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A Relation Between Chung's and Strassen's Laws of the Iterated Logarithm

E. Csáki

Mathematical Institute of the Hungarian Academy of Sciences,
Budapest, Reáltanoda u. 13–15, Hungary H-1053

Summary. Let $W(t)$ be a standard Wiener process and let $f(x)$ be a function from the compact class in Strassen's law of the iterated logarithm. We investigate the lim inf behavior of the variable

$$\sup_{0\leqq x\leqq 1} |W(xT)(2T\operatorname{loglog} T)^{-1/2} - f(x)|,$$

suitably normalized as $T\to\infty$.

This extends Chung's result valid for $f(x)\equiv 0$, stating that $\liminf_{T\to\infty}[\sup_{0\leqq x\leqq 1} |(2T\operatorname{loglog} T)^{-1/2} W(xT)|(\operatorname{loglog} T)^{-1}] = \pi/4$ a.s.

1. Introduction

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $W(t)=W(t,\omega)$ $(\omega\in\Omega)$ be a standard Wiener process defined on it. We also consider the space $C[0,1]$ with the sup metric $\|\ \|$ and let us denote by P_W the Wiener measure defined on the Borel sets of $C[0,1]$. It is well known that there is a close relation between the measures P and Pw, i.e. if B is any Borel set in $C[0,1]$ and $A=\{\omega\colon W(t,\omega)\in B\}\in\mathscr{A}$, then

$$P_W(B)=P(A). \tag{1.1}$$

Let $S\subset C[0,1]$ be the class of functions defined in Strassen's law of the iterated logarithm [7], i.e. $f(x)\in S$ $(0\leqq x\leqq 1)$ if and only if $f(0)=0$, $f(x)$ is absolutely continuous and $\int_0^1 f'^2(x)\,dx\leqq 1$. Denote by S^ε the ε-neighbourhood of S, i.e. $g(x)\in S^\varepsilon (0\leqq x\leqq 1)$ means that there exists an $f(x)\in S$ such that $\|f(x) - g(x)\| < \varepsilon$.

The proof of Strassen's law of the iterated logarithm usually breaks up into two parts:

(i) The first part consists in showing that for almost all $\omega\in\Omega$ and for all $\varepsilon>0$ there exists a $T_0=T_0(\omega)$ such that

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$$\frac{W(xT)}{(2T \operatorname{loglog} T)^{1/2}} \in S^{\varepsilon} \tag{1.2}$$

whenever $T \geqq T_0$.

(ii) The second part consists in showing that there exists an $\Omega_0 \subset \Omega$ with $P(\Omega_0)=1$ such that for all $\omega \in \Omega_0$, for all $\varepsilon>0$ and for all $f(x) \in S$, the inequality

$$\sup_{0 \leqq x \leqq 1} \left| \frac{W(xT)}{(2T \operatorname{loglog} T)^{1/2}} - f(x) \right| < \varepsilon \tag{1.3}$$

holds true at least for an increasing sequence of T tending to infinity.

Concerning part (i) Bolthausen [2] investigated the problem how can the constant ε be replaced by a function $\varepsilon(T)$ in (1.2) so that the assertion in (i) remains true. He shows that $\varepsilon(T) = (\operatorname{loglog} T)^{-\alpha}$ with $\alpha < 1/2$ will do, but $\varepsilon(T) = (\operatorname{loglog} T)^{-1}$ will not. The problem is open for $1/2 \leqq \alpha < 1$.

Our concern in this paper is to investigate the analoguous problem for part (ii), i.e. we want to replace ε by $\varepsilon(T)$ in (1.3). To be precise, we choose $f(x) \in S$ and fix it. Our aim is to determine the best rate $\varepsilon(T)$ in the sense that

$$P\left(\sup_{0 \leqq x \leqq 1} \left| \frac{W(xT)}{(2T \operatorname{loglog} T)^{1/2}} - f(x) \right| < (1+c)\, \varepsilon(T) \text{ i.o.} \right) = 1 \tag{1.4}$$

but

$$P\left(\sup_{0 \leqq x \leqq 1} \left| \frac{W(xT)}{(2T \operatorname{loglog} T)^{1/2}} - f(x) \right| < (1-c)\, \varepsilon(T) \text{ i.o.} \right) = 0 \tag{1.5}$$

for any $c>0$. Here and in what follows i.o. (infinitely often) means that the inequality in the bracket occurs for a sequence of T increasing to infinity.

One can not reasonably expect to give a universal result for all $f(x) \in S$. Indeed, the best $\varepsilon(T)$ will depend on $f(x)$. Also, the exceptional set of measure 0 in our results may depend on $f(x)$. Unfortunately we can not give a complete solution to the problem described above, i.e. we can give the best rate $\varepsilon(T)$ only for $f(x) \in S$, satisfying certain further restrictions.

For the function $f(x) \equiv 0$ $(0 \leqq x \leqq 1)$ Chung's law of the iterated logarithm [4] says that

$$P\left(\sup_{0 \leqq x \leqq 1} \left| \frac{W(xT)}{(2T \operatorname{loglog} T)^{1/2}} \right| < \frac{(1+c)\pi}{4 \operatorname{loglog} T} \text{ i.o.} \right) = 1 \text{ or } 0 \tag{1.6}$$

according as $c>0$ or $c<0$. In this way our results can be regarded as extensions of Chung's LIL, establishing also a connection with Strassen's LIL.

In Theorem 1 we give universal results, valid for all $f(x) \in S$, by determining upper and lower bounds for the best rate $\varepsilon(T)$. In Theorems 2 and 3 basically two cases will be treated:

case (i): $$\int_0^1 f'^2(x)\, dx < 1, \tag{1.7}$$

case (ii): $$\int_0^1 f'^2(x)\, dx = 1. \tag{1.8}$$

In case (i) we solve our problem provided $f'(x)$ is of bounded variation, while case (ii), i.e. the case of extremal functions of S seems to be more difficult. In this case we can give a solution only if $f(x)$ is piecewise linear. In particular, we can treat the functions $f(x)=x$ $(0\leqq x\leqq 1)$ and $f(x)=-x$ $(0\leqq x\leqq 1)$.

In Sect. 2 some preliminary results will be presented in the form of lemmas. Sect. 3 contains our main results.

2. Preliminary Lemmas

Our results are based on the translation formula for Wiener integrals due to Cameron and Martin [3] (see also Skorokhod [6]), stated as

Lemma 1. *Let $W(x)$ $(0\leqq x\leqq 1)$ be a standard Wiener process; $\psi(x)\in C[0,1]$, $\psi(0)=0$, and suppose $\psi(x)$ is absolutely continuous with $\int_0^1 \psi'^2(x)\,dx<\infty$. Then*

$$P(\|W(x)-\psi(x)\|<z)=e^{-\frac{1}{2}\int_0^1 \psi'^2(x)\,dx}\int_{\{\|W(x)\|<z\}} e^{-\int_0^1 \psi'(x)\,dW(x)}\,dP_W. \tag{2.1}$$

From Lemma 1 we obtain the following inequalities:

Lemma 2. *Under the conditions of Lemma 1, the following inequalities hold:*

$$\begin{aligned} &e^{-\frac{1}{2}\int_0^1 \psi'^2(x)\,dx}\,P(\|W(x)\|<z)\\ &\qquad\leqq P(\|W(x)-\psi(x)\|<z)\leqq P(\|W(x)\|<z). \end{aligned} \tag{2.2}$$

If, furthermore, $\psi'(x)$ is of bounded variation and $T_0^1[\psi']$ denotes its total variation over the interval $[0,1]$, then

$$\begin{aligned} &P(\|W(x)-\psi(x)\|<z)\\ &\qquad\leqq P(\|W(x)\|<z)\exp\left(-\tfrac{1}{2}\int_0^1 \psi'^2(x)\,dx+z(|\psi'(1)|+T_0^1[\psi'])\right). \end{aligned} \tag{2.3}$$

Proof. The first inequality in (2.2) is an easy consequence of (2.1) and Jensen's inequality. For the second inequality in (2.2) we may refer to Anderson [1].

To show (2.3), we use the following estimation: on the set $\{\|W(x)\|<z\}$ we have

$$\begin{aligned} \left|\int_0^1 \psi'(u)\,dW(u)\right| &= \left|\psi'(1)\,W(1)-\int_0^1 W(x)\,d\psi'(x)\right|\\ &\leqq z(|\psi'(1)|+T_0^1[\psi']), \end{aligned} \tag{2.4}$$

hence (2.3) follows from (2.1) and (2.4). Thus Lemma 2 is proved.

Explicit expressions are well known for $P(\|W(x)\|<z)$. In the next lemma we give the distribution of $\|W(x)-\gamma x\|$, where γ is a real constant. The given distribution is suitable to obtain also an asymptotic expression near zero. A different-but of course equivalent-expression for $P(\|W(x)-\gamma x\|<z)$ is given in Skorokhod [6].

Lemma 3. *For real γ and $z \geqq 0$ we have*

$$P(\|W(x)-\gamma x\|<z)$$
$$=4\pi e^{-\frac{\gamma^2}{2}} ch(\gamma z) \sum_{r=0}^{\infty} \frac{(-1)^r(2r+1)}{4\gamma^2 z^2+(2r+1)^2\pi^2} e^{-\frac{(2r+1)^2\pi^2}{8z^2}}$$

$$=\frac{4\pi e^{-\frac{\gamma^2}{2}} ch(\gamma z)}{4\gamma^2 z^2+\pi^2} e^{-\frac{\pi^2}{8z^2}}+R(z), \tag{2.5}$$

where

$$|R(z)| \leqq \frac{\frac{4}{\pi} e^{|\gamma| z-\frac{\gamma^2}{2}-\frac{9\pi^2}{8z^2}}}{1-e^{-\frac{\pi^2}{8z^2}}}. \tag{2.6}$$

Proof. By applying Lemma 1, and by evaluating the Wiener integral, we have

$$P(\|W(x)-\gamma x\|<z)=e^{-\frac{\gamma^2}{2}} \int_{\{\|W(x)\|<z\}} e^{-\gamma W(1)} dP_W$$
$$=e^{-\frac{\gamma^2}{2}} \int_{-z}^{z} e^{-\gamma y} P(\|W(x)\|<z, W(1)=y)\, dy. \tag{2.7}$$

We use the following formula (see e.g. Feller [5]):

$$P(-b<W(t)<2z-b \text{ for } 0 \leqq t<T, W(T)=y)$$
$$=\frac{1}{z} \sum_{r=1}^{\infty} e^{-\frac{r^2\pi^2}{8z^2}T} \sin\frac{r\pi b}{2z} \sin\frac{r\pi(y+b)}{2z}, \tag{2.8}$$

where the probability $P(A, W(T)=y)$ is understood as

$$\lim_{\Delta y \to 0} (P(A, y \leqq W(T)<y+\Delta y)/\Delta y).$$

From (2.7) and (2.8) we obtain,

$$P(\|W(x)-\gamma x\|<z)$$
$$=e^{-\frac{\gamma^2}{2}} \int_{-z}^{z} e^{-\gamma y} \frac{1}{z} \sum_{r=1}^{\infty} e^{-\frac{r^2\pi^2}{8z^2}} \sin\frac{r\pi}{2} \sin\frac{r\pi(y+z)}{2z}\, dy$$
$$=e^{-\frac{\gamma^2}{2}} \sum_{r=1}^{\infty} (-1)^r e^{-\frac{(2r+1)^2\pi^2}{8z^2}} \frac{1}{z} \int_{-z}^{z} e^{-\gamma y} \sin\frac{(2r+1)\pi(y+z)}{2z}\, dy. \tag{2.9}$$

On integrating the last expression, we get (2.5).

To prove (2.6), we use the following inequalities:

$$ch(\gamma z) \leqq e^{|\gamma| z}, \tag{2.10}$$

$$\frac{2r+1}{4\gamma^2 z^2+(2r+1)^2\pi^2} \leqq \frac{1}{\pi^2}. \tag{2.11}$$

Hence

$$|R(z)|=\left|4\pi e^{-\frac{\gamma^2}{2}} ch(\gamma z)\sum_{r=1}^{\infty}\frac{(-1)^r(2r+1)}{4\gamma^2 z^2+(2r+1)^2\pi^2} e^{-\frac{(2r+1)^2\pi^2}{8z^2}}\right|$$

$$\leqq \frac{4}{\pi} e^{-\frac{\gamma^2}{2}+|\gamma|z}\sum_{r=9}^{\infty} e^{-\frac{r\pi^2}{8z^2}}=\frac{\frac{4}{\pi} e^{-\frac{\gamma^2}{2}+|\gamma|z-\frac{9\pi^2}{8z^2}}}{1-e^{-\frac{\pi^2}{8z^2}}}. \tag{2.12}$$

This proves Lemma 3.

Now consider piecewise linear functions $\psi(x)$. Assume that

$$\psi'(x)=\gamma_i, \quad a_{i-1}<x<a_i; \quad i=1,\dots,k, \tag{2.13}$$

where $a_0=0<a_1<\dots<a_k=1$; $\gamma_0=\gamma_{k+1}=0$; $\gamma_i \neq \gamma_{i-1}$, $i=2,3,\dots,k$; $\psi(0)=0$ and $\psi(x)$ is a continuous broken line. Put $\lambda=\min_{1\leqq i\leqq k}(a_i-a_{i-1})$.

Lemma 4. *For $\psi(x)$ defined above, we have*

$$P(\|W(x)-\psi(x)\|<z)$$

$$=e^{-\int_0^1 \psi'^2(x)\,dx-\frac{\pi^2}{8z^2}}\frac{4\pi\, ch(\gamma_k z)}{4\gamma_k^2 z^2+\pi^2}\prod_{i=2}^{k}\frac{\pi^2 sh(z(\gamma_i-\gamma_{i-1}))}{z(\gamma_i-\gamma_{i-1})(\pi^2+z^2(\gamma_i-\gamma_{i-1})^2)}$$

$$+R_1(z)\, e^{-\int_0^1 \psi'^2(x)\,dx}, \quad z\geqq 0 \tag{2.14}$$

where

$$|R_1(z)|\leqq\frac{k 2^k e^{-\frac{\pi^2}{8z^2}(1+3\lambda)}}{(1-e^{-\frac{\lambda\pi^2}{8z^2}})^k}\prod_{i=1}^{k}\frac{sh z(\gamma_{i+1}-\gamma_i)}{z(\gamma_{i+1}-\gamma_i)}. \tag{2.15}$$

Proof. Again, by virtue of Lemma 1 we have to evaluate the Wiener integral

$$I=\int_{\{\|W(x)\|<z\}} e^{-\int_0^1 \psi'(x)\,dW(x)}\,dP_W$$

$$=\int_{\{\|W(x)\|<z\}} e^{-\sum_{i=1}^{k}\gamma_i(W(a_i)-W(a_{i-1}))}\,dP_W$$

$$=\int\dots\int_{\prod_{i=1}^{k}\{|y_i|<z\}}\prod_{i=1}^{k} P(|W(x)|<z,\, a_{i-1}\leqq x<a_i,\, W(a_{i-1})=y_{i-1},\, W(a_i)=y_i)$$

$$\times e^{-\gamma_i(y_i-y_{i-1})}\,dy_i, \tag{2.16}$$

where $y_0=0$. From (2.8),

$$I=\int\dots\int_{\prod_{i=1}^{k}\{|y_i|<z\}}\prod_{i=1}^{k}\frac{1}{z}\sum_{r_i=1}^{\infty} e^{-\frac{r_i^2\pi^2}{8z^2}(a_i-a_{i-1})}$$

$$\times \sin\frac{r_i\pi(y_{i-1}+z)}{2z}\sin\frac{r_i\pi(y_i+z)}{2z} e^{-\gamma_i(y_i-y_{i-1})}\,dy_i=I_1+I_2, \tag{2.17}$$

where in I_1 we consider the terms of summation corresponding to $r_i=1$, $i=1, \ldots, k$, while I_2 involves all the other terms. Hence

$$I_1 = \int_{\prod_{i=1}^{k}\{|y_i|<z\}} \cdots \int \frac{1}{z^k} e^{-\frac{\pi^2}{8z^2}} \left(\sin\frac{\pi(y_1+z)}{2z}\right)^2 \cdots$$
$$\times \left(\sin\frac{\pi(y_{k-1}+z)}{2z}\right)^2 \sin\frac{\pi(y_k+z)}{2z}$$
$$\times e^{y_1(\gamma_2-\gamma_1)+\ldots+y_{k-1}(\gamma_k-\gamma_{k-1})-y_k\gamma_k} dy_1 \ldots dy_k. \tag{2.18}$$

Substituting $(y_i+z)/2z=u_i$ in the above integral, we get

$$I_1 = 2^k e^{-\frac{\pi^2}{8z^2}+z\gamma_1} \prod_{i=1}^{k-1} \int_0^1 (\sin \pi u_i)^2 e^{2zu_i(\gamma_{i+1}-\gamma_i)} du_i$$
$$\times \int_0^1 \sin \pi u_k e^{-2zu_k\gamma_k} du_k. \tag{2.19}$$

On integrating out, we obtain the first term on the right hand side of (2.14). To estimate $I_2=R_1(z)$, we use the inequality

$$\left|\sum_{r=j}^{\infty} e^{-\frac{r^2\pi^2}{8z^2}(a_i-a_{i-1})} \sin\frac{r\pi(y_{i-1}+z)}{2z} \sin\frac{r\pi(y_i+z)}{2z}\right|$$
$$\leqq \frac{e^{-\frac{j^2\pi^2}{8z^2}(a_i-a_{i-1})}}{1-e^{-\frac{\lambda\pi^2}{8z^2}}} \tag{2.20}$$

for $j=1$ and $j=2$.

Notice that in I_2 exactly one summation is from $r_i=2$ to ∞ and all the other summations are from $r_i=1$ to ∞. Therefore from (2.20)

$$|I_2| = |R_1(z)|$$
$$\leqq \int_{\prod_{i=1}^{k}\{|y_i|<z\}} \cdots \int \frac{e^{-\frac{\pi^2}{8z^2}}}{(1-e^{-\frac{\lambda\pi^2}{8z^2}})^k} \sum_{i=1}^{k} e^{-\frac{3\pi^2}{8z^2}(a_i-a_{i-1})}$$
$$\times \frac{1}{z^k} e^{-\sum_{i=1}^{k} y_i(\gamma_i-\gamma_{i-1})} dy_1 \ldots dy_k$$
$$\leqq \frac{k2^k e^{-\frac{\pi^2}{8z^2}(1+3\lambda)}}{(1-e^{-\frac{\lambda\pi^2}{8z^2}})^k} \prod_{i=1}^{k} \frac{sh(\gamma_{i+1}-\gamma_i)z}{(\gamma_{i+1}-\gamma_i)z}, \tag{2.21}$$

proving (2.15).

The proof of Lemma 4 is complete.

The following property of Strassen's function is well known [7]:

Lemma 5. *If $f(x)\in S$ and $0\leqq a\leqq 1$, then*

$$|f(u)-f(au)| \leqq (u(1-a))^{1/2} \leqq (1-a)^{1/2}. \tag{2.22}$$

Assume that $\{T_n\}$ is an increasing sequence and define the following variables:

$$U_n = \sup_{\frac{T_{n-1}}{T_n} \leqq x \leqq 1} \left| W(xT_n) - W(T_{n-1}) - \left(f(x) - f\left(\frac{T_{n-1}}{T_n}\right)\right)(2T_n \operatorname{loglog} T_n)^{1/2} \right|, \quad (2.23)$$

$$V_n = \sup_{0 \leqq x \leqq \frac{T_{n-1}}{T_n}} |W(xT_n)|, \quad (2.24)$$

$$Z(T) = \sup_{0 \leqq x \leqq 1} |W(xT) - f(x)(2T \operatorname{loglog} T)^{1/2}|, \quad (2.25)$$

$$Z_n = Z(T_n), \quad (2.26)$$

$$Z_n^{(1)} = \inf_{T_n \leqq T < T_{n+1}} Z(T), \quad (2.27)$$

where $f(x) \in S$.

Lemma 6. *By using the notations* (2.23)–(2.27), *the following inequalities hold:*

$$Z_n \leqq U_n + V_n + (2T_{n-1} \operatorname{loglog} T_n)^{1/2}, \quad (2.28)$$

$$Z_n \leqq Z_n^{(1)} + (2T_{n+1} \operatorname{loglog} T_{n+1})^{1/2} - (2T_n \operatorname{loglog} T_n)^{1/2} + (2(T_{n+1} - T_n) \operatorname{loglog} T_n)^{1/2}. \quad (2.29)$$

Proof. We prove first (2.28). Choose an $x (0 \leqq x \leqq 1)$.

For $0 \leqq x \leqq T_{n-1}/T_n$, from Lemma 5 we get

$$\begin{aligned} &|W(xT_n) - f(x)(2T_n \operatorname{loglog} T_n)^{1/2}| \\ &\leqq |W(xT_n)| + (2xT_n \operatorname{loglog} T_n)^{1/2} \\ &\leqq V_n + (2T_{n-1} \operatorname{loglog} T_n)^{1/2} \\ &\leqq U_n + V_n + (2T_{n-1} \operatorname{loglog} T_n)^{1/2}. \end{aligned} \quad (2.30)$$

For $T_{n-1}/T_n < x \leqq 1$, by using Lemma 5 again, we obtain

$$\begin{aligned} &|W(xT_n) - f(x)(2T_n \operatorname{loglog} T_n)^{1/2}| \\ &\leqq \left| W(xT_n) - W(T_{n-1}) - \left(f(x) - f\left(\frac{T_{n-1}}{T_n}\right)\right)(2T_n \operatorname{loglog} T_n)^{1/2} \right. \\ &\qquad \left. + W(T_{n-1}) - f\left(\frac{T_{n-1}}{T_n}\right)(2T_n \operatorname{loglog} T_n)^{1/2} \right| \\ &\leqq U_n + |W(T_{n-1})| + \left| f\left(\frac{T_{n-1}}{T_n}\right)(2T_n \operatorname{loglog} T_n)^{1/2} \right| \\ &\leqq U_n + V_n + (2T_{n-1} \operatorname{loglog} T_n)^{1/2}. \end{aligned} \quad (2.31)$$

(2.30) and (2.31) yield (2.28).

To show (2.29), define τ_n as the point where $Z(T)$ takes its infinium on the interval $[T_n, T_{n+1})$. Thus $Z_n^{(1)} = Z(\tau_n)$. Let $u (0 \leqq u \leqq 1)$ be arbitrary and put $x = u T_n/\tau_n$. Then $0 \leqq x \leqq T_n/\tau_n \leqq 1$ and we obtain

$$\begin{aligned}
&|W(uT_n)-f(u)(2T_n \operatorname{loglog} T_n)^{1/2}| \\
&= \left| W(x\tau_n)-f\left(\frac{x\tau_n}{T_n}\right)(2T_n \operatorname{loglog} T_n)^{1/2} \right| \\
&\leqq |W(x\tau_n)-f(x)(2\tau_n \operatorname{loglog} \tau_n)^{1/2}| \\
&\quad +|f(x)|((2\tau_n \operatorname{loglog} \tau_n)^{1/2}-(2T_n \operatorname{loglog} T_n)^{1/2}) \\
&\quad +\left| f(x)-f\left(\frac{x\tau_n}{T_n}\right)\right| (2T_n \operatorname{loglog} T_n)^{1/2} \\
&\leqq Z(\tau_n)+(2T_{n+1} \operatorname{loglog} T_{n+1})^{1/2}-(2T_n \operatorname{loglog} T_n)^{1/2} \\
&\quad +(2(T_{n+1}-T_n) \operatorname{loglog} T_n)^{1/2}, \qquad (2.32)
\end{aligned}$$

where in the last step we applied Lemma 5 again and the fact that $T_n \leqq \tau_n < T_{n+1}$. Since u is arbitrary, (2.32) yields (2.29).

This completes the proof of Lemma 6.

Put

$$W_1(u)=\frac{W(T_{n-1}+u(T_n-T_{n-1}))-W(T_{n-1})}{(T_n-T_{n-1})^{1/2}}, \qquad (2.33)$$

$$f_1(u)=\left(\frac{T_n}{T_n-T_{n-1}}\right)^{1/2}\left(f\left(\frac{T_{n-1}+u(T_n-T_{n-1})}{T_n}\right)-f\left(\frac{T_{n-1}}{T_n}\right)\right) \qquad (2.34)$$

for $0\leqq u\leqq 1$.

It is readily checked that $W_1(u)$ is again a standard Wiener process with $W_1(0)=0$, and $f_1(u)\in S$. Furthermore

$$\frac{U_n}{(T_n-T_{n-1})^{1/2}}=\sup_{0\leqq u\leqq 1}|W_1(u)-f_1(u)(2\operatorname{loglog} T_n)^{1/2}|. \qquad (2.35)$$

These facts will be used in the proofs of theorems in Sect. 3.

In Lemma 6 the sequence $\{T_n\}$ was an arbitrary increasing sequence, however in the proofs of our theorems we will use two particular sequences whose properties are stated in the next Lemma.

Lemma 7. (i) *If* $T_n=n^n$, $n\geqq 1$, *and* V_n *is defined by* (2.24), *then for any* $\kappa\geqq 0$ *we have*

$$\lim_{n\to\infty}(T_{n-1}/T_n)^{1/2}(\operatorname{loglog} T_n)^{\kappa}=0, \qquad (2.36)$$

$$P(\lim_{n\to\infty} V_n T_n^{-1/2}(\operatorname{loglog} T_n)^{\kappa}=0)=1. \qquad (2.37)$$

(ii) *If* $T_n=\exp\left(\frac{n}{(\log n)^3}\right)$, $n\geqq 20$, *then for any* $0\leqq\kappa\leqq 1/2$, *we have*

$$\lim_{n\to\infty}\frac{(T_{n+1}\operatorname{loglog} T_{n+1})^{1/2}-(T_n \operatorname{loglog} T_n)^{1/2}}{T_{n+1}^{1/2}}(\operatorname{loglog} T_{n+1})^{\kappa}=0, \qquad (2.38)$$

$$\lim_{n\to\infty}\left(\frac{T_{n+1}-T_n}{T_n}\right)^{1/2}(\operatorname{loglog} T_n)^{1/2}(\operatorname{loglog} T_{n+1})^{\kappa}=0, \qquad (2.39)$$

furthermore

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = 1. \tag{2.40}$$

Proof. (i) Let $T_n = n^n$, then it is easy to see that $T_n/T_{n-1} \sim en$ and $\log\log T_n \sim \log n$, hence (2.36) follows. For V_n we have by the law of the iterated logarithm,

$$P(\limsup_{n\to\infty} V_n(2T_{n-1} \log\log T_{n-1})^{-1/2} \leqq 1) = 1 \tag{2.41}$$

which together with (2.36) implies (2.37).

(ii) Let $T_n = \exp(n(\log n)^{-3})$, $n \geqq 20$. Then

$$1 \leqq \frac{T_{n+1}}{T_n} = \exp\left(\frac{n+1}{(\log(n+1))^3} - \frac{n}{(\log n)^3}\right) \leqq \exp\left(\frac{1}{(\log n)^3}\right), \tag{2.42}$$

and

$$\log\log T_n \sim \log n \tag{2.43}$$

from which (2.38), (2.39) and (2.40) follow easily.

We note that in part (ii) T_n is defined for $n \geqq 20$ only, because $\{n(\log n)^{-3}\}$ is increasing from $n=20$. For $n<20$ T_n may be defined so that the sequence $\{T_n\}$ is increasing but arbitrary otherwise.

3. Main Results

In this section we state and prove three theorems. The first of them provides universal results valid for any $f(x) \in S$, i.e. upper and lower bounds are given for the best rate $\varepsilon(T)$. Then we give the best rates for $f(x) \in S$ satisfying certain additional conditions. The cases $\int_0^1 f'^2(x)\,dx < 1$ and $\int_0^1 f'^2(x)\,dx = 1$ are treated separately in Theorem 2 and in Theorem 3, resp.

Theorem 1. *For any $f(x) \in S$ and $c>0$ we have*

$$P\left(\sup_{0\leqq x\leqq 1} \left|\frac{W(xT)}{(2T\log\log T)^{1/2}} - f(x)\right| < \frac{c}{(\log\log T)^{1/2}} \text{ i.o.}\right) = 1 \tag{3.1}$$

and

$$P\left(\sup_{0\leqq x\leqq 1} \left|\frac{W(xT)}{(2T\log\log T)^{1/2}} - f(x)\right| < \frac{\frac{\pi}{4}(1-c)}{\log\log T} \text{ i.o.}\right) = 0. \tag{3.2}$$

Proof. To prove (3.1), we choose $T_n = n^n$ and show that for arbitrary $c>0$,

$$P(Z_n < c\,T_n^{1/2} \text{ i.o.}) = 1, \tag{3.3}$$

where Z_n is defined by (2.26).

By using the inequality (2.28), the limit relations (2.36) with $\kappa = 1/2$ and (2.37) with $\kappa = 0$, it suffices to verify (3.3) with Z_n replaced by U_n and since U_n are independent, is suffices to show that

$$\sum_n P(U_n < c\,T_n^{1/2}) = \infty. \tag{3.4}$$

Referring to (2.35) and to Lemma 2, we get

$$\begin{aligned}
P(U_n < c T_n^{1/2}) &\geqq P(U_n < c(T_n - T_{n-1})^{1/2}) \\
&= P\Big(\sup_{0\leqq u\leqq 1} |W_1(u) - f_1(u)(2\,\mathrm{loglog}\,T_n)^{1/2}| < c\Big) \\
&\geqq \exp\left(-\mathrm{loglog}\,T_n \int_0^1 f_1'^2(u)\,du\right) P(\|W_1(u)\| < c) \\
&\geqq \frac{P(\|W_1(u)\| < c)}{\log T_n} = \frac{P(\|W_1(u)\| < c)}{n \log n}
\end{aligned} \tag{3.5}$$

from which (3.4) follows. This proves (3.1). To show (3.2) is suffices to establish

$$P\left(Z_n^{(1)} < \frac{\pi}{4}(1-c)\left(\frac{2T_{n+1}}{\mathrm{loglog}\,T_{n+1}}\right)^{1/2} \text{ i.o.}\right) = 0, \tag{3.6}$$

where $T_n = \exp(n(\log n)^{-3})$, $n \geqq 20$. In fact we show

$$\sum_n P\left(Z_n < \frac{\pi}{4}(1-c)\left(\frac{2T_{n+1}}{\mathrm{loglog}\,T_{n+1}}\right)^{1/2}\right) < \infty, \tag{3.7}$$

which together with the inequality (2.29) and the limit relations (2.38) and (2.39) with $\kappa = 1/2$, will imply (3.7). Note that Z_n and $Z_n^{(1)}$ are defined by (2.26) and (2.27), resp.

By using (2.2), (2.5) and (2.6) with $\gamma = 0$, we obtain for any $0 < c < 1$, and for n large enough,

$$\begin{aligned}
&P\left(Z_n < \frac{\pi}{4}(1-c)\left(\frac{2T_{n+1}}{\mathrm{loglog}\,T_{n+1}}\right)^{1/2}\right) \\
&\leqq P\left(\frac{\sup_{0\leqq x\leqq 1}|W(xT_n)|}{T_n^{1/2}} < \frac{\pi}{4}(1-c)\left(\frac{2T_{n+1}}{T_n\,\mathrm{loglog}\,T_{n+1}}\right)^{1/2}\right) \\
&= \frac{4}{\pi}\exp\left(-\frac{T_n\,\mathrm{loglog}\,T_{n+1}}{(1-c)^2\,T_{n+1}}\right) + O\left(\exp\left(-\frac{9T_n\,\mathrm{loglog}\,T_{n+1}}{(1-c)^2\,T_{n+1}}\right)\right) \\
&\leqq K\left(\frac{(\log n)^3}{n}\right)^{\frac{1}{1-c}},
\end{aligned} \tag{3.8}$$

where in the last step we used that for n large, $T_n/T_{n+1} \geqq 1/(1-c)$.

This shows (3.7) and as explained above, the proof of Theorem 1 is complete.

As mentioned already, our set of probability one for which (3.1) is valid, may depend on $f(x)$ and therefore our Theorem 1 does not imply part (ii) of Strassen's theorem. More precisely, it follows that (1.3) occurs i.o. with probability one, i.e. for all $\omega \in \Omega_f$ with $P(\Omega_f) = 1$, but Ω_f may depend on $f(x) \in S$, while Strassen's Ω_0 does not. It is not hard however to complete the proof of part (ii) of Strassen's theorem, one has to choose only a countable subset $(f_1, f_2 \dots)$ of S, dense in S. Then it is easy to see that $\Omega_0 = \bigcap_{i=1}^{\infty} \Omega_{f_i}$ will do as Strassen's Ω_0. Our results, of course, do not concern part (i) of Strassen's theorem.

Theorem 2. *If* $f(x)\in S$, $\int_0^1 f'^2(x)\,dx=\alpha<1$, $c>0$, *then*

$$P\left(\sup_{0\leq x\leq 1}\left|\frac{W(xT)}{(2T\operatorname{loglog}T)^{1/2}}-f(x)\right|<\frac{\pi(1+c)}{4(1-\alpha)^{1/2}\operatorname{loglog}T}\text{ i.o.}\right)=1. \tag{3.9}$$

If, furthermore, $f'(x)$ *is of bounded variation, then*

$$P\left(\sup_{0\leq x\leq 1}\left|\frac{W(xT)}{(2T\operatorname{loglog}T)^{1/2}}-f(x)\right|<\frac{\pi(1-c)}{4(1-\alpha)^{1/2}\operatorname{loglog}T}\text{ i.o.}\right)=0. \tag{3.10}$$

Proof. As in the proof of Theorem 1, to show (3.9), choose $T_n=n^n$ and prove

$$P\left(Z_n<\frac{\pi(1+c)(2T_n)^{1/2}}{4(1-\alpha)^{1/2}(\operatorname{loglog}T_n)^{1/2}}\text{ i.o.}\right)=1, \tag{3.11}$$

where Z_n is defined by (2.26). By using (2.28), (2.36) with $\kappa=1$ and (2.37) with $\kappa=1/2$, it suffices to verify that

$$\sum_n P\left(U_n<\frac{\pi(2T_n)^{1/2}}{4(1-\alpha)^{1/2}(\operatorname{loglog}T_n)^{1/2}}\right)=\infty. \tag{3.12}$$

From (2.35) and (2.2),

$$\begin{aligned}
&P\left(U_n<\frac{\pi(2T_n)^{1/2}}{4(1-\alpha)^{1/2}(\operatorname{loglog}T_n)^{1/2}}\right)\\
&=P\left(\frac{U_n}{(T_n-T_{n-1})^{1/2}}<\frac{\pi(2T_n)^{1/2}}{4(1-\alpha)^{1/2}(T_n-T_{n-1})^{1/2}(\operatorname{loglog}T_n)^{1/2}}\right)\\
&=P\Big(\sup_{0\leq u\leq 1}|W_1(u)-f_1(u)(2\operatorname{loglog}T_n)^{1/2}|\\
&\quad<\frac{\pi(2T_n)^{1/2}}{4(1-\alpha)^{1/2}(T_n-T_{n-1})^{1/2}(\operatorname{loglog}T_n)^{1/2}}\Big)\\
&\geqq\exp\left(-\operatorname{loglog}T_n\int_0^1 f_1'^2(u)\,du\right)P\left(\|W_1(u)\|<\frac{\pi 2^{1/2}}{4(1-\alpha)^{1/2}(\operatorname{loglog}T_n)^{1/2}}\right).
\end{aligned} \tag{3.13}$$

From (2.34), we obtain

$$\begin{aligned}
\int_0^1 f_1'^2(u)\,du&=\frac{T_n}{T_n-T_{n-1}}\int_0^1 f'^2\left(\frac{T_{n-1}+u(T_n-T_{n-1})}{T_n}\right)du\\
&=\int_{\frac{T_{n-1}}{T_n}}^1 f'^2(x)\,dx\leqq\int_0^1 f'^2(x)\,dx=\alpha,
\end{aligned} \tag{3.14}$$

Therefore by applying Lemma 3 with $\gamma=0$,

$$\begin{aligned}
&P\left(U_n<\frac{\pi(2T_n)^{1/2}}{4(1-\alpha)^{1/2}(\operatorname{loglog}T_n)^{1/2}}\right)\\
&\quad\geqq(\log T_n)^{-\alpha}\left(\frac{\pi}{4}e^{-(1-\alpha)\operatorname{loglog}T_n}+O(e^{-9(1-\alpha)\operatorname{loglog}T_n})\right)\\
&\quad=\frac{\pi}{4n\log n}+O((n\log n)^{8\alpha-9}),
\end{aligned} \tag{3.15}$$

hence (3.12) follows. This proves (3.9). For (3.10), similarly to the proof of Theorem 1, we have to verify that

$$\sum_n P\left(Z_n < \frac{\pi(1-c)}{4(1-\alpha)^{1/2}} \left(\frac{2T_{n+1}}{\log\log T_{n+1}}\right)^{1/2}\right) < \infty, \tag{3.16}$$

where $T_n = \exp(n(\log n)^{-3})$, $n \geqq 20$.

By using (2.3), (2.5) and (2.6) with $\gamma=0$, we get for n large enough,

$$\begin{aligned}
&P\left(Z_n < \frac{\pi(1-c)}{4(1-\alpha)^{1/2}} \left(\frac{2T_{n+1}}{\log\log T_{n+1}}\right)^{1/2}\right) \\
&\quad = P\left(\sup_{0\leqq x\leqq 1} \left|\frac{W(xT_n)}{T_n^{1/2}} - f(x)(2\log\log T_n)^{1/2}\right| \right. \\
&\quad\quad \left. < \frac{\pi(1-c)}{4(1-\alpha)^{1/2}} \left(\frac{2T_{n+1}}{T_n \log\log T_{n+1}}\right)^{1/2}\right) \\
&\quad \leqq \exp\left(-\alpha \log\log T_n + \frac{\pi(1-c)}{2(1-\alpha)^{1/2}} \left(\frac{T_{n+1}\log\log T_n}{T_n \log\log T_{n+1}}\right)(|f'(1)| + T_0^1[f'])\right) \\
&\quad\quad \times P\left(\sup_{0\leqq x\leqq 1} \left|\frac{W(xT_n)}{T_n^{1/2}}\right| < \frac{\pi(1-c)}{2(2(1-\alpha))^{1/2}} \left(\frac{T_{n+1}}{T_n \log\log T_{n+1}}\right)^{1/2}\right) \\
&\quad \leqq \frac{K}{(\log T_n)^\alpha} \exp\left(-\frac{1-\alpha}{(1-c)^2} \frac{T_n}{T_{n+1}} \log\log T_{n+1}\right) \\
&\quad \leqq \frac{K}{(\log T_n)^\alpha (\log T_{n+1})^{\frac{1-\alpha}{1-c}}} \leqq \frac{K}{(\log T_n)^{1+\frac{c(1-\alpha)}{1-c}}} \\
&\quad \leqq K\left(\frac{(\log n)^3}{n}\right)^{1+\frac{c(1-\alpha)}{1-c}}
\end{aligned} \tag{3.17}$$

with some constant K. This shows (3.16), completing the proof of Theorem 2.

It is an open problem, whether (3.10) is true without the condition that $f'(x)$ is of bounded variation. The main problem is to give a good asymptotic value, or at least an appropriate estimation for $P(\|W(x)-\psi(x)\|<z)$, as in Lemma 2. The case $\int_0^1 f'^2(x)\,dx = 1$ seems to be more difficult, we can give the best rate only if $f(x)$ is piecewise linear. So let $f(x)$ be a continuous broken line with $f(0)=0$, and

$$f'(x) = \beta_i, \qquad a_{i-1} < x < a_i \qquad (i=1,\dots,k), \tag{3.18}$$

where $a_0 = 0 < a_1 < \dots < a_{k-1} < a_k = 1$.

Theorem 3. *If* $f(x)$ *is defined as above and* $\int_0^1 f'^2(x)\,dx = 1$, *then*

$$P\left(\sup_{0\leqq x\leqq 1} \left|\frac{W(xT)}{(2T\log\log T)^{1/2}} - f(x)\right| < \frac{c}{(\log\log T)^{2/3}} \text{ i.o.}\right) = 1 \textit{ or } 0 \tag{3.19}$$

according as $c>\pi^{2/3}2^{-5/3}B^{-1/3}$ *or* $c<\pi^{2/3}2^{-5/3}B^{-1/3}$ *where* $B=|\beta_2-\beta_1|+\ldots+|\beta_k-\beta_{k-1}|+|\beta_k|$.

Proof. First let $c>\pi^{2/3}2^{-5/3}B^{-1/3}$ and define $T_n=n^n$. We show that

$$\sum_n P\left(U_n<\frac{c(2T_n)^{1/2}}{(\operatorname{loglog} T_n)^{1/6}}\right)=\infty. \tag{3.20}$$

By (2.35), we get

$$\begin{aligned} P&\left(U_n<\frac{c(2T_n)^{1/2}}{(\operatorname{loglog} T_n)^{1/6}}\right)\\ &=P\Big(\sup_{0\leqq u\leqq 1}|W_1(u)-f_1(u)(2\operatorname{loglog} T_n)^{1/2}|\\ &\qquad<\frac{c(2T_n)^{1/2}}{(T_n-T_{n-1})^{1/2}(\operatorname{loglog} T_n)^{1/6}}\Big), \end{aligned} \tag{3.21}$$

where $f_1(u)$ (see (2.34)) is again a continuous broken line with $f(0)=0$, $\int_0^1 f_1'^2(u)\,du\leqq 1$ and

$$f_1'(u)=\begin{cases}\beta_1\left(\dfrac{T_n-T_{n-1}}{T_n}\right)^{1/2}, & 0<u<\dfrac{a_1T_n-T_{n-1}}{T_n-T_{n-1}}\\ \beta_i\left(\dfrac{T_n-T_{n-1}}{T_n}\right)^{1/2}, & \dfrac{a_{i-1}T_n-T_{n-1}}{T_n-T_{n-1}}<u<\dfrac{a_iT_n-T_{n-1}}{T_n-T_{n-1}}, \quad i=2,\ldots,k.\end{cases} \tag{3.22}$$

We may assume that n is large enough to have $T_{n-1}/T_n<a_1$. Applying Lemma 4 with

$$\gamma_i=\beta_i\left(\frac{T_n-T_{n-1}}{T_n}\right)^{1/2}(2\operatorname{loglog} T_n)^{1/2}, \quad i=1,\ldots,k, \tag{3.23}$$

$$z=c\left(\frac{2T_n}{T_n-T_{n-1}}\right)^{1/2}(\operatorname{loglog} T_n)^{-1/6}, \tag{3.24}$$

we obtain further

$$\begin{aligned} P&\left(U_n<\frac{c(2T_n)^{1/2}}{(\operatorname{loglog} T_n)^{1/6}}\right)\\ &=\exp\left(-\operatorname{loglog} T_n\int_0^1 f_1'^2(u)\,du-\frac{\pi^2(T_n-T_{n-1})(\operatorname{loglog} T_n)^{1/3}}{16c^2T_n}\right.\\ &\quad\times\left[\frac{4\pi\, ch(2\beta_k c(\operatorname{loglog} T_n)^{1/3})}{16\beta_k^2c^2(\operatorname{loglog} T_n)^{2/3}+\pi^2}\right.\\ &\quad\times\left.\prod_{i=2}^k\frac{\pi^2\, sh(2c(\beta_i-\beta_{i-1})(\operatorname{loglog} T_n)^{1/3})}{2c(\beta_i-\beta_{i-1})(\operatorname{loglog} T_n)^{1/3}(\pi^2+4c^2(\beta_i-\beta_{i-1})^2(\operatorname{loglog} T_n)^{2/3})}+R_n\right], \end{aligned} \tag{3.25}$$

where

$$|R_n| \leq \frac{k 2^k \exp\left(-\frac{3\lambda\pi^2(T_n - T_{n-1})(\log\log T_n)^{1/3}}{16c^2 T_n}\right)}{\left(1-\exp\left(-\frac{\lambda\pi^2(T_n - T_{n-1})(\log\log T_n)^{1/3}}{16c^2 T_n}\right)\right)^k} \times \prod_{i=2}^{k+1} \frac{sh(2c(\beta_i - \beta_{i-1})(\log\log T_n)^{1/3})}{2c(\beta_i - \beta_{i-1})(\log\log T_n)^{1/3}}. \tag{3.26}$$

By comparing the first term in the squared bracket on the right-hand side of (3.25) with the upper bound of $|R_n|$ given by (3.26) it is easily seen that R_n compared to the first term, tends to zero, thus for sufficiently large n, with some constant K we have

$$P\left(U_n < \frac{c(2T_n)^{1/2}}{(\log\log T_n)^{1/6}}\right) \geq K \exp\left(-\log\log T_n - \frac{\pi^2}{16c^2}(\log\log T_n)^{1/3}\right) \times \frac{ch(2c\beta_k(\log\log T_n)^{1/3}) \prod_{i=2}^{k} sh(2c|\beta_i - \beta_{i-1}|(\log\log T_n)^{1/3})}{(\log\log T_n)^{k-1/3}}. \tag{3.27}$$

Since $sh(A(\log\log T_n)^{1/3}) \sim \frac{1}{2}\exp(A(\log\log T_n)^{1/3})$, as $n \to \infty$, if $A > 0$ and $ch u \geq \frac{1}{2}\exp(|u|)$, we get

$$P\left(U_n < \frac{c(2T_n)^{1/2}}{(\log\log T_n)^{1/6}}\right) \geq \frac{K_1}{n \log n} \frac{\exp((2cB - \pi^2/16c^2)(\log\log T_n)^{1/3})}{(\log\log T_n)^{k-1/3}} \geq \frac{K_2}{n \log n}, \tag{3.28}$$

because $2cB - \frac{\pi^2}{16c^2} > 0$. This shows (3.20).

The proof of the first half of Theorem 3 can be completed by using the inequlaity (2.28) and the limit relations (2.36) and (2.37) with $\kappa = 2/3$ and $\kappa = 1/6$, resp.

Assume now that $c < \pi^{2/3} 2^{-5/3} B^{-1/3}$ and let $T_n = \exp(n(\log n)^{-3})$, $n \geq 20$.

We show that

$$\sum_n P\left(Z_n < \frac{c(2T_{n+1})^{1/2}}{(\log\log T_{n+1})^{1/6}}\right) < \infty. \tag{3.29}$$

By applying Lemma 4, it can be seen after some calculations that for n large enough,

$$\begin{aligned} &P\left(Z_n<\frac{c(2T_{n+1})^{1/2}}{(\log\log T_{n+1})^{1/6}}\right) \\ &=P\left(\sup_{0\leq x\leq 1}\left|\frac{W(xT_n)}{T_n^{1/2}}-f(x)(2\log\log T_n)^{1/2}\right| \right.\\ &\left.<\frac{c(2T_{n+1})^{1/2}}{T_n^{1/2}(\log\log T_{n+1})^{1/6}}\right) \\ &\leq\frac{K}{\log T_n}\,\frac{\exp(-K_1(\log\log T_n)^{1/3})}{(\log\log T_n)^{k-1/3}} \end{aligned} \tag{3.30}$$

with some K and K_1. Hence (3.29) follows and the proof of Theorem 3 can be completed by referring to the inequality (2.29) and to the limit relations (2.38) and (2.39) both with $\kappa=1/6$.

References

1. Anderson, T.W.: The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Amer. Math. Soc. **6**, 170–176 (1955)
2. Bolthausