}(0, 0, \dots);$$

here $\mathscr{L}$ denotes the operator assigning the distribution to a random variable (which, in this case, is a sequence of random variables); the limit is to be interpreted as pointwise convergence on real-valued bounded measurable continuous functions, which are defined on the space H of all sequences of elements of E having uniform topology.

Given almost sure convergence we can replace H by a separable space and ask for the limit behavior of $\mathscr{L}(N(n)(\xi_n, \xi_{n+1}, \dots))$, $N(n)$ being suitable norming factors $\big(\lim_{n\to\infty} N(n) = \infty\big)$. In the following the answer is given for the case that ξ_n is the error D_n of the n-th sequence of independent uniformly distributed observations. By means of Theorem 1 below it will for instance be possible to reduce the problem of finding the limit distribution of the maximum error that will occur after the n-th observation to a simpler (combinatorial) problem. Although this distribution F is not known explicitly to the author, the present method yields the estimates

$$e^{-2\alpha^2} \leqq 1 - F([0, \alpha]) \leqq 2 e^{-2\alpha^2} \qquad (\alpha \geqq 0)$$

(the left inequality being trivial).

There are not many results in multidimensional fluctuation theory as yet so that explicit applications of Theorem 1 are not abundant. However, we mention that from a theorem of Hobby and Pyke [2] together with Theorem 1 it follows that the asymptotic distribution of the $t \geqq 1$ maximizing the Hilbertian norm of

$$D_{[nt]} - \tfrac{1}{2} D_n$$

has density $\alpha \rightsquigarrow \alpha^{-2}$. Another application is to the asymptotic joint distribution of the medians $\operatorname{med}(n)$ of the first n of a sequence of independent uniformly

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distributed variables: the distributions of the (linearly interpolated) processes $t \rightsquigarrow \sqrt{n}(\operatorname{med}(nt)-\frac{1}{2})$ $(t \geqq 1)$ converge weakly to $\mathscr{L}(t \rightsquigarrow \zeta(t)/2t)$ (ζ = Brownian motion) in the space of all continuous functions on $[1, \infty)$ vanishing at ∞ (uniform topology).

An estimate of the speed of convergence will be given by Theorem 2. The author believes that it cannot be improved over $O(n^{-\frac{1}{4}})$.

1. Let $(X_i: i=1,2,\ldots)$ be a sequence of independent random variables which are uniformly distributed over $[0,1]$, and let

$$F_n(s) := \frac{1}{n}\{\text{number of } X_i \leqq s \ (i \leqq n)\}$$

be their n-th empirical distribution function ($s \in [0,1]$). We put $D_n(s) := F_n(s) - s$. Then, as is well-known, $D_n \in \mathscr{D}[0,1]$ (= the space of real-valued functions on the unit interval without discontinuities of the second kind, endowed with the Skorokhod topology) with probability 1 and $\lim_{n\to\infty} \mathscr{L}(\sqrt{n} D_n) = \mathscr{L}(s \rightsquigarrow \zeta(s) - s\zeta(1))$, where ζ denotes Brownian motion over $[0,1]$ with $\zeta(0)=0$. Moreover

$$P[\lim_{n\to\infty} \|D_n\|_\infty = 0] = 1 \qquad (\|\ \|_\infty \text{ being uniform norm}).$$

The topology of $\mathscr{D}[0,1]$ can be described by a metric $\rho_{\mathscr{D}}$ such that

$$\rho_{\mathscr{D}}(y_1, y_2) \leqq \|y_1(s) - y_2(s)\|_\infty \qquad (y_1, y_2 \in \mathscr{D}[0,1]).$$

This metric will be used to define a metric on the space $\mathscr{C}_0$ of all continuous mappings x from $[0,\infty)$ to $\mathscr{D}[0,1]$ which satisfy

(i) $x(0)=0$;

(ii) $\lim_{t\to\infty} \frac{\|x(t)\|_\infty}{t} = 0$:

namely, define

$$\rho(x_1, x_2) := \sup_{t \geqq 0} \frac{\rho_{\mathscr{D}}(x_1(t), x_2(t))}{t \vee 1}.$$

We form the process

$$t \rightsquigarrow \Delta_n(t) := \sqrt{n}\, t\, D_{nt}$$

where D_{nt} is obtained from the $(D_k: k=1,2,\ldots)$ by linear interpolation:

$$D_t(s) = (1-(t-[t]))\, D_{[t]}(s) + (t-[t])\, D_{[t]+1}(s).$$

Theorem 1. *There exists a process Δ, $\Delta \in \mathscr{C}_0$ with probability 1, its distribution being uniquely determined by the following two conditions:*

(i) *Δ has independent increments $\Delta(t+h)-\Delta(t)$;*

(ii) $\mathscr{L}(\Delta(t+h)-\Delta(t)) = \mathscr{L}(s \rightsquigarrow \sqrt{h}(\zeta(s)-s\zeta(1)))$ $(h>0)$.

$(t,s) \rightsquigarrow \Delta(t)_s$ *is a Gaussian process over* $[0,\infty)\times[0,1]$, *continuous with probability* 1, *having covariance function*

$$E\Delta(t)_s\, \Delta(t')_{s'} = t s(1-s') \qquad (s<s', t \leqq t').$$

For Δ the inversion formula holds:

$$\mathcal{L}\left(t \rightsquigarrow t\Delta\left(\frac{1}{t}\right)\right) = \mathcal{L}(t \rightsquigarrow \Delta(t)),$$

where the value for $t=0$ of the process on the left is understood to be 0.

$\Delta_n \in \mathcal{C}_0$ *with probability* 1.

$\lim_{n\to\infty} \mathcal{L}(\Delta_n) = \mathcal{L}(\Delta)$ *in weak* topology.*

To clarify the relation between Theorem 1 and the previous section we state the following immediate

Corollary. *Let $\mathcal{C}_1$ be the space of all continuous mappings x from $[1,\infty)$ to $\mathcal{D}[0,1]$ having $\lim_{t\to\infty} \|x(t)\|_\infty = 0$, endowed with uniform topology. $\mathcal{C}_1$ is isomorphic to $\mathcal{C}_{01}$, the space of all continuous mappings x from $[0,1]$ to $\mathcal{D}[0,1]$ with $x(0)=0$, the isomorphism being induced by the point transformation T: $t \rightsquigarrow t^{-1}$.*

$$\begin{aligned}\lim_{n\to\infty} \mathcal{L}(t \rightsquigarrow \sqrt{n}\, D_{nt}\colon t \geqq 1) &= \mathcal{L}(t \rightsquigarrow \Delta(t)/t\colon t \geqq 1)\\ &= T\mathcal{L}(t \rightsquigarrow \Delta(t)\colon 0 \leqq t \leqq 1)\end{aligned}$$

where the convergence takes place in weak sense for measures on $\mathcal{C}_1$.*

The proof of Theorem 1 can be obtained by a straightforward generalization of the proof of Theorem 1 of [6]. However, some additional attention must be paid to keep the s-part of the "paths" smooth. In the following we sketch the main steps of the argument.

