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Inequalities for the Expectation of Δ-Monotone Functions

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Summary. For some subsets of the set of all Δ-monotone functions on $[0,1]^n$ we characterize distribution functions F, G such that $E_F f \leqq E_G f$ for all f within these subsets. Furthermore, we determine sharp upper and lower bounds of integrals of functions in these subsets w.r.t. all distributions with fixed marginals and give some applications of these results.

1. Introduction

For functions $f\colon [0,1]^n \to R^1$ the multivariate difference operator Δ is defined by

$$\Delta_x^y f = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{0,1\}^n} (-1)^{\sum_{i=1}^n \varepsilon_i} f(\varepsilon_1 x_1 + (1-\varepsilon_1) y_1, \ldots, \varepsilon_n x_n + (1-\varepsilon_n) y_n)$$

where $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in[0,1]^n$. f is called Δ-monotone if $\Delta_x^y f \geqq 0$ for all $x \leqq y$. f is called Δ-antitone if $-f$ is Δ-monotone.

If f is Δ-monotone and right continuous, then f determines a measure μ on $[0,1]^n \mathscr{B}^n$ by $\mu((x,y])=\Delta_x^y f$. Examples of Δ-monotone functions are absolutely continuous functions f with

$$\frac{\partial^n}{\partial x_1,\ldots,\partial x_n} f \geqq 0 \text{ a.s.,} \qquad f(x_1,\ldots,x_n)=\min\{x_i, 1\leqq i \leqq n\}$$

and for $t\in[0,1]^n$

$$f_t(x)=1_{[t,1]}(x),$$

while for $n\geqq 3$

$$g_t(x)=1_{[0,t]}(x)$$

is (in contrast to $n=2$) not Δ-monotone.

We consider the following two subsets M_1, M_2 of all Δ-monotone functions.

$$M_1 = M_1([0,1]^n)=\{f\colon [0,1]^n \to R^1;$$

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f right continuous, $f(\dots, t_{i_1}, \dots, t_{i_k}, \dots)$ is
Δ-monotone on $[0,1]^{n-k}, \forall k \leqq n-1, \forall t_{i_1}, \dots, t_{i_k} \in [0,1]\}$
$M_2 = \{f: [0,1]^n \to R^1; f$ right continuous,
f Δ-monotone and $f(x)=0$ if there is an $i \leqq n$ with $x_i = 0\}$.

M_1 contains the set of all n-dim. distribution functions. The elements of M_2 are not necessarily nondecreasing.

In the present paper we prove characterizations of distributions F, G on $[0,1]^n$ such that $E_F f \leqq E_G f$ for all $f \in M_1$ as well as for all $f \in M_2$. Furthermore, we derive bounds for the expectation of f in the class of all distributions with fixed marginals. Similar questions have been considered in the case $n=2$ by Cambanis, Simons and Stout (1976) for the class M of all Δ-monotone functions. In contrast to comparison of distributions w.r.t. M_1, M_2, $\int f\,dF \leqq \int f\,dG$ for all $f \in M$ implies, that F, G have the same marginal distribution functions (df's).

2. Characterizations of M_1

Let

$$G_m = \{(i_1/2^m, \dots, i_n/2^m); \quad 0 \leqq i_j \leqq 2^m,\ 1 \leqq j \leqq n\}$$

be the lattice with side length 2^{-m}. For a nondecreasing function $f: G_m \to R^1$ with $f(0)=0$ define $z_0 \in G_m$ such that

a) $$f(z_0) = \min\{f(z);\ z \in G_m, f(z) > 0\}$$

and (1)

b) $$z \in G_m,\ z \leqq z_0,\ z \neq z_0 \ \text{implies}\ f(z)=0.$$

Furthermore, define

$$M_1(G_m) = \{f: G_m \to R^1; f(\dots, t_{i_1}, \dots, t_{i_k}, \dots)$$
$$\text{is } \Delta\text{-monotone for all } k \leqq n-1, t_{i_j} \in \{l/2^m;\ 0 \leqq l \leqq 2^m\},\ 1 \leqq j \leqq k\},$$

where a function $f: G_m \to R^1$ is called Δ-monotone if $\Delta_x^y f \geqq 0$ for all $x, y \in G_m, x \leqq y$.

