Z. Wahrscheinlichkeitstheorie verw. Gebiete
54, 331-340 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Generalization of Poisson Point Processes with Application to a Classical Limit Theorem

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1. Introduction

Let $\{X_{nk};\ 1 \leqq k \leqq n,\ n \geqq 1\}$ be a triangular array of uniformly asymptotically negligible random variables which are independent in each row. Suppose that $\sum_{k=1}^{n} X_{nk} - \beta_n$ converges in distribution for some constants β_n. Define point processes Φ_n, $n \geqq 1$, by $\Phi_n = \sum_{k=1}^{n} \delta_{X_{nk}}$, where δ_a is the Dirac measure located at a. In the ordinary sense the sequence $\{\Phi_n\}$ does not converge in distribution because $\Phi_n(U)$ tends to infinity for every neighborhood U of the origin. However it is possible to modify the definition of point processes so that $\{\Phi_n\}$ is convergent under some reasonable assumptions.

For our purpose we first introduce the concept of generalized measures. Let $\mathscr{F}_0$ be the space of functions f on R satisfying (i) f, f' and f'' are bounded continuous and vanish at infinity, and (ii) $f(0)=0$. With a suitable norm $\mathscr{F}_0$ is a separable Banach space. A positive continuous linear functional on $\mathscr{F}_0$ is called a generalized measure.

In terms of generalized measures we can define generalized random measures (g.r.m.) and generalized point processes (g.p.p.). It should be noted that g.r.m's defined in this paper are different from those considered before by several authors including Dennler [1] and Nawrotzki [8]. It turns out that g.p.p.'s introduced here are closely connected to processes with interchangeable increments (ich. incr.) on the interval [0, 1], which has been investigated extensively by Kallenberg [4-6].

It is proved that for every generalized measure ϕ there exists a g.p.p. which may be called a generalized Poisson point process (g.P.p.p.) with intensity ϕ. There exists a natural one-to-one mapping from the class of all distributions of g.P.p.p.'s onto the class of all infinitely divisible distributions on R. It is shown that the limit distribution of a sequence $\{\Phi_n\}$ of g.p.p.'s generated by a triangular array $\{X_{nk}\}$ is essentially generalized Poisson. A necessary and sufficient con-

0044-3719/80/0054/0331/$02.00

dition for the convergence in distribution of such a sequence $\{\Phi_n\}$ is given. This result corresponds to the well-known classical limit theorem for sums $S_n = \sum_{k=1}^{n} X_{nk}$ and provides a direct and intuitively appealing way to understand the role of Lévy measures in the classical limit theorem.

In § 2 we prove a representation theorem for generalized measures which is interesting in itself because of its close relation to the generators of Lévy processes. In § 3 we state a uniqueness theorem and a convergence theorem for g.r.m.'s. In § 4 we show the existence of g.P.p.p. with a given intensity ϕ. In § 5 the relation between g.p.p.'s and the processes with ich. incr. is described. In § 6 we give some results on the convergence of a sequence $\{\Phi_n\}$ of g.p.p.'s generated by a triangular array. For simplicity we restrict ourselves to g.p.p.'s on R. However our results are valid for R^d without any essential change.

2. Generalized Measures

We write $\mathscr{F}$ for the space of all real-valued functions on the real line R which are bounded continuous with their first two derivatives and satisfying $f(0)=0$. Denote by $\mathscr{F}_0$ the subspace of $\mathscr{F}$ consisting of all functions vanishing at infinity with their first two derivatives. We define a norm of $f \in \mathscr{F}_0$ by

$$\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty\},$$

where $\|f\|_\infty = \sup\{|f(x)|;\, x \in R\}$. With this norm $\mathscr{F}_0$ is a separable Banach space.

We call a continuous linear functional ϕ on $\mathscr{F}_0$ positive and write $\phi \geqq 0$ if $\phi(f) \geqq 0$ for every $f \geqq 0$, $f \in \mathscr{F}_0$. A positive continuous linear functional ϕ on $\mathscr{F}_0$ is called a generalized measure. The class of all generalized measures is denoted by M.