Proof (Theorem 1). Let us first point out that $\Delta_n(t)$ is asymptotically equal in probability to

$$n^{-\frac{1}{2}} \sum_{i=1}^{[nt]} (1_{[X_i \leqq s]} - s),$$

i.e. to a sum of independent random vectors. The finite-dimensional distributions of Δ_n therefore converge to the corresponding marginals of $\mathcal{L}(\Delta)$, and the theorem will be proven if $(\mathcal{L}(\Delta_n)\colon n=1,2,\ldots)$ can be shown to be tight. In order to show that $\underline{P}[\Delta_n \in \mathrm{ns}\,{}^*\mathcal{C}_0] = 1$ for all infinite n (cp. [5, 6]) in an enlargement of $\mathcal{C}_0$, or, which amounts to the same, to establish the existence of a set $S \subset \mathrm{ns}\,{}^*\mathcal{C}_0$ (depending on n) with $P[\Delta_n \in S] \geqq 1-\varepsilon$ for given standard $\varepsilon > 0$, we apply the inequality

$$\begin{aligned}&P\Big[\sup_{\substack{t',t'' \in \{0,1/n,\ldots\} \cap [0,T]\\ |t''-t'|<\delta}} \|\Delta_n(t'') - \Delta_n(t')\|_\infty \geqq 4\gamma\Big]\\ &\quad \leqq 2\frac{T}{\delta} \sup_{\substack{t',t'' \in \{0,1/n,\ldots\} \cap [0,T]\\ |t''-t'|<\delta}} P[\|\Delta_n(t'') - \Delta_n(t')\|_\infty \geqq \gamma] \qquad \left(\delta \geqq \frac{1}{n}\right),\end{aligned}$$

which, given a finite positive integer m and a standard $T_m > 0$, permits to find a standard $\delta_m > 0$ satisfying

$$P\left[\sup_{\substack{t',t'' \in \{0,1/n,\ldots\} \cap [0,T_m]\\ |t''-t'|<\delta_m}} \|\Delta_n(t'') - \Delta_n(t')\|_\infty \geqq \frac{4}{m}\right] \leqq \frac{\varepsilon}{3} 2^{-m}.$$

Let m_0 be the largest (necessarily infinite) integer m, for which this inequality holds. Furthermore, to assure smoothness in s, let $(t_l\colon l=1,2,\ldots)$ be a standard dense subset of $[0,\infty)$; then, by tightness of $(\mathscr{L}(s \rightsquigarrow \Delta_n(t_l)_s)\colon n=1,2,\ldots)$ there is a standard compact $K_l \subset \mathscr{D}[0,1]$ such that $P[\Delta_n(t_l) \notin {}^*K_l] \leqq \frac{\varepsilon}{3} 2^{-l}$ for all $l \leqq l_0$ (l_0 infinite).

Delaying the choice of $T_m \geqq 1$ we define

$$S = \bigcap_{m=1}^{m_0} \left\{x\colon \sup_{\substack{t',t'' \in [0,T_m] \\ |t''-t'|<\delta_m}} \|x(t'')-x(t')\|_\infty \leqq \frac{4}{m}\right\}$$
$$\cap \left\{x\colon \sup_{t \geqq T_m} \left\|\frac{x(t)}{t}\right\|_\infty \leqq \frac{1}{m}\right\} \cap \bigcap_{l=1}^{l_0} \{x\colon x(t_l) \in {}^*K_l\}.$$

Then $S \subset \operatorname{ns} {}^*\mathscr{C}_0$ if $T_m \to \infty$ $(m \to \infty)$ as follows from the simple lemma below.

Lemma. *An s-continuous image of a topological Hausdorff space in the sense of Robinson* [8] *consists entirely of near-standard points, provided the same is true for the image of an s-dense subset (internal or not) of the original space.*

In order to obtain $P[\Delta_n \in S] \geqq 1-\varepsilon$ it suffices therefore to choose T_m standard and so large that

$$P\left[\sup_{t \geqq T_m} \left\|\frac{\Delta_n(t)}{t}\right\|_\infty \geqq \frac{1}{m}\right] \leqq \frac{\varepsilon}{3} 2^{-m}.$$

Substituting $T_m = n_m/n$ we obtain

$$\begin{aligned} P\left[\sup_{t \geqq n_m/n} \left\|\frac{\Delta_n(t)}{t}\right\|_\infty \geqq \frac{1}{m}\right] &= P\left[\|k D_k\|_\infty \geqq \frac{k}{m\sqrt{n}} \text{ for some } k \geqq n_m\right] \\ &\leqq \sum_{j=1}^{\infty} P\left[\max_{k \leqq n_m 2^j} \|k D_k\|_\infty \geqq \frac{n_m 2^{j-1}}{m\sqrt{n}}\right]. \end{aligned} \tag{1}$$

$S_k = k D_k$ being a sum of independent random variables the generalized inequality of Kolmogorov is applicable saying that

$$P[\max_{k \leqq k_0} \|S_k\|_\infty \geqq 2M] \leqq 2P[\|S_{k_0}\|_\infty \geqq M]$$

provided that $P[\|S_k\|_\infty \geqq M] \leqq \frac{1}{2}$ for $k \leqq k_0$ (which is certainly the case for $M = n_m 2^{j-1}/m\sqrt{n}$ according to the inequality of Dvoretzky-Kiefer-Wolfowitz [1, 4]:

$$P[\|k D_k\|_\infty \geqq M] \leqq c_1 \exp\{-c_2 M^2/k\} \leqq c_1 \exp\{-c_2 \sqrt{n}\, 2^j/4m\}$$

(c_1, c_2 constants)). This inequality therefore yields

$$\begin{aligned} &P\left[\max_{k \leqq n_m 2^j} \|k D_k\|_\infty \geqq \frac{n_m 2^{j-1}}{m\sqrt{n}}\right] \\ &\qquad \leqq 2c_1 \exp\left(-\frac{c_2 2^{j-2}}{m^2} \frac{n_m}{n}\right) \leqq 2c_1 c_2^{-1} m^2 \frac{n}{n_m}, \end{aligned}$$

whence (1) is majorized by

$$2c_1 c_2^{-1} m^2 T_m^{-1}.$$

Now put $T_m = 6c_1 c_2^{-1} \varepsilon^{-1} m^2 2^m$.

This concludes the proof of the main statements of Theorem 1. As to the inversion formula we content ourselves with mentioning that it is proved by computing covariances (for the simpler case of Brownian motion on the line cp. [2], problem 3).