Lemma 1. *For $f \in M_1(G_m)$ with $f(0)=0$ define*

$$f_1(z) = f(z) - f(z_0)\, 1_{[z_0, 1]}(z), \qquad z \in G_m$$

where z_0 is defined as in (1). *Then f_1 is an element of $M_1(G_m)$.*

Proof. To prove Δ-monotonicity of a function f on G_m it is sufficient to consider points x, y with $y_i - x_i \in \{0, 1/2^m\}$ since all other Δ-differences can be composed by Δ-differences of this type.

Defining $G(z_0)$ to be the set of all minimal elements of $\{z \in G_m; z_0 \leqq z\}$ we have

$$\Delta_x^y 1_{[z_0,1]} = 1 \qquad \text{for } x, y \text{ with } y_i - x_i \in \{0, 1/2^m\}$$

if and only if $[x, y]$ contains only the point y from $G(z_0)$. In all other cases we have

$$\Delta_x^y 1_{[z_0, 1]} = 0$$

and, therefore, in these cases

$$\Delta_x^y f_1 = \Delta_x^y f - f(z_0) \Delta_x^y 1_{[z_0, 1]} = \Delta_x^y f \geqq 0.$$

If $[x, y]$ contains exactly one point from $G(z_0)$ we decrease successively those components of y which are greater than the corresponding components of z_0 and, simultaneously, the same components of x. In this way we get a sequence

$$y \geqq y_1 \geqq \ldots \geqq y_r = z_0$$

and

$$x \geqq x_1 \geqq \ldots \geqq x_r \quad \text{with} \quad x_r \leqq z_0, \quad x_r \neq z_0.$$

Now by definition of $M_1(G_m)$ we have

$$\Delta_x^y f \geqq \Delta_{x_1}^{y_1} f \geqq \ldots \geqq \Delta_{x_r}^{z_0} f = f(z_0)$$

and, therefore,

$$\Delta_x^y f_1 = \Delta_x^y f - f(z_0) \geqq 0.$$

We are now ready to prove the following characterization of M_1.

Theorem 2. *$f \in M_1$ if and only if there exist $\alpha \in R^1$, $\alpha_{i,j} \geqq 0$, $t_{i,j} \in [0,1]^n$, $1 \leqq i \leqq m_j$, $m_j \in \mathbb{N}$, $j \in \mathbb{N}$ such that the sequence of functions*

$$f_j(x) = \alpha + \sum_{i=1}^{m_j} \alpha_{i,j} 1_{[t_{i,j}, 1]}(x), \qquad x \in [0,1]^n, \tag{2}$$

converges pointwise to $f(x)$. Furthermore, f_j can be chosen nonincreasing in j.

Proof. The elements f_j defined in (2) are in M_1; so only one direction remains to be shown since

$$\lim_{j \to \infty} \Delta_x^y f_j = \Delta_x^y (\lim_{j \to \infty} f_j).$$

Let $\alpha = f(0) \in R^1$ and denote the restriction of f on G_m again by f. Then $f^1(x) = f(x) - \alpha$, $x \in G_m$, is in $M_1(G_m)$ and there exists at least one point $x \in G_m$ with $f^1(x) = 0$. If $f^1 \not\equiv 0$, then let $z_0 \in G_m$ be a point with property (1) which exists since elements of M_1 are nondecreasing. The function

$$f^2(x) = f^1(x) - f^1(z_0) 1_{[z_0, 1]}(x), \qquad x \in G_m$$

is by Lemma 1 in $M_1(G_m)$ and has at least two points x in G_m with $f^2(x) = 0$. Going on by induction we get a sequence f^l of functions in $M_1(G_m)$, $1 \leqq l \leqq m_j + 2$ and $z_l \in [0,1]^n$, $0 \leqq l \leqq m_j - 1$ such that

a) $$f^{l+1}(x) = f^l(x) - f^l(z_{l-1}) 1_{[z_{l-1}, 1]}(x), \qquad x \in G_m, \ 1 \leqq l \leqq m_j + 1$$

and

b) $$f^{m_j + 2} \equiv 0.$$

This implies

$$f(x)=\alpha+f^1(x)=\alpha+f^2(x)+f^1(z_0)\,1_{[z_0,1]}(x)$$
$$=\alpha+\sum_{l=1}^{m_j} f^l(z_{l-1})\,1_{[z_{l-1},1]}(x), \qquad x\in G_m.$$