Example 2.1. Let $\delta_0'(f) = f'(0)$ and $\delta_0''(f) = f''(0)$, $f \in \mathscr{F}_0$. If $f \geqq 0$, $f \in \mathscr{F}_0$, then $f'(0) = 0$ and $f''(0) \geqq 0$. Therefore $a\delta_0' \in M$ for $a \in R$ and $b\delta_0'' \in M$ for $b \in R^+ = [0, \infty)$.

Example 2.2. Let M' denote the class of all Borel measures μ on $R' = R \setminus \{0\}$ satisfying $\mu(v) < \infty$, where $v(x) = x^2/(1+x^2)$ and $\mu(f) = \int_{R'} f\,d\mu$. For $\mu \in M'$ a linear functional ϕ defined by $\phi(f) = \mu(f - f'(0)u)$, $f \in \mathscr{F}_0$, is in M, where u is a fixed element of $\mathscr{F}_0$ such that

$$u'(0) = 1 \quad \text{and} \quad u''(0) = 0. \tag{2.1}$$

In fact suppose $\|f\| < \varepsilon$, $f \in \mathscr{F}_0$, and let $g = f - f'(0)u$. Then $g'(0) = 0$, $\|g\| < \varepsilon(1 + \|u\|)$, $g(y) = (1/2)\,g''(\theta y)\,y^2$, $0 < \theta < 1$, and therefore $|g(y)| \leqq (1/2)\,\varepsilon(1 + \|u\|)\,y^2$. Thus $|g(y)| \leqq \varepsilon(1 + \|u\|)\,v(y)$ and $|\phi(f)| \leqq \mu(|g|) \leqq \varepsilon(1 + \|u\|)\,\mu(v)$. This shows that ϕ is continuous. Obviously $\phi \geqq 0$.

Throughout the rest u is a fixed element of $\mathscr{F}_0$ satisfying (2.1).

Theorem 2.1. *Every $\phi \in M$ is uniquely represented as follows:*

$$\phi(f) = af'(0) + (b^2/2)f''(0) + \mu(f - f'(0)u), \quad f \in \mathscr{F}_0, \tag{2.2}$$

where $a \in R$, $b \in R^+$ *and* $\mu \in M'$. *Conversely every functional* ϕ *defined by* (2.2) *with* $a \in R$, $b \in R^+$ *and* $\mu \in M'$ *is an element of* M.

Remarks. (i) In the representation (2.2) the value of a depends on the choice of u. (ii) By (2.2) the domain of definition of a positive continuous linear functional ϕ on $\mathscr{F}_0$ can be extended to the space of all bounded complex-valued Baire functions f having continuous first and second derivatives in a neighborhood of the origin and satisfying $f(0)=0$. In particular $\phi(f)$ is defined for every $f \in \mathscr{F}$.

Proof of Theorem 2.1. The last half was proved in Examples 2.1 and 2.2. To prove the first half suppose $\phi \in M$. Let $\mathscr{F}_0^{(n)}$ be the class of functions $f \in \mathscr{F}_0$ supported by $E_n = (-\infty, -1/n) \cup (1/n, \infty)$, $n \geqq 1$, and $\mathscr{F}_0' = \bigcup_{n=1}^{\infty} \mathscr{F}_0^{(n)}$. By a standard argument using Riesz representation theorem we find that there exists a unique Borel measure μ on R' such that $\phi(f) = \mu(f)$ for $f \in \mathscr{F}_0'$. For $f \geqq 0$, $f \in \mathscr{F}_0$, let $g \in \mathscr{F}_0^{(2n)}$ be such that $g = f$ on E_n and $0 \leqq g \leqq f$ on R. Then $\phi(f) \geqq \phi(g) = \mu(g) \geqq \int_{E_n} f\,d\mu$. By letting $n \to \infty$ we have

$$\mu(f) \leqq \phi(f) \qquad \text{for } f \geqq 0,\ f \in \mathscr{F}_0. \tag{2.3}$$