2. In the following we determine the probability that the process $t \rightsquigarrow \sqrt{n} D_{nt}$ $(t \geqq 1)$ lies between two bounds (i.e. continuous functions from $[1, \infty)$ to $\mathscr{C}[0,1]$) g_1 and g_2. In order to cover a variety of interesting special cases we state a theorem of greater generality first.

Let L denote the Prokhorov-distance in the space of distributions on $\mathscr{C}_0$, i.e.

$$L(Q_1, Q_2) = \max_{i,j=1,2} \inf\{\varepsilon\colon Q_i(F) < Q_j(F^\varepsilon) + \varepsilon \text{ for all closed } F \subset \mathscr{C}_0\}.$$

Theorem 2. $L(\mathscr{L}(\Delta_n), \mathscr{L}(\Delta)) = o(n^{-\frac{1}{8}+\varepsilon})$ $(\varepsilon > 0)$.

Proof. As a first step we construct a mapping $\pi_n\colon \mathscr{D}[0,1] \rightsquigarrow \mathscr{D}[0,1]$ having the property $\mathscr{L}(\pi_n \Delta(1)) = \mathscr{L}(\Delta_n(1))$ and

$$P\left[\|\pi_n(\Delta(1)) - \Delta(1)\|_\infty \geqq C\alpha \frac{\log n}{n^{\frac{1}{4}}}\right] \leqq C_l \alpha^{-l}/n$$

for $l \geqq 4$, $\alpha \geqq 1$ and C independent of α, l, n.

To this end let us denote by $X_1 \leqq X_2 \leqq \cdots \leqq X_n$ the ordered n-tupel of independent uniformly distributed random variables. We put

$$\Delta_n(1)_s = \frac{1}{\sqrt{n}} \sum_{i=1}^n (1_{[X_i \leqq s]} - s),$$

whence $\Delta_n(1)_{X_k} = \frac{1}{\sqrt{n}}(k - nX_k)$. Now the X_k, as is well-known, can be represented as exponentials of sums of independent random variables.

$\xi_{n-k} := -\log X_k$ have the following properties:

(i) $0 \leqq \xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_n$;

(ii) $\xi_1, \xi_2 - \xi_1, \ldots, \xi_n - \xi_{n-1}$ are independent;

(iii) $P[\xi_{i+1} - \xi_i < \alpha] = \int_0^\alpha (n-i)\, e^{-(n-i)u}\, du \qquad (i = 0, 1, \ldots, n-1;\ \xi_0 = 0)$.

As

$$E(\xi_{i+1} - \xi_i) = (n-i)^{-1} \quad \text{and} \quad \operatorname{var}(\xi_{i+1} - \xi_i) = (n-i)^{-2} \qquad (i = 0, 1, \ldots, n-1),$$

$$\Xi(k) = \sum_{i=1}^k (\xi_i - \xi_{i-1} - (n-i+1)^{-1}) \qquad (= 0 \text{ if } k = 0)$$

is a centered partial sum of independent random variables having

$$E\Xi(k)^2 = \sum_{i=1}^{k} (n-i+1)^{-2}.$$

Let ζ denote the process defined by

$$\zeta(s) = (s+1)\,\Delta(1)_{(s+1)^{-1}} \qquad (s \geqq 0)$$

which is a Brownian motion. By a theorem of Skorokhod [9] there exist n independent non-negative random variables τ_i $(i=1,\dots,n)$ having $E\tau_i = n\operatorname{var}(\xi_i - \xi_{i-1})$ and $E\tau_i^k \leqq M_k n^k E(\xi_i - \xi_{i-1} - (n-i+1)^{-1})^{2k}$ $(k>1,\ M_k$ absolute constants) such that

$$\sqrt{n}\,\Xi(k) \quad \text{and} \quad \zeta\left(\sum_{i=1}^{k} \tau_i\right)$$

have the same joint distribution.

Now define π_n by putting

$$\pi_n(\Delta(1))_s = \Delta_n(1)_s$$

where the underlying X_i are the

$$\exp\left\{-\frac{1}{\sqrt{n}}\,\zeta\left(\sum_{i=1}^{n-k} \tau_i\right) - \sum_{i=1}^{n-k} (n-i+1)^{-1}\right\}.$$

Clearly $\mathscr{L}(\pi_n \Delta(1)) = \mathscr{L}(\Delta_n)$.

To make plausible that $\Delta_n(1)$ is close to $\Delta(1)$ observe that by the law of large numbers $\sum_{i=1}^{n-k} \tau_i$ is close to its expectation $\sum_{i=1}^{n-k} E\tau_i$ which is approximately equal to $\frac{n}{k} - 1$ such that $\Delta_n(1)_{X_k}$ is close to

$$\sqrt{n}\left(\frac{k}{n} - \exp\left\{-\frac{1}{\sqrt{n}}\,\zeta\left(\frac{n}{k} - 1\right) - \log\frac{n}{k} + \gamma(n,k)\right\}\right)$$

$$\left(\gamma(n,k) = \sum_{i=1}^{k} (n-i+1)^{-1} - \log\frac{n}{k} \text{ being small}\right)$$

which again is close to

$$\sqrt{n}\,\frac{k}{n}\left(1 - \exp\left\{-\frac{1}{\sqrt{n}}\,\zeta\left(\frac{n}{k} - 1\right)\right\}\right)$$

and therefore to $\frac{k}{n}\,\zeta\left(\frac{n}{k} - 1\right) = \Delta(1)_{k/n}$.

The rigorous proof follows. We put $\lambda(n) = C n^{-\frac{1}{4}} \log n$ with a positive constant C that will always be assumed to be large enough to allow the estimates given below. Furthermore, to avoid massing of subscripts, we agree to simply omit them denoting all positive absolute constants by C (so that two C's need not be identical). Let α denote a variable with range $[1, \infty)$.

In the following we majorize $P[\|\Delta_n(1)-\Delta(1)\|_\infty>\alpha\lambda(n)]$ by a sum of eight terms $I, II, \ldots, VIII$ (in the order they are listed down below) each of which will then be replaced by an upper estimate. For brevity write $\mu(n)=\sqrt{n}\log n$.