Since $f^l(0)=0$, we have $f^l(z_{l-1})\geqq 0$. With $\alpha_{i,j}=f^i(z_{i-1})$ and $t_{i,j}=z_{i-1}$ we get:

$$f_m=f^{m_j+1} \quad \text{is of type (2) and } f(x)=f_m(x), \qquad x\in G_m.$$

By definition of f the lattice approximation f_m is nonincreasing in m. Right continuity of f implies that

$$f(x)=\lim_{m\to\infty}\inf\{f(y);\, y\in G_m, y\geqq x\}.$$

So there exists a sequence $y_m\downarrow x$, $y_m\in G_m$ such that

$$f(x)=\lim_{m\to\infty} f(y_m)=\lim_{m\to\infty} f_m(y_m)=\lim_{m\to\infty} f_m(x).$$

3. Inequalities for the Expectation of Δ-Monotone Functions

For a df. F on $[0,1]^n$ define

$$h_F(t)=P_F([t,1]), \qquad t\in[0,1]^n$$

where P_F is the probability measure associated with F.

Theorem 3. *Let F, G be* df's *on* $[0,1]^n$. *Then*

a) $E_F f\leqq E_G f$ *for all* $f\in M_1$ *if and only if* $h_F\leqq h_G$
b) $h_F\leqq h_G$ *implies that* $E_F f\leqq E_G f$ *holds for all* $f\in M_2$.

Proof. a) is immediate from Theorem 2, since it is sufficient to consider functions of the type $1_{[t,1]}(x)$.

b) For the proof of b) we use the following integration by parts formula.

Let f, g be real functions on $[0,1]^n$ such that $\int(\Delta_0^x f)\,dg(x)$ exists (as weak net integral) then $\int(\Delta_x^1 g)\,df(x)$ exists and both integrals are equal.

For $n=2$ this formula has been proved by Hildenbrand (1963) pg. 127. The proof for general n can be given along similar lines.

For $f\in M_2$ we have $f(x)=\Delta_0^x f$ and, therefore,

$$E_F f=\int f\,dF=\int(\Delta_0^x f)\,dF=\int(\Delta_x^1 F)\,df=\int h_F\,df\leqq\int h_G\,df=E_G f.$$

Remark 1. a) For $n=2$ and when F, G have the same marginals the condition $h_F\leqq h_G$ is equivalent to $F\leqq G$ so that the statement of Theorem 3 a) for this special case is contained in Theorem 1 of Cambanis, Simons and Stout (1976).

b) The ordering of the survival functions h_F can be managed in a number of examples by means of Schur convex functions. Consider Nevius, Proschan and Sethuraman (1977), Sect. 3.5 of Marshall and Olkin (1974) and Tong (1977).

c) By an approximation argument the statement of Theorem 3 holds true also for functions $f\colon R^n \to R^1$ which have the properties of functions in M_1 and which are integrable w.r.t. F, G.

Denote by $\mathscr{H}(F_1, \ldots, F_n)$ the set of all n-dimensional df's F with marginals $F_1, \ldots, F_n$ and define

$$\bar{F}(x) = \min_{1 \le i \le n} F_i(x_i), \qquad \underline{F}(x) = \left(\sum_{i=1}^{n} F_i(x_i) - (n-1) \right)_+ \tag{3}$$

for $x = (x_1, \ldots, x_n)$ where $a_+ = \max\{a, 0\}$ and

$$h_1(x) = 1 - \max_{1 \le i \le n} F_i(x_i -)$$

and (4)

$$h_2(x) = \left(1 - \sum_{i=1}^{n} F_i(x_i -) \right)_+ .$$

Furthermore, define

$$z_i' = \inf\{x; F_i(x) > 0\}$$

and (5)

$$z_i'' = \sup\{x; F_i(x) < 1\}.$$

The proof of the following lemma can be derived from Theorems 2, 3 of Dall'Aglio (1972) which give a characterization of $\mathscr{H}(F_1, \ldots, F_n)$ by $\underline{F}$, $\bar{F}$.

Lemma 4. a) *$F \in \mathscr{H}(F_1, \ldots, F_n)$ if and only if $h_2 \leqq h_F \leqq h_1$.*

b) *$h_1 = h_{\bar{F}}$ and $\bar{F} \in \mathscr{H}(F_1, \ldots, F_n)$. $\underline{F} \in \mathscr{H}(F_1, \ldots, F_n)$ if and only if*

1) $n = 2$

or (6)

2) *$n \geqq 3$, at least three of the F_i are non degenerate and*

$$\sum_{i=1}^{n} F_i(z_i' -) \geqq n - 1 \quad \text{or} \quad \sum_{i=1}^{n} F_i(z_i'' -) \leqq 1.$$

If $\underline{F} \in \mathscr{H}(F_1, \ldots, F_n)$, then $h_2 = h_{\underline{F}}$. In all cases distinct from (6) *there is no $H \in \mathscr{H}(F_1, \ldots, F_n)$ with $h_H \leqq h_F$, $\forall F \in \mathscr{H}(F_1, \ldots, F_n)$.*

From Theorem 3, Lemma 4 and Remark 1 we get immediately the following result. For $n = 2$ consider Cambanis, Simons and Stout (1976).