This shows $\mu \in M'$. Let $\phi_n(f) = \phi(f) - \int_{E_n} f\,d\mu, f \in \mathscr{F}_0$, $n \geqq 1$. By (2.3) we have $\phi_n \in M$ and $\phi_n(f) = 0$ for $f \in \mathscr{F}_0^{(n)}$. We shall show that if $f \in \mathscr{F}_0$ is such that $f(x) = o(x^2)$ as $x \to 0$ then $\lim_n \phi_n(f) = 0$. In fact choose $\gamma \in C^2(R)$ such that $\gamma(x) = 0$ for $x \in E_1$, $\gamma(x) = 1$ for $x \in E_2$ and $0 \leqq \gamma(x) \leqq 1$ on R. Define γ_n by $\gamma_n(x) = \gamma(nx/2)$. Then $f \cdot \gamma_n \in \mathscr{F}_0$, $\|f \cdot \gamma_n\| \to 0$ and therefore $\phi_n(f) = \phi_n(f \cdot \gamma_n) = \phi(f \cdot \gamma_n) - \int_{E_n} f \cdot \gamma_n\,d\mu \to 0$. Thus we have for every $f \in \mathscr{F}_0$

$$\lim_n \phi_n(f - f'(0)\,u - f''(0)\,u^2/2) = 0. \tag{2.4}$$

On the other hand by the definition of ϕ_n we have

$$\lim_n \phi_n(f - f'(0)\,u) = \phi(f - f'(0)\,u) - \mu(f - f'(0)\,u), \tag{2.5}$$

for $f \in \mathscr{F}_0$. By letting $f(x) = (u(x))^2$ in (2.5) we find that the finite limit $b^2 = \lim_n \phi_n(u^2) \geqq 0$ exists. It follows from (2.4) and (2.5) that (2.2) holds with $a = \phi(u)$. This proves the theorem.

A measure μ on R' is called integer-valued if $\mu(A)$ is equal to an integer or $+\infty$ for every Borel set A of R'. The class of all integer-valued measures $\mu \in M'$ is denoted by N'. The class of all $\phi \in M$ such that the measure μ in representation (2.2) belongs to N' is denoted by N.

We introduce in M the coarsest topology with respect to which every mapping $\phi \to \phi(f)$, $f \in \mathscr{F}$, is continuous. Thus for $\phi, \phi_1, \phi_2, \ldots$ in M $\phi_n \to \phi$ iff $\phi_n(f) \to \phi(f)$ for every $f \in \mathscr{F}$. By the same argument as in A7.4 of [7] we can show that N is a closed subset of M. Furthermore suitable modification of the proof of A7.7 of [7] yields the following:

Theorem 2.2. *M and N are Polish spaces.*

In the space M' we introduce the coarsest topology with respect to which every mapping $\phi \to \phi(f)$, $f \in \mathscr{F}'$, is continuous where $\mathscr{F}'$ is the class of all $f \in \mathscr{F}$ vanishing in a neighborhood of the origin. Then it is easy to see that the mapping from M onto the product space $R \times R_+ \times M'$ which sends ϕ to $(a, b^2 + \mu(u^2), \mu)$ is a homeomorphism, where a, b^2 and μ are those in the representation (2.2). The restriction of this mapping to N is a homeomorphism of N onto $R \times R_+ \times N'$.

Let $\mu \in N'$ be the measure appearing in the representation (2.2) of $\phi \in N$. Then $\mu\{x; |x| > c\} = 0$ for large c. This shows that if a function g on R has continuous second derivative in a neighborhood of the origin and satisfies $g(0) = 0$ then we can define $\phi(g)$, $\phi \in N$, by (2.2). Furthermore if g is continuous then the mapping $\phi \to \phi(g)$ is continuous on N.

3. Generalized Random Measures

Let $\mathscr{M}$ be the σ-algebra of Borel sets of M. Since M is Polish this coincides with the smallest σ-algebra making all mappings $\phi \to \phi(f)$, $f \in \mathscr{F}$, measurable ([7] Lemma 4.1). Let $\mathscr{N} = N \cap \mathscr{M}$.