$$
\begin{aligned}
&P[\|\Delta_n(1)-\Delta(1)\|_\infty>\alpha\lambda(n)]\\
&\quad\leqq P\Big[\sup_{\substack{k\leqq n\\ X_k<s\leqq X_{k+1}}}|\Delta_n(1)_s-\Delta_n(1)_{X_k}|>\alpha\lambda(n)\Big]\\
&\quad+P\left[\sup_k\left|X_k-\frac{k}{n}\right|>C\alpha\frac{\log n}{\sqrt{n}}\right]\\
&\quad+P\left[\sup_{\mu(n)\leqq k\leqq n}\sqrt{n}\left|\frac{k}{n}(1-e^{-\Xi(n-k)+\gamma(n,k)})-\frac{k}{n}\Xi(n-k)\right|>\alpha\lambda(n)\right]\\
&\quad+P\left[\sup_{\mu(n)\leqq k\leqq n}\left|\sqrt{n}\frac{k}{n}\Xi(n-k)-\frac{k}{n}\zeta\left(\sum_{i=1}^{n-k}E\tau_i\right)\right|>\alpha\lambda(n)\right]\\
&\quad+P\left[\sup_{\mu(n)\leqq k\leqq n}\left|\frac{k}{n}\zeta\left(\sum_{i=1}^{n-k}E\tau_i\right)-\frac{k}{n}\zeta\left(\frac{n}{k}-1\right)\right|>\alpha\lambda(n)\right]\\
&\quad+P\left[\sup_{\substack{|s-k/n|\leqq C\alpha\frac{\log n}{\sqrt{n}}\\ \mu(n)\leqq k\leqq n}}\left|\frac{k}{n}\zeta\left(\frac{n}{k}-1\right)-s\zeta\left(\frac{1}{s}-1\right)\right|>\alpha\lambda(n)\right]\\
&\quad+P\Big[\sup_{s\leqq\mu(n)/n}|\Delta_n(1)_s|>\alpha\lambda(n)\Big]+P\Big[\sup_{s\leqq\mu(n)/n}|\Delta(1)_s|>\alpha\lambda(n)\Big].
\end{aligned}
$$

We now show that $N\leqq C_l\alpha^{-l}n^{-1}$ for $N=\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII}$ (l any positive integer $\geqq 4$).

$$
\begin{aligned}
\mathrm{I}\quad&\leqq 2P\Big[\sup_{\substack{k\geqq n/2\\ X_k<s\leqq X_{k+1}}}|\Delta_n(1)_s-\Delta_n(1)_{X_k}|>\alpha\lambda(n)\Big]\\
&\leqq 2P\Big[\sup_{k\geqq n/2}\sqrt{n}\,|X_{k+1}-X_k|>\alpha\lambda(n)\Big]\\
&=2P\left[\sup_{k\geqq n/2}\sqrt{n}\,e^{-\Xi(n-k)}\left|\frac{k+1}{n}e^{\gamma(n,k+1)}-\frac{k}{n}e^{\gamma(n,k)}\right|>\alpha\lambda(n)\right]\\
&\leqq 2P\Big[\sup_{k\geqq n/2}n^{-\frac{1}{2}}e^{-\Xi(n-k)}>\alpha\lambda(n)\Big]\\
&\leqq 2P\Big[\sup_{k\geqq n/2}[-\Xi(n-k)]>\log(\alpha\sqrt{n}\,\lambda(n))\Big]\\
&\leqq 2\Big(\inf_{k\geqq n/2}P[\Xi(n-k)\geqq 0]\Big)^{-1}P[\Xi(n/2)<-\log(\alpha\sqrt{n}\,\lambda(n))]
\end{aligned}
$$

(according to a well-known inequality; we assume n to be even for simplicity only)

$\leqq CP[\Xi(n/2)<-\log(\alpha\sqrt{n}\,\lambda(n))]$ (after the central limit theorem)

$\leqq C\exp\{-\sqrt{n}\log(\alpha\sqrt{n}\,\lambda(n))\}$ by Cebysev's inequality (and hence $\leqq C(\alpha\sqrt{n})^{-\sqrt{n}}$), as the moment-generating function $E\exp\{u\sqrt{n}\,\Xi(n/2)\}$ is bounded for $u=1$: namely, a direct computation shows it to be equal to

$$\frac{n(n-1)\dots n/2}{(n-\sqrt{n})\dots(n/2-\sqrt{n})}\exp\left\{-\sqrt{n}\sum_{k=n/2}^{n}\frac{1}{k}\right\}\sim\exp\left\{n\sum_{k=0}^{n/2}(n-k)^{-2}\right\}\leqq C.$$

$$\text{II}\qquad \leqq 2P\left[\sup_{k\geqq n/2}\left|X_k-\frac{k}{n}\right|>C\alpha\frac{\log n}{\sqrt{n}}\right]$$

$$\leqq 2P\left[\sup_{k\geqq n/2}|1-e^{-\Xi(n-k)+\gamma(n,k)}|>C\alpha\frac{\log n}{\sqrt{n}}\right]$$

$$\leqq 2P\left[\sup_{k\geqq n/2}|\Xi(n-k)|>C\alpha\frac{\log n}{\sqrt{n}}\right]$$ (as $\Xi(n-k)\geqq-\gamma(n,k)+\log k/n$ and $\sup_{k\geqq n/2}\gamma(n,k)=O(1/n)$).

$$\leqq C e^{-\alpha\log n}\leqq C/n^{\alpha}$$ according to the above argument.

$$\text{III}\qquad \leqq P\left[\sup_{\mu(n)\leqq k\leqq n}\frac{k}{\sqrt{n}}f(\Xi(n-k)-\gamma(n,k))>\alpha\lambda(n)-\sup_{\mu(n)\leqq k\leqq n}\frac{k}{\sqrt{n}}\gamma(n,k)\right]$$

(where $f(t)=|1-e^{-t}-t|$)

$$\leqq P\left[\sup_{\mu(n)\leqq k\leqq n}\frac{k}{\sqrt{n}}[f\wedge 1](\Xi(n-k)-\gamma(n,k))>\alpha\lambda(n)\right]$$

(as $\sup_{\mu(n)\leqq k\leqq n}k\gamma(n,k)\leqq C$)

$$\leqq P\left[\sup_{\mu(n)\leqq k\leqq n}\frac{k}{\sqrt{n}}(\Xi(n-k)-\gamma(n,k))^2>\alpha\lambda(n)\right]$$

$$\leqq P\left[\sup_{n^{\frac{2}{3}}\log n\leqq k\leqq n^{\frac{5}{6}}\log n}(\Xi(n-k)-\gamma(n,k))^2>\frac{\alpha\lambda(n)}{n^{\frac{1}{6}}\log n}\right]$$
$$+P\left[\sup_{n^{\frac{2}{3}}\log n\leqq k\leqq n^{\frac{5}{6}}\log n}(\Xi(n-k)-\gamma(n,k))^2>\frac{\alpha\lambda(n)}{n^{\frac{1}{3}}\log n}\right]$$
$$+P\left[\sup_{n^{\frac{5}{6}}\log n\leqq k\leqq n}(\Xi(n-k)-\gamma(n,k))^2>\frac{\alpha\lambda(n)}{n^{\frac{1}{2}}}\right]$$

the first term of which will be estimated now (the others do not present additional problems); it is

$$\leqq P\left[\sup_{n^{\frac{1}{2}}\log n\leqq k\leqq n^{\frac{2}{3}}\log n}(\Xi(n-k))^2>C\alpha n^{-\frac{5}{12}}\right]$$

(as $(a-a_0)^2>A$ & $a_0\leqq\sqrt{A/4}$ implies $a^2>A/4$; observe that for $k\geqq\mu(n)$ $\gamma(n,k)\leqq C(\sqrt{n}\log n)^{-1}\leqq C n^{-\frac{5}{24}}$)

$$\leqq P\left[\sup_{n^{\frac{1}{2}}\log n\leqq k\leqq n^{\frac{2}{3}}\log n}\mu(n)\,\Xi(n-k)^2>C\alpha\log^2 n\right].$$

But as the moment-generating function of $\sqrt{\mu(n)}\,\Xi(n-\mu(n))$ is bounded for $u=1$, this is $\leqq C\exp\{-\sqrt{C\alpha\log^2 n}\}\leqq C/n^{\sqrt{\alpha}}$ by an argument similar to the above.