Theorem 5. a) *$E_{\bar{F}} f = \sup\{E_F f;\ F \in \mathscr{H}(F_1, \ldots, F_n)\}$ for all $f \in M_1 \cup M_2$.*

b) *Under condition* (6) *we have $E_{\underline{F}} f = \inf\{E_F f;\ F \in \mathscr{H}(F_1, \ldots, F_n)\}$ for all $f \in M_1$.*

c) *$\int h_2 \, df \leqq \inf\{E_F f;\ F \in \mathscr{H}(F_1, \ldots, F_n)\}$, for all $f \in M_2$.*

By Theorem 5 and Lemma 4 we get a sharp upper bound for $\{E_F f;\ F \in \mathscr{H}(F_1, \ldots, F_n)\}$ for $f \in M_1 \cup M_2$ while a sharp lower bound for all $f \in M_1 \cup M_2$ only exists under condition (6).

We now want to derive sharp lower bounds for a subset of M_1; the lower bound depending on the elements of this subset.

Define M_n^2 to be the set of all functions f: $[0,1]^n \to R$ with the following properties a), b)

a) There exist $f_i \in M_1([0,1]^2)$, $1 \leqq i \leqq n-1$ such that

$$f(x_1, \ldots, x_n) = f_{n-1}(f_{n-2}(\ldots f_2(f_1(x_1, x_2), x_3), \ldots), x_n),$$

for $x_1, \ldots, x_n \in [0,1]$.

b) For two df's F, G on $[0,1]^2$ $h_F \leqq h_G$ implies $h_{f_i(F)} \leqq h_{f_i(G)}$ $1 \leqq i \leqq n-1$, where $f_i(F)$ is the df. of the image of F under f_i.

An example for an element of M_n^2 is $f(x_1, \ldots, x_n) = \min\{x_i, 1 \leqq i \leqq n\}$.

The following lemma can be proved by a modification of the proof of Lemma 1, pg. 216 of Ferguson (1967). For a real df. F and a random variable X with df. F define

$$F(x, \lambda) = P(X < x) + \lambda P(X = x), \qquad x \in R^1, \ \lambda \in [0,1].$$

Lemma 6. *Let X, U be real, independent random variables. Let X have df. F_1 and let U be $R(0,1)$-distributed. Let, furthermore, F_2 be a real df. and define*

$$Y = F_2^{-1}(1 - F_1(X, U)).$$

Then the random variable (X, Y) has the df.

$$\underline{F}(x, y) = (F_1(x) + F_2(y) - 1)_+.$$

Now let $F_1, \ldots, F_n$ be n real df.s, let $f \in M_n^2$ with associated f_i, $1 \leqq i \leqq n-1$ and let $U_1, \ldots, U_{n-1}$ be stoch. independent $R(0,1)$-distributed. Then define inductively random variables $V_1, \ldots, V_n$ by

$$V_1 = F_1^{-1}(U_1), \qquad V_2 = F_2^{-1}(1 - U_1). \tag{7}$$

Let $V_1, \ldots, V_l$, $1 < l < n$ be defined and let L_l be the df of

$$H_l(V_1, \ldots, V_l) = f_{l-1}(f_{l-2}(\ldots (f_1(V_1, V_2), V_3), \ldots), V_l)$$

then define

$$V_{l+1} = F_{l+1}^{-1}(1 - L_l(H_l(V_1, \ldots, V_l), U_{l+1})).$$

Theorem 7. *Let f, U_i, V_i, F_i be defined as above, then*

a) *$(V_1, \ldots, V_n)$ is a random variable with df. $F_0 \in \mathscr{H}(F_1, \ldots, F_n)$.*

b) $h_{f(F_0)} \leqq h_{f(F)}$, $\quad \forall F \in \mathscr{H}(F_1, \ldots, F_n)$.

c) $E_{F_0} f = \inf\{E_F f;\ F \in \mathscr{H}(F_1, \ldots, F_n)\}$.