A random element defined on a probability space $(\Omega, \mathscr{B}, P)$ taking values in the space $(M, \mathscr{M})$ is called a generalized random measure (g.r.m.). A g.r.m. Φ is called a generalized point process (g.p.p.) if $\Phi \in N$ a.s. By Theorem 2.1 we have the following canonical representation of g.r.m. or g.p.p. Φ:

$$\Phi(f) = Af'(0) + B^2 f''(0)/2 + \Psi(f - f'(0)\,u), \qquad f \in \mathscr{F}_0,$$

where A and B are random variables, $B \geqq 0$, and Ψ is a random measure or a point process on R' resp. (i.e. M'-valued random element or N'-valued random element resp.) satisfying $\Psi(v) < \infty$ a.s. The triple (A, B, Ψ) may be called the canonical random element of Φ. The characteristic functional C_Φ of a g.r.m. Φ is defined by

$$C_\Phi(f) = \int_\Omega \exp(i\Phi(f))\,dP = E\exp(i\Phi(f)), \qquad f \in \mathscr{F}.$$

The distribution of Φ is uniquely determined by C_Φ.

The following two theorems are proved by the same argument as in Theorems 3.1 and 4.2 of [7] resp.

Theorem 3.1. *For two g.r.m.'s Φ_1 and Φ_2 the following three statements are equivalent:*

(i) $\Phi_1 \overset{d}{=} \Phi_2$, (ii) $\Phi_1(f) \overset{d}{=} \Phi_2(f)$, $f \in \mathscr{F}_0$,

(iii) $C_{\Phi_1}(f) = C_{\Phi_2}(f)$, $f \in \mathscr{F}_0$,

where we write $\overset{d}{=}$ for equality in distribution.

Theorem 3.2. *Let Φ and Φ_n, $n \geqq 1$, be g.r.m.'s. Then the following three statements are equivalent:*

(i) $\Phi_n \xrightarrow{d} \Phi$, (ii) $\Phi_n(f) \xrightarrow{d} \Phi(f)$, $f \in \mathscr{F}$,

(iii) $C_{\Phi_n}(f) \to C_\Phi(f)$, $f \in \mathscr{F}$,

where we write $\xrightarrow{d}$ *for convergence in distribution.*

Remark. Let (A, B, Ψ) and (A_n, B_n, Ψ_n) be canonical random elements of Φ and Φ_n resp. Then the statements (i)–(iii) of Theorem 3.2 are equivalent to

(iv) $(A_n, B_n^2 + \Psi_n(u^2), \Psi_n) \xrightarrow{d} (A, B^2 + \Psi(u^2), \Psi)$.

Furthermore when Φ and Φ_n are g.p.p.'s these statements are also equivalent to

(v) $(\Phi_n(h), \Phi_n(h^2), \Psi_n) \xrightarrow{d} (\Phi(h), \Phi(h^2), \Psi)$,

where $h(x) = x$, $x \in R$.

4. Generalized Poisson Point Processes

Theorem 4.1. *For every* $\phi \in M$ *there exists a g.p.p.* Φ *with characteristic functional*

$$C_\Phi(f) = \exp[\phi(e^{if} - 1)], \quad f \in \mathscr{F}. \tag{4.1}$$

Remarks. (i) We call Φ the generalized Poisson point process (g.P.p.p.) with intensity ϕ. The distribution $P\Phi^{-1}$ of Φ will be denoted by P_ϕ. (ii) By letting $f = th$, $t \in R$, in (4.1) we find that the random variable $X = \Phi(h)$ has infinitely divisible characteristic function

$$E e^{it\Phi(h)} = C_\Phi(th) = \exp[\phi(e^{ith} - 1)], \quad t \in R. \tag{4.2}$$

Conversely if X is an infinitely divisible random variable then there exists a g.P.p.p. Φ such that $X \stackrel{d}{=} \Phi(h)$. The formula (4.2) is no other than the well-known Lévy-Khintchine formula.

Example 4.1. If $\phi = \delta_0'$ then $C_\Phi(tf) = \exp(itf'(0))$, $f \in \mathscr{F}$, $t \in R$. Therefore $\Phi(f) = f'(0)$ a.s. If $\phi = \delta_0''/2$ then $C_\Phi(tf) = \exp[if''(0)t/2 - (f'(0))^2 t^2/2]$, $f \in \mathscr{F}$, $t \in R$, and therefore $\Phi(f)$ is Gaussian with mean $f''(0)/2$ and variance $|f'(0)|^2$.