To obtain an estimate of IV, we need the following lemma which we will prove first.

Lemma. *For each integer* $l \geqq 4$ *there exists a constant* C_l *such that*

$$P\left[\left|\sum_{i=1}^{k}\left(\tau_i-\frac{n}{(n-i+1)^2}\right)\right|>\frac{2^{\frac{5}{2}}\alpha n\log n}{(n-k)^{\frac{3}{2}}} \text{ for some } k\leqq n-\mu(n)\right]\leqq C_l\frac{\log n}{\mu(n)^{l/2-1}\alpha^l}.$$

Proof. As a first step we show

$$P\left[\sup_{k\leqq n-q}\left|\sum_{i=1}^{k}\rho_i\right|>2\alpha\log n/q^{\frac{3}{2}}\right]\leqq\frac{C_l}{q^{l/l-1}\alpha^l}, \tag{2}$$

where $q\leqq\frac{n}{2}$ and $\rho_i=\frac{1}{n}\left(\tau_i-\frac{n}{(n-i+1)^2}\right)$.

ρ_i has the same distribution as $\frac{\hat{\tau}-1}{(n-i+1)^2}$, where $\hat{\tau}$ is a positive random variable stopping Brownian motion ζ in the sense of Skorokhod [9] such that

$$d\mathscr{L}(\zeta(\hat{\tau}))=e^{-u-1}du \qquad (u\geqq 0).$$

Thus $E\rho_i^l=E(\hat{\tau}-1)^l/(n-i+1)^{2l}$ $(l\geqq 1)$ whence

$$\operatorname{var}\frac{1}{n}\sum_{i=1}^{n-q}\tau_i=C\sum_{i=1}^{n-q}(n+1-i)^{-4}\sim C/q^3$$

for small values of q/n. Therefore it is reasonable to consider the "normalized" variables

$$\eta_k:=q^{\frac{3}{2}}\rho_k.$$

Now truncate:

$$\eta_k':=\begin{cases}\eta_k & (|\eta_k|\leqq\alpha)\\ 0 & \text{otherwise},\end{cases}$$

$$\eta_k'':=\eta_k-\eta_k'.$$

Hence

$$\begin{aligned}
P&\left[\sup_{k\leqq n-q}\left|\sum_{i=1}^{k}\eta_i\right|>2\alpha\log n\right]\\
&\leqq P\left[\sup_{k\leqq n-q}\left|\sum_{i=1}^{k}\eta_i'\right|>2\alpha\log n\right]+P\left[\sup_{k\leqq n-q}|\eta_k''|>0\right]\\
&\leqq\sum_{k=1}^{n-q}P\left[\left|\sum_{i=1}^{k}\eta_i'\right|>2\alpha\log n\right]+\sum_{k=1}^{n-q}P[|\eta_k|>\alpha]\\
&\leqq\sum_{k=1}^{n-q}\left\{\frac{1}{n^{2\alpha}}E\exp\left\{\sum_{i=1}^{k}\eta_i'\right\}+\frac{1}{n^{2\alpha}}E\exp\left\{-\sum_{i=1}^{k}\eta_i'\right\}\right\}+\sum_{k=1}^{n-q}E\eta_k^l\alpha^{-l}
\end{aligned}$$

the second term of which is

$$\begin{aligned}
&=E(\hat{\tau}-1)^l q^{3l/2}\sum_{i=1}^{n-q}(n-i+1)^{-2l}\alpha^{-l}\\
&\leqq C_l q^{-l/2+1}\alpha^{-l}
\end{aligned}$$

for $q \leqq n/2$, as can be seen by an integral approximation of the sum. As $q \leqq n$ it remains to show that

$$E \exp\left\{\sum_{i=1}^{k} \eta_i'\right\} \leqq C e^{\alpha}$$

(and similarly for the parallel term). But

$$\begin{aligned} E \exp\left\{\sum_{i=1}^{k} \eta_i'\right\} &= \prod_{i=1}^{k} E e^{\eta_i'} \\ &= \prod_{i=1}^{k} E\left(1+\eta_i' + \tfrac{1}{2}(\eta_i')^2 e^{\theta_i' \eta_i'}\right) \qquad (\theta_i' \in (0,1)) \\ &\leqq \prod_{i=1}^{k} (1+|E\eta_i'| + \tfrac{1}{2} e^{\alpha} E \eta_i^2) \end{aligned}$$

and

$$\begin{aligned} |E\eta_i'| &\leqq E|\eta_i''| \\ &\leqq (E|\eta_i''|^2 P[|\eta_i''|>0])^{\frac{1}{2}} \\ &\leqq (E|\eta_i|^2 E|\eta_i|^4)^{\frac{1}{2}} \alpha^{-2} \\ &\leqq \frac{C q^{\frac{9}{2}}}{\alpha^2 (n-i+1)^6} \, (E|\eta_i|^2 \leqq C q^3 (n-i+1)^{-4}) \end{aligned}$$

together imply

$$\begin{aligned} \log E e^{\sum_{i=1}^{k} \eta_i'} &\leqq \sum_{i=1}^{k} \log(1+|E\eta_i'| + \tfrac{1}{2} e^{\alpha} E\eta_i^2) \\ &\leqq \sum_{i=1}^{n-q} (|E\eta_i'| + \tfrac{1}{2} e^{\alpha} E \eta_i^2) \\ &\leqq C \frac{q^{\frac{9}{2}}}{\alpha^2 q^5} + C e^{\alpha} \frac{q^3}{q^3} \leqq C e^{\alpha} \end{aligned}$$

whence (2) follows.