Proof. a) By Lemma 1, pg. 216 of Ferguson (1967)

$$W_l = L_{l-1}(H_{l-1}(V_1, \ldots, V_{l-1}), U_l)$$

is $R(0,1)$-distributed and, therefore, $V_l = F_l^{-1}(1 - W_l)$ has df. F_l.

b) We prove b) by induction in n. For $n=2$ b) follows from Theorem 5 since $(F_1^{-1}(U_1),\ F_2^{-1}(1-U_1))$ has df. $\underline{F}$. Assume b) to be true for n and so by assumption $h_{H_n(F_0)} \leqq h_{H_n(F)}$ for all $F \in \mathscr{H}(F_1, \ldots, F_n)$. Let X_i have df. F_i, $1 \leqq i \leqq n+1$ and denote the df's of

$$W = (H_n(V_1, \ldots, V_n), V_{n+1})$$

and

$$Z = (H_n(X_1, \ldots, X_n), X_{n+1})$$

by F, G respectively. By the assumption of the induction $H_n(V_1, \ldots, V_n)$ is stochastically smaller than $H_n(X_1, \ldots, X_n)$ and, therefore, by means of Lemma 1, pg. 216 of Ferguson (1967) one can construct a $\tilde{F} \in \mathscr{H}(L_n, F_{n+1})$ with $h_{\tilde{F}} \leqq h_G$. By Lemma 4 we get $h_F \leqq h_{\tilde{F}}$ and so by property b) of elements in M_n^2 this implies

$$h_{f_n(F)} \leqq h_{f_n(G)}$$

which gives the induction step.

c) is immediate from b.

4. Examples

a) Let $X = (X_1, \ldots, X_n)$ be a n-dim. random variable and define

$$\mathrm{sp}(x) = \max\{x_i;\, 1 \leqq i \leqq n\} - \min\{x_i;\, 1 \leqq i \leqq n\},$$

the span of x. $\mathrm{sp}(x)$ is a Δ-antitone function. So by Theorem 5 (generalized to R^n) we have

$$E_{\tilde{F}}\,\mathrm{sp} = \inf\{E_F\,\mathrm{sp};\, F \in \mathscr{H}(F_1, \ldots, F_n)\}.$$

This result has been proved by Schaefer (1976). It has applications in dynamic programming.

b) Let $X_1, \ldots, X_n$ be n real random variables, $X_i \geqq 0$, with df's $F_1, \ldots, F_n$. Let $\alpha_i > 0$, $1 \leqq i \leqq n$ be real numbers with $\sum_{i=1}^{n} 1/\alpha_i = 1$. By Theorem 5 the best lower bound for $\prod_{i=1}^{n} \|X_i\|_{\alpha_i}$ obtainable by Hölders inequality is given by $E \prod_{i=1}^{n} F_i^{-1}(U)$, where U is $R(0,1)$-distributed. Let especially $(a_{i,j})_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq k}}$ be real numbers $a_{i,j} \geqq 0$ such that $a_{i,1} \leqq \ldots \leqq a_{i,k}$, $1 \leqq i \leqq n$, and let F_i be the discrete uniform distribution on $\{a_{i,1}, \ldots, a_{i,k}\}$, $1 \leqq i \leqq n$, then $F_i(x) = j/k$ for $a_{i,j} \leqq x < a_{i,j+1}$, $1 \leqq j \leqq k$ $(a_{i,k+1} = \infty)$ and, therefore,

$$E \prod_{i=1}^{n} F_i^{-1}(U) = \frac{1}{k} \sum_{j=1}^{k} \prod_{i=1}^{n} a_{i,j}.$$

On the other hand,

$$\prod_{i=1}^{n} \|X_i\|_{\alpha_i} = \frac{1}{k} \prod_{i=1}^{n} \left(\sum_{j=1}^{k} a_{i,j}^{\alpha_i} \right)^{1/\alpha_i}.$$

So Theorem 5 implies the following extension of Hölders inequality (cf. Beckenbach, Bellman (1965), pg. 20)

$$\sum_{j=1}^{k} \prod_{i=1}^{n} a_{i,j} \leqq \prod_{i=1}^{n} \left(\sum_{j=1}^{k} a_{i,j}^{\alpha_i} \right)^{1/\alpha_i}. \tag{8}$$

c) Let $a_i \in R_+$, $1 \leqq i \leqq n$ be real numbers and let $\alpha_j \in R_+$, $1 \leqq j \leqq n-1$ be real numbers with $\sum_{j=1}^{n} 1/\alpha_j = 1$. Furthermore, let $F_1 = \ldots = F_{n-1}$ be the discrete uniform df. on $\{a_1, \ldots, a_n\}$. A distribution with marginals $F_1, \ldots, F_{n-1}$ is given f.e. by

$$P\{(a_{i_1}, \ldots, a_{i_{n-1}})\} = \begin{cases} 0 & \text{if there are } l,\, k \leqq n-1,\ l \neq k \text{ and } i_l = i_k. \\ \dfrac{1}{n!} & \text{else} \end{cases}$$