Proof of Theorem 4.1. Let a, b and μ be those in the representation (2.2) of ϕ. Let Ψ be a Poisson point process on R' with intensity μ and let X be an $N(0,1)$ random variable independent of Ψ. Define $\Phi(f)$, $f \in \mathscr{F}$, by

$$\Phi(f) = (a + bX) f'(0) + b^2 f''(0)/2 + \int [f \, d\Psi - f'(0) u \, d\mu],$$

where the last term on the right is the so-called centered Poisson shower integral [6], that is

$$\int [f \, d\Psi - f'(0) u \, d\mu] = \lim_n \Big[\int_{E_n} f \, d\Psi - \int_{E_n} f'(0) u \, d\mu\Big],$$

where $\{E_n\}$ is a sequence of Borel sets of R' such that $\mu(E_n)<\infty$, $E_n \uparrow R'$. The existence of this limit is implied by $\mu(f^2)<\infty$ and $\mu(f-f'(0)u)<\infty$. It is easy to verify that

$$\begin{aligned} C_\Phi(f) &= \lim_n \exp\left[iaf'(0)+ibf''(0)/2-b(f'(0))^2/2 \right. \\ &\quad \left. + \int_{E_n} (e^{if}-1-if'(0)u)\,d\mu\right] \\ &= \exp\left[iaf'(0)+ibf''(0)/2-b(f'(0))^2/2 \right. \\ &\quad \left. +\mu(e^{if}-1-if'(0)u)\right] \\ &= \exp\left[\phi(e^{if}-1)\right]. \end{aligned}$$

Let Θ be a g.r.m. with the distribution Q. The characteristic functional

$$C(f;\phi)=\exp\left[\phi(e^{if}-1)\right]$$

of the g.P.p.p. with distribution P_ϕ determines an $\mathscr{M}$-measurable function of ϕ for each $f\in\mathscr{F}$. Therefore by the same argument as in Lemma 1.7 of [7] the mixture of P_ϕ with respect to Q exists. Thus we can define a generalized Cox process Φ directed by Θ whose characteristic functional is given by

$$C_\Phi(f)=E\exp\left[\Theta(e^{if}-1)\right]=C_\Theta(-i(e^{if}-1)), \quad f\in\mathscr{F}. \tag{4.3}$$

Obviously the distribution of Θ is uniquely determined by that of Φ. The following theorem is easily proved from (4.3) and Theorem 3.2.

Theorem 4.2. *For each $n\geq 1$ let Φ_n be a generalized Cox process directed by g.r.m. Θ_n. Then $\Phi_n \xrightarrow{d} \Phi$ for some g.p.p. Φ iff $\Theta_n \xrightarrow{d} \Theta$ for some g.r.m. Θ. In this case Φ is a generalized Cox process directed by Θ.*

5. Processes with Interchangeable Increments

Let $D_0[0,1]$ and $D_0(R_+)$ be the class of functions on $[0,1]$ and R_+ resp. which are right-continuous with left hand limits and which start at zero. The topology of these spaces are Skorohod J_1 topology and its natural extension. Let X be a random process on $[0,1]$ separated by binary rationals and having interchangeable increments (ich. incr.). We assume that every random process starts at zero. Kallenberg [4] obtained a representation theorem for X. In terms of g.p.p. his result is stated as follows ([4], Theorem 2.1): To every g.p.p. Φ with representation

$$\Phi(f)=Af'(0)+B^2f''(0)/2+\Psi(f-f'(0)h), \quad f\in\mathscr{F},$$

we can associate a process X in $D_0[0,1]$ with ich. incr. represented as

$$X(t)=At+BW(t)+\sum_j \beta_j[1_+(t-\tau_j)-t], \quad t\in[0,1], \tag{5.1}$$

where $\Psi=\sum_j \delta_{\beta_j}$, W is the standard Brownian bridge on $[0,1]$, $\{\tau_j\}$ is a sequence of i.i.d. random variables uniformly distributed over $[0,1]$, 1_+ is the indicator of

R_+, and finally Φ, W and $\{\tau_j\}$ are independent. Conversely every process X with ich. incr. separated by binary rationals has an equivalent version in $D_0[0,1]$ represented as (5.1) with Φ, W and $\{\tau_j\}$ subjected to the above conditions. Φ may be called the canonical g.p.p. of X.

In view of Theorem 3.2 a convergence criterion for processes with ich. incr. (Theorem 2.3 of [4]) is stated as follows:

Theorem 5.1. *For $n \geqq 1$ let Φ_n be the canonical g.p.p. of a process X_n in $D_0[0,1]$ with ich. incr. In order that $X_n \xrightarrow{d}$ some X it is necessary and sufficient that $\Phi_n \xrightarrow{d}$ some Φ. In this case X has ich. incr. and Φ is the canonical g.p.p. of X.*

The process with ich. incr. whose canonical g.p.p. is a g.P.p.p. with intensity ϕ is equivalent to a Lévy process X such that

$$E \exp[itX(1)] = \exp[\phi(e^{ith} - 1)].$$

It is shown in [4] that every process X in $D_0(R_+)$ with ich. incr. is a mixture of Lévy processes. In terms of g.p.p. Theorem 3.1 of [4] is stated as follows:

Theorem 5.2. *The canonical g.p.p. of a process X in $D_0(R_+)$ with ich. incr. is a generalized Cox process directed by a g.r.m. Θ. Conversely every generalized Cox process directed by a g.r.m. Θ is the canonical g.p.p. of a process X in $D_0(R_+)$ with ich. incr.*

The g.r.m. Θ is called the canonical g.r.m. of X. The following theorem is a restatement of Theorem 3.3. of [4].

Theorem 5.3. *For each $n \geqq 1$ let Θ_n be the canonical g.r.m. of a process X_n in $D_0(R_+)$ with ich. incr. Then $X_n \xrightarrow{d}$ some X iff $\Theta_n \xrightarrow{d}$ some Θ. In this case X is a process in $D_0(R_+)$ with ich. incr. and has the canonical g.r.m. Θ.*

6. Application to a Classical Limit Theorem

Let $\{X_{nk};\ 1 \leqq k \leqq n,\ n \geqq 1\}$ be a triangular array of uniformly asymptotically negligible (u.a.n.) random variables, i.e. X_{nk}, $1 \leqq k \leqq n$, are independent for each n and

$$\lim_{n} \max_{1 \leqq k \leqq n} P\{|X_{nk}| > \varepsilon\} = 0, \tag{6.1}$$

for every $\varepsilon > 0$. Let Φ_n, $n \geqq 1$, denote the g.p.p. defined by $\Phi_n = \sum_{k=1}^{n} \delta_{X_{nk}}$ and let $S_n = \Phi_n(h) = \sum_{k=1}^{n} X_{nk}$. Define $\mu_n \in M'$ by $\mu_n = \sum_{k=1}^{n} F_{nk}$, where F_{nk} is the distribution of X_{nk}. Obviously the measure μ_n is identified with an element of M which will be denoted by ϕ_n.

Theorem 6.1. *Suppose $Eu(X_{nk}) = 0$, $1 \leqq k \leqq n$, $n \geqq 1$. Then the following three statements are equivalent:*

(i) $\Phi_n \xrightarrow{d}$ some Φ, (ii) $S_n \xrightarrow{d}$ some S, (iii) $\phi_n \to$ some ϕ in M. In this case Φ is a g.P.p.p. with intensity ϕ and $S \overset{d}{=} \Phi(h)$ has the characteristic function $\exp[\phi(e^{ith} - 1)]$.

Proof. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iii). Immediate from a well-nown result ([2], p. 585, [3] p. 116).

(iii) $\Rightarrow$ (i). Suppose $\phi_n \to \phi$ and ϕ has the representation

$$\phi(f) = af'(0) + b^2 f''(0)/2 + \mu(f - f'(0)u), \qquad f \in \mathscr{F}.$$

It suffices to show that $\Phi_n(f) \xrightarrow{d} \Phi(f)$ for $f \in \mathscr{F}$, where Φ is a g.P.p.p. with intensity ϕ. Notice that $\Phi_n(f) = \sum_{k=1}^n Y_{nk}$, where $Y_{nk} = f(X_{nk})$. Since f is bounded $\{Y_{nk};\ 1 \leqq k \leqq n,\ n \geqq 1\}$ is a u.a.n. array of uniformly bounded random variables. It is obvious that

$$\mu_n f^{-1} \to \mu f^{-1} \quad \text{in } M', \tag{6.2}$$

$$\sum_{k=1}^n EY_{nk} = \sum_{k=1}^n F_{nk}(f) = \phi_n(f) \to \phi(f), \tag{6.3}$$

and

$$\sum_{k=1}^n EY_{nk}^2 = \sum_{k=1}^n F_{nk}(f^2) = \phi_n(f^2) \to \phi(f^2). \tag{6.4}$$

The assumptions (6.1) and (iii) imply that

$$\lim_n \max_{1 \leqq k \leqq n} |F_{nk}(f)| = 0$$

and

$$\sup_n \sum_{k=1}^n |F_{nk}(f - f'(0)u)| < \infty$$

resp. and therefore we have

$$\sum_{k=1}^n (F_{nk}(f))^2 \leqq \max_{1 \leqq k \leqq n} |F_{nk}(f)| \sum_{k=1}^n |F_{nk}(f - f'(0)u)| \to 0.$$

Together with (6.4) this shows

$$\sum_{k=1}^n \mathrm{Var}(Y_{nk}) \to \phi(f^2) = b^2 (f'(0))^2 + \mu(f^2). \tag{6.5}$$

It is well-known ([2] p. 585, [3] p. 100) that (6.2), (6.3) and (6.5) imply that $\Phi_n(f) = \sum_{k=1}^n Y_{nk} \xrightarrow{d}$ some Y. The characteristic function of Y is given by

$$\begin{aligned} Ee^{itY} &= \exp[it\phi(f) - b^2(f'(0))^2 t^2/2 + \mu(e^{itf} - 1 - itf)] \\ &= \exp[\phi(e^{itf} - 1)] = E[\exp(i\Phi(tf))]. \end{aligned}$$

This proves the theorem.

Let $c_{nk} = Eu(X_{nk})$. It follows from (6.1) that

$$\lim_n \max_{1 \leqq k \leqq n} |c_{nk}| = 0.$$

Write $X_{nk}^* = X_{nk} - c_{nk}$ and $F_{nk}^* = F_{nk}(\cdot + c_{nk})$. Define $\phi_n^* \in M$ and a g.p.p. Φ_n^* by $\phi_n^* = \sum_{k=1}^n F_{nk}^*$ and $\Phi_n^* = \sum_{k=1}^n \delta_{X_{nk}^*}$. Then we have the following:

Corollary 6.1. *In addition to* (2.1) *we suppose that u satisfies* $u(x)=x$ *in a neighborhood of the origin. Then the following three statements are equivalent:*

(i) $\Phi_n^* \xrightarrow{d}$ *some* Φ,

(ii) *there exists a sequence* $\{\beta_n\}$ *such that* $S_n - \beta_n \xrightarrow{d}$ *some* S,

(iii) $\phi_n^* \to$ *some* ϕ *in* M.

Proof. The proofs of implications (i) ⇒ (ii) and (ii) ⇒ (iii) are the same as the proof of Theorem 6.1. To prove (iii) ⇒ (i) we shall show $\sum_{k=1}^n (F_{nk}^*(f))^2 \to 0, f \in \mathscr{F}$. It is easy to see that

$$|u(x+c) - u(x) - c| \leqq (1 + \|u\|)|c|, \qquad c \in R.$$

Furthermore by the additional assumption on u

$$u(x + c_{nk}) - u(x) - c_{nk} = 0, \qquad \text{for } |x| < \delta \text{ and } n \geqq n_0,\ 1 \leqq k \leqq n.$$

Since $F_{nk}^*(u) = F_{nk}(u(\cdot + c_{nk}) - u(\cdot) - c_{nk})$ we have $\sum_{k=1}^n |F_{nk}^*(u)| \to 0$ and therefore $\sum_{k=1}^n F_{nk}^*(u)^2 \to 0$. By an argument similar to the proof of the theorem we obtain

$$\lim_n \sum_{k=1}^n F_{nk}^*(f)^2 = \lim_n \sum_{k=1}^n [F_{nk}^*(f - f'(0)u) + f'(0) F_{nk}^*(u)]^2$$
$$= f'(0)^2 \lim_n \sum_{k=1}^n F_{nk}^*(u)^2 = 0.$$

The rest of the proof is contained in the proof of the theorem.

Theorem 6.2. *Suppose* $u(x) = x$ *in a neighborhood of the origin. In order that* $\Phi_n \xrightarrow{d}$ *some* Φ *it is necessary and sufficient that* (i) $\phi_n^* \to$ *some* ϕ, (ii) $\sum_{k=1}^n c_{nk} \to$ *some* m, *and* (iii) $\sum_{k=1}^n c_{nk}^2 \to$ *some* q. *In this case the distribution of* Φ *is* $P_\phi * \delta_{\phi_0}$, *where* $*$ *denotes the convolution and* $\phi_0 \in M$ *is such that* $\phi_0(f) = mf'(0) + qf''(0)/2$.

Remark. It follows from the theorem that $\Phi_n \xrightarrow{d}$ some Φ iff $S_n \xrightarrow{d}$ some S and $T_n = \sum_{k=1}^n X_{nk}^2 \xrightarrow{d}$ some T.

Proof of Theorem 6.2. To show the necessity of the conditions suppose $\Phi_n \xrightarrow{d} \Phi$. Then (i) is immediate because $\sum_{k=1}^n X_{nk} = \Phi_n(h) \xrightarrow{d} \Phi(h)$. It follows from

$$\Phi_n(u) = \sum_{k=1}^n u(X_{nk}) \xrightarrow{d} \Phi(u) \quad \text{and} \quad \Phi_n(u^2) = \sum_{k=1}^n u^2(X_{nk}) \xrightarrow{d} \Phi(u^2)$$

that

$$\sum_{k=1}^n c_{nk} = \sum_{k=1}^n Eu(X_{nk}), \quad \sum_{k=1}^n \operatorname{Var} u(X_{nk}) \quad \text{and} \quad \sum_{k=1}^n Eu^2(X_{nk})$$

converge and therefore

$$\sum_{k=1}^{n} c_{nk}^2 = \sum_{k=1}^{n} Eu^2(X_{nk}) - \sum_{k=1}^{n} \operatorname{Var} u(X_{nk})$$

converges as $n \to \infty$.

Now suppose that the conditions (i)–(iii) hold. Let Φ^* be a g.P.p.p. with intensity ϕ. It suffices to prove that

$$\Phi_n(f) \xrightarrow{d} \Phi^*(f) + mf'(0) + qf''(0)/2, \qquad f \in \mathscr{F}. \tag{6.6}$$

By the mean value theorem in calculus we have

$$[f(X_{nk}^* + c_{nk}) - f(c_{nk})] - [f(X_{nk}^*) - f(0)] = c_{nk} Z_{nk}, \tag{6.7}$$

where

$$\begin{aligned} Z_{nk} &= f'(X_{nk}^* + \theta_{nk} c_{nk}) - f'(\theta_{nk} c_{nk}) \\ &= X_{nk}^* f''(\theta_{nk}^* X_{nk}^* + \theta_{nk} c_{nk}), \end{aligned} \tag{6.8}$$

with $0 < \theta_{nk} < 1$ and $0 < \theta_{nk}^* < 1$. Summing (6.7) over k we obtain

$$\Phi_n(f) = \sum_{k=1}^{n} f(X_{nk}) = \sum_{k=1}^{n} f(X_{nk}^*) + \sum_{k=1}^{n} f(c_{nk}) + \sum_{k=1}^{n} c_{nk} Z_{nk}. \tag{6.9}$$

By Corollary 6.1 the first term on the right of (6.9) converges to $\Phi^*(f)$ in distribution. It is easy to see that the second term converges to $mf'(0) + (q/2)f''(0)$. From (6.8) we have $|Z_{nk}| \leq \min(2\|f\|, \|f\| \cdot |X_{nk}^*|)$. Thus by (6.1) $\sup_k EZ_{nk}^2 \to 0$. This implies $\sum_{k=1}^{n} c_{nk}^2 EZ_{nk}^2 \to 0$ and therefore $\sum_{k=1}^{n} c_{nk} Z_{nk} \to 0$ in probability. This proves (6.6) and therefore the theorem.

Acknowledgement. The author wishes to thank the referee for his helpful comments.

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Received December 12, 1977; in final form June 1, 1980