The statement of the lemma now follows by summing up:

$$\begin{aligned} &P\left[\left|\sum_{i=1}^{k} \rho_i\right| > \frac{2^{\frac{5}{2}} \alpha \log n}{(n-k)^{\frac{3}{2}}} \text{ for some } k \leqq n-\mu(n)\right] \\ &\leqq \sum_{\substack{[\log \mu(n)/\log 2] \\ \leqq j < [\log n/\log 2]}} P\left[\left|\sum_{i=1}^{k} \rho_i\right| > \frac{2^{\frac{5}{2}} \alpha \log n}{(n-k)^{\frac{3}{2}}} \text{ for some } k \in [n-2^{j+1}, n-2^j]\right] \\ &\quad + P\left[\sup_{k \leqq n/2} \left|\sum_{i=1}^{k} \rho_i\right| > \frac{2\alpha \log n}{(n/2)^{\frac{3}{2}}}\right] \\ &\leqq \sum_{\substack{[\log \mu(n)/\log 2] \\ \leqq j < [\log n/\log 2]}} P\left[\sup_{k \leqq n-2^j} \left|\sum_{i=1}^{k} \rho_i\right| > \frac{2\alpha \log n}{(2^j)^{\frac{3}{2}}}\right] + \frac{C_l}{2^{l/2-1} \alpha^l} \\ &\leqq C_l \sum_{\cdots} (2^{-j})^{l/2-1} \alpha^{-l} + \frac{C_l}{2^{l/2-1} \alpha^l} \leqq C_l \sum_{\cdots} \frac{1}{\mu(n)^{l/2-1} \alpha^l} \leqq \frac{C_l \log n}{\mu(n)^{l/2-1} \alpha^l}. \end{aligned}$$

According to this lemma, therefore, it suffices to consider the quantity

$$P\Bigg[\sup_{\left|s-\sum_{i=1}^{n-k} n(n-i+1)^{-2}\right| \leqq 2^{\frac{5}{2}} \alpha n \log n / k^{\frac{3}{2}}} \left|\zeta\left(\sum_{i=1}^{n-k} n(n-i+1)^{-2}\right)-\zeta(s)\right|>\alpha \frac{n}{k} \lambda(n) \quad \text{for some } k \in[\mu(n), n]\Bigg]. \tag{3}$$

It can be considerably simplified by the use of

$$\left|\sum_{i=1}^{n-k} \frac{n}{(n-i+1)^2}-\left(\frac{n}{k}-1\right)\right| \leqq C \frac{n}{k^2},$$

which follows from Euler's summation formula. Observe that $\frac{n}{k^2} \leqq \alpha \frac{n \log n}{k^{\frac{3}{2}}}$, which allows us to replace (3) by the larger quantity

$$P\Bigg[\sup_{\left|s-\frac{n}{k}+1\right| \leqq \frac{8 \alpha n \log n}{k^{\frac{3}{2}}}}\left|\zeta\left(\frac{n}{k}-1\right)-\zeta(s)\right|>\alpha \frac{n}{k} \lambda(n) \text{ for some } k \in[\mu(n), n]\Bigg]$$

which is

$$\leqq P\Bigg[\sup_{\substack{\left|s-\frac{n}{k}+1\right| \leqq C \alpha \frac{\log n}{\sqrt{n}} \\ n/3 \leqq k \leqq n'}}\left|\zeta\left(\frac{n}{k}-1\right)-\zeta(s)\right|>\alpha \lambda(n)\Bigg]$$

$$+P\Bigg[|\zeta(r)-\zeta(s)|>\alpha(r+1) \lambda(n) \text{ for some } s, r \geqq 1 \text{ such that } |s-r| \leqq \frac{8 \alpha(r+1)^{\frac{3}{2}} \log n}{\sqrt{n}}\Bigg]$$

$$=A+B\left(\text{here } \frac{n}{k}-1 \text{ was substituted by } r\right).$$

$$A \leqq C \frac{\sqrt{n}}{\alpha \log n} P\left[\zeta\left(C \alpha \frac{\log n}{\sqrt{n}}\right)>\alpha \lambda(n)\right]$$

$$\leqq \frac{C n^{\frac{1}{4}}}{\alpha^{\frac{3}{2}} \lambda(n) \sqrt{\log n}} \exp \left\{-\frac{C \alpha \lambda(n)^2 \sqrt{n}}{\log n}\right\}$$

$$=\frac{C n^{\frac{1}{2}}}{\alpha^{\frac{3}{2}}(\log n)^{\frac{3}{2}}} \exp \{-C \alpha \log n\} \leqq \frac{C}{n^\alpha}.$$

$$B \leqq P\left[|\zeta(r)-\zeta(s)|>\alpha r \lambda(n) \text{ for some } s, r \geqq 1 \text{ such that } |s-r| \leqq C \alpha r^{\frac{3}{2}} n^{-\frac{1}{2}} \log n\right]$$

$$\leqq P\Bigg[\sup_{\substack{s, r \geqq 1 \\ |s-r| \leqq C \alpha r^{\frac{3}{2}} n^{-\frac{1}{2}} \log n}}\left|\frac{1}{r} \zeta(r)-\frac{1}{s} \zeta(s)\right|>\alpha \lambda(n)\Bigg]$$

$$+P\Bigg[\left|\frac{1}{s}-\frac{1}{r}\right||\zeta(s)|>\alpha \lambda(n) \text{ for some } s, r \in[1, \sqrt{n}] \text{ such that } |s-r| \leqq C \alpha r^{\frac{3}{2}} n^{-\frac{1}{2}} \log n\Bigg]$$

$$=B'+B''.$$

By introducing the Brownian motion $\hat{\zeta}(s)=s\zeta(1/s)$ B' becomes

$$\leqq P\Big[\sup_{\substack{|s-r|\leqq C\alpha r^{\frac{1}{2}} n^{-\frac{1}{2}}\log n\\ s,r\leqq 1}} |\hat{\zeta}(s)-\hat{\zeta}(r)|>\alpha\lambda(n)\Big]\leqq\frac{C}{n^{\alpha}} \quad \text{(as above).}$$

$$B''\leqq P\big[|\zeta(s)|>\alpha s^{\frac{1}{2}}\lambda(n)\, n^{\frac{1}{2}}\log^{-1} n \text{ for some } s\in[1,\sqrt{n}]\big]$$

$$\leqq P\Big[\sup_{s\in[1,n^{\frac{1}{2}}\log^{-2} n]} |\zeta(s)|>\alpha\lambda(n)\, n^{\frac{1}{2}}\log^{-1} n\Big]$$

$$+P\Big[\sup_{s\in[n^{\frac{1}{2}}\log^{-2} n,\, n^{\frac{1}{2}}]} |\zeta(s)|>\alpha\lambda(n)\, n^{\frac{3}{4}}\log^{-2} n\Big],$$

both of which are clearly $\leqq C/n^{\alpha}$.

From the same argument it follows that $V\leqq C/n^{\alpha}$. Also VI can be treated similarly; we omit these details.

As $\frac{\mu(n)}{n}=\frac{\log n}{\sqrt{n}}$, a function occurring in II, the problem of giving an estimate of VII can be reduced to the estimation of $VIII$. Putting $\Delta(1)_s=\zeta_0(s)-s\,\zeta_0(1)$ (ζ_0 some Brownian motion having $\zeta_0(0)=0$) we obtain (using the symmetry $s\rightsquigarrow 1-s$)

$$\text{VIII} \qquad \leqq P\Big[\sup_{0\leqq s\leqq \log n/\sqrt{n}} |\zeta_0(s)|>\alpha\lambda(n)\Big]+P\big[\zeta_0(1)>C\alpha n^{\frac{1}{4}}\big]$$

$$\leqq P\Big[\sup_{0\leqq s\leqq 1} |\zeta_0(s)|>\alpha\sqrt{\log n}\Big]+C/n^{\alpha}\leqq C/n^{\alpha}.$$

In a second step of the proof we consider the processes Δ_n and Δ for $t\leqq 1$. We are going to construct a mapping Π_n: $\mathscr{C}_{01}\rightsquigarrow\mathscr{C}_{01}$ having the properties

(i) $\mathscr{L}(\Pi_n\Delta)=\mathscr{L}(\tilde{\Delta}_n)$ where

$$\tilde{\Delta}_n(t)=\begin{cases}\Delta_n(t) & \text{for } t=i/\kappa(n) \quad (i=1,\dots,\kappa(n))\\ \text{linearly interpolated otherwise;}\end{cases}$$

here $\kappa(n)\leqq n$ will be determined later (we agree to skip minor difficulties arising from the fact that $i/\kappa(n)$ may not be of the form j/n);

(ii) $P[\|\Pi_n\Delta-\Delta\|_{\infty}>\Lambda(n)]\leqq C_{\varepsilon}/n$ where $\Lambda(n)=C_{\varepsilon}/n^{1/6-\varepsilon}(\varepsilon>0)$.

To this end let

$$w(i,n):=\Pi_n\Delta\left(\frac{i+1}{\kappa(n)}\right)-\Pi_n\Delta\left(\frac{i}{\kappa(n)}\right)=\tilde{\pi}_n\left(\Delta\left(\frac{i+1}{\kappa(n)}\right)-\Delta\left(\frac{i}{\kappa(n)}\right)\right)$$

($i=1,\dots,\kappa(n)-1$), where $\tilde{\pi}_n$ is a mapping constructed for $\Delta(1/\kappa(n))$ in the same way as π_n was for $\Delta(1)$. (According to our convention we deal with $n/\kappa(n)$ as if these numbers were integers.) This defines Π_n.

To show (ii) we note that

$$P[\sup_t \|\Pi_n\Delta(t)-\Delta(t)\|_{s,\infty}>\Lambda(n)]\leqq CP[\|\Pi_n\Delta(1)-\Delta(1)\|_{\infty}>\Lambda(n)]$$

which in turn is

$$\begin{aligned}&\leqq C\sum_{j=1}^{\kappa'(n)} P\left[\left|\sum_{i=1}^{\kappa(n)} w(i,n)(s=j/\kappa'(n))\right|>\Lambda(n)\right]\\&\quad+CP\big[\sup_{|s''-s'|\leqq 1/\kappa'(n)} |\Delta_n(1)_{s''}-\Delta_n(1)_{s'}|>\Lambda(n)\big]\\&\quad+CP\big[\sup_{|s''-s'|\leqq 1/\kappa'(n)} |\Delta(1)_{s''}-\Delta(1)_{s'}|>\Lambda(n)\big];\end{aligned} \tag{4}$$

here $\kappa'(n)$ is a function of the same order as $\kappa(n)$.

Now from the result of our first step it follows that

$$E\,|w(i,n)_{s=j/\kappa(n)}|^l \leqq C_l\,\lambda\left(\frac{n}{\kappa(n)}\right)^l \kappa(n)^{-l/2} \qquad (l=4,6,\ldots).$$

Observing that in the multinomial expansion of $\sum_{i=1}^{\kappa(n)} w(i,n)_s$ the monomials having some exponent equal to 1 have expectation 0 (such that the number of monomials with non-vanishing expectation is of order $\kappa(n)^{l/2}$) we conclude

$$E\left(\sum_{i=1}^{\kappa(n)} w(i,n)_s\right)^l \leqq C_l\,\lambda\left(\frac{n}{\kappa(n)}\right)^l.$$

Consequently, by Cebysev's inequality, the first term in (4) is

$$\leqq C_l\,\Lambda(n)^{-l}\,\frac{(\log n)^l}{n^{l/4}}\,\kappa(n)^{l/4}\,\kappa'(n). \tag{5}$$

Moreover, putting $\kappa(n)=\log n/\Lambda(n)^2$, we obtain C_l/n as a bound for the second and third term. Therefore $\Lambda(n)$ makes $(4)\leqq C_\varepsilon/n$ $(\varepsilon>0)$ if l is chosen to be large enough.

Observe that from (5) we have for $\beta>1$

$$P[\|\Pi_{\beta n}\Delta-\Delta\|_\infty>\beta^{\frac{1}{2}}\Lambda(\beta n)]\leqq\frac{C_\varepsilon}{\beta n}$$

with the same C_ε, provided that $\kappa(\beta n)$ is substituted by $\beta\kappa(n)$. This leads to the third step of the proof. We extend the mapping Π_n in a natural way to the whole space $\mathscr{C}_0$. Then

$$\begin{aligned}&P\left[\sup_{t\geqq 0}\frac{1}{t\vee 1}\,\|\Pi_n\Delta(t)-\Delta(t)\|_\infty>\Lambda(n)\right]\\&\quad\leqq P\left[\sup_{t\geqq 1}\frac{1}{t}\,\|\Pi_n\Delta(t)-\Delta(t)\|_\infty>\Lambda(n)\right]\\&\qquad+P\big[\sup_{t\leqq 1}\|\Pi_n\Delta(t)-\Delta(t)\|_\infty>\Lambda(n)\big]\\&\quad\leqq\sum_{k=1}^{\infty} P\big[\sup_{t\leqq 2^k} 2^{-k/2}\,\|\Pi_n\Delta(t)-\Delta(t)\|_\infty>2^{k/2}\Lambda(n)\big]+\frac{C_\varepsilon}{n}\\&\quad\leqq C_\varepsilon\sum_{k=1}^{\infty}2^{-k}n^{-1}+\frac{C_\varepsilon}{n}=\frac{2C_\varepsilon}{n},\end{aligned}$$

whence

$$P[\rho(\Pi_n \Delta, \Delta) > \Lambda(n)] \leqq \frac{C_\varepsilon}{n},$$

implying

$$L(\mathscr{L}(\Pi_n \Delta), \mathscr{L}\Delta) = o(n^{-1/6+\varepsilon}) \qquad (\varepsilon > 0),$$

(cp. Lemma 1.2 of [7]). The proof is now concluded by the observation that $L(\mathscr{L}(\Pi_n \Delta), \mathscr{L}\Delta_n) = o(n^{-1/6+\varepsilon})$ which follows from the Dvoretzky-Kiefer-Wolfowitz inequality.

3. In this section we return to the consideration of $t^{-1}\Delta(t)$ for $t \geqq 1$. We are interested in comparing the probabilities of

$$Z_n = \left[g_1\left(\frac{k}{n}, s\right) < \sqrt{n}\, D_k(s) < g_2\left(\frac{k}{n}, s\right) \text{ for all } k \geqq n,\ s \in [0,1]\right]$$

and

$$Z = [g_1(t, s) < t^{-1}\Delta(t) < g_2(t, s) \text{ for all } (t, s) \in [1, \infty) \times [0, 1]].$$

Here g_i $(i = 1, 2)$ denote real-valued continuous functions on $[1, \infty) \times [0, 1]$ subject to the condition

$$\liminf_{t^{-1}s(1-s)\to 0} g_2(t, s) > 0,$$

$$\limsup_{t^{-1}s(1-s)\to 0} g_1(t, s) < 0.$$

We note that the following theory can easily be extended to the case that the "bounds" g_1 are defined on $[0, \infty) \times [0, 1]$.

Theorem 3. $|PZ_n - PZ| = o(n^{-1/6+\varepsilon})$ $(\varepsilon > 0)$.

Proof. It suffices to show that $PZ^\delta - PZ = O(\delta)$; for then the statement appears to be a consequence of Theorem 2. Without losing generality we put $g_1 \equiv -\infty$ and then omit the subscript of g_2. It is sufficient to show that the random variable

$$\Delta \rightsquigarrow G(\Delta) := \max_{\substack{t \geqq 1 \\ \frac{1}{2} \leqq s \leqq 1}} (t^{-1}\Delta(t) - g(t))$$

has a density bounded in a neighborhood of 0. We compute this density.

$$\begin{aligned} P[G(\Delta) < a_0] &= \int P[G(u(1-s) + \Gamma(t, s)) < a_0]\, \varphi(u)\, du \\ &= \iint 1_{[G(u(1-s)+\Gamma(t,s)) < \alpha_0]}\, \varphi(u)\, du\, dP, \end{aligned}$$

where $\Gamma(t, s) = \zeta(s - \frac{1}{2}) - s\zeta(\frac{1}{2}) + \Delta(t-1)_s$ for some Brownian motion ζ $(\zeta(0) = 0)$, φ is the normal density with variance $\frac{1}{2}$. Given Γ $\alpha = G(u(1-s) + \Gamma)$ is a strictly increasing function of u, hence invertible:

$$u = G^{-1}(\alpha | \Gamma).$$

To each α we make correspond the coordinates (t_α, s_α) of the maximum α of

$$(t, s) \rightsquigarrow \frac{G^{-1}(\alpha|\Gamma)(1-s) + \Gamma(t, s)}{t} - g(t, s)$$

(since these coordinates may not be unique, let t_α be the smallest t-coordinate and take s_α to be maximal under this condition). Then

$$\frac{1}{h} P[\alpha < G(\Delta) < \alpha + h] = \frac{1}{h} \iint_{\alpha}^{\alpha+h} \varphi\big(G^{-1}(\bar{\alpha} \mid \Gamma)\big)\, F_\Gamma(d\bar{\alpha})\, dP$$

(with $F_\Gamma = \mathscr{L}(G(u \mid \Gamma) \mid du)$).

$$\leqq \frac{C}{h} \int F_\Gamma\{[\alpha, \alpha+h]\}\, dP$$

$$\leqq C \int \frac{t_\alpha}{1-s_\alpha}\, dP, \qquad \text{for } F_\Gamma\{[\alpha, \alpha+h]\} \leqq \frac{C h t_\alpha}{1-s_\alpha},$$

as follows from

$$\max\{t^{-1}[(G^{-1}(\alpha \mid \Gamma) + h t_\alpha (1-s_\alpha)^{-1})(1-s) + \Gamma(t,s)] - g(t,s)\}$$
$$\geqq t_\alpha^{-1}[G^{-1}(\alpha \mid \Gamma)(1-s_\alpha) + \Gamma(t_\alpha, s_\alpha)] - g(t_\alpha, s_\alpha) + h = \alpha + h.$$

Thus $\dfrac{d\mathscr{L}(G(\Delta))}{d\alpha}$ exists and is majorized by $C E t_\alpha (1-s_\alpha)^{-1}$ provided the latter is finite. But

$$E t_\alpha (1-s_\alpha)^{-1} = \sum_{i,j=1}^{\infty} E[t_\alpha (1-s_\alpha)^{-1} \mid 2^{-i-1} < 1-s_\alpha \leqq 2^i \,\&\, 2^{j-1} \leqq t_\alpha < 2^j]$$
$$\leqq \sum_{i,j=1}^{\infty} 2^{i+j+1} P[2^{-i-1} < 1-s_\alpha \leqq 2^{-i} \,\&\, 2^{j-1} \leqq t_\alpha < 2^j],$$

the boundedness of which follows from the inequality

$$P[\eta/2 < 1-s_\alpha < \eta \,\&\, \delta/2 < t_\alpha < \delta] < C e^{-C\delta/\eta} \qquad (\eta > 0, \delta \geqq 2)$$

to be proved now: its left side is

$$\leqq P\Big[\sup_{\substack{0 \leqq 1-s \leqq \eta \\ 1 \leqq t \leqq \delta}} \Gamma(t,s) > b\delta/4\Big]$$
$$+ P[G^{-1}(\alpha \mid \Gamma) > b\delta/4\eta]$$
$$= \mathrm{I} + \mathrm{II} \qquad \big(b := \liminf_{t^{-1} s(1-s) \to 0} g(t,s) > 0\big).$$

I $\qquad \leqq P\Big[\sup_{0 \leqq 1-s \leqq \eta} \zeta(s) > b\delta/12\Big]$

$$+ P[\zeta(\tfrac{1}{2}) > b\delta/12\eta]$$
$$+ P\Big[\sup_{\substack{1 \leqq t \leqq \delta \\ 0 \leqq 1-s \leqq \eta}} \Delta(t-1)_s > b\delta/12\Big]$$

$$\leqq C e^{-C(b\delta)^2/\eta} + C P\left[\sup_{0 \leqq 1-s \leqq \eta} \Delta(\delta-1)_s \geqq \frac{b\delta}{24}\right] \leqq C e^{-C\delta/\eta}.$$

II $\qquad \leqq P\left[\zeta(\tfrac{1}{2}) \geqq \dfrac{b\delta}{4\eta} - 2\alpha - 2g(1, \tfrac{1}{2})\right] \leqq C e^{-C\left(\frac{b\delta}{\eta}\right)^2}.$

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