Theorem 5 and Hölder's inequality imply

$$\sum_{i=1}^{n} \prod_{j \neq i} a_j \leqq \sum_{j=1}^{n} a_j^{n-1} \leqq \prod_{j=1}^{n-1} \Big(\sum_{i=1}^{n} a_i^{\alpha_j}\Big)^{1/\alpha_j}. \tag{9}$$

By the second inequality in (9) the product of the α_j-norms of F_i is minimal for $\alpha_j = \frac{1}{n-1}$, $1 \leqq j \leqq n-1$. This result is also implied by Theorem 1 of Tong (1977) who proves his result by means of Schur-convex functions.

d) For $F_1 = \ldots = F_n$ the df. of the $R(0,1)$-distribution Theorem 5 yields the following lower bound

$$\inf\Big\{E_F \prod_{i=1}^{n} x_i;\, F \in \mathscr{H}(F_1, \ldots, F_n)\Big\} \geqq \int \ldots \int \Big(1 - \sum_{i=1}^{n} x_i\Big)_+ d \prod_{i=1}^{n} x_i$$
$$= \int_0^1 \int_0^{z_1} \ldots \int_0^{z_{n-1}} z_n \, d \prod_{i=1}^{n} z_i = \frac{1}{(n+1)!}.$$

For $n=3$ we get:

$$\tfrac{1}{24} \leqq EX_1 X_2 X_3 \leqq \tfrac{1}{4}$$

for all X_i, $1 \leqq i \leqq 3$ which are $R(0,1)$-distributed. The author is not aware of $R(0,1)$-distributed X_i, $1 \leqq i \leqq 3$ with $EX_1 X_2 X_3 < \frac{1}{16}$. The value $\frac{1}{16}$ is attained for:

$$X_1 = U, \quad X_2 = 1 - U, \; X_3 = 2|U - \tfrac{1}{2}|,$$

where U is $R(0,1)$-distributed. (These are the random variables V_1, V_2, V_3 from Theorem 7. The result of Theorem 7 is not applicable in this case).

e) Let $X_1, \ldots, X_n$ be real random variables with df's $F_1, \ldots, F_n$ and define

$$Z_n = \max_{1 \leqq i \leqq n} X_i \quad \text{and} \quad W_n = \min_{1 \leqq i \leqq n} X_i.$$

Then Theorem 7 is applicable for

$$f_1(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} \text{ resp.}$$
$$f_2(x_1, \ldots, x_n) = -\max\{x_1, \ldots, x_n\}.$$

We obtain the following sharp upper and lower bounds for W_n

$$\min(V_1, \ldots, V_n) \leqq_s W_n \leqq_s \min\{F_1^{-1}(U), \ldots, F_n^{-1}(U)\} \tag{10}$$

where $\leqq_s$ is the stoch. ordering, U is $R(0,1)$-distributed and V_i, $1 \leqq i \leqq n$ are constructed as in Theorem 7. We have by Lemma 6

$$P(\min(V_1, \ldots, V_n) \leqq t) = \min\left\{1, \sum_{i=1}^{n} F_i(t-)\right\}. \tag{11}$$

For Z_n we obtain the following sharp bounds

$$1 - \min_{1 \leqq i \leqq n} (1-F_i)^{-1}(t-) \geqq P(Z_n \leqq t) \geqq \left(\sum_{i=1}^{n} F_i(t-) - (n-1)\right)_+. \tag{12}$$

In the case of continuous df's the right inequality in (12) has been proved by Lai and Robbins (1978), (3.4). Lai and Robbins give in the case of $F_1 = \ldots = F_n$ - the df. of a $R(0,1)$-distribution - a nice geometric construction for random variables $V_1, \ldots, V_n$ such that $\max\{V_1, \ldots, V_n\}$ has the df. given in the right hand side of (12). A similar construction is possible for the left hand side of (10), (11). It shows f.e. that $\min\{V_1, \ldots, V_n\} \leqq 1/n$ for all $n \in \mathbb{N}$.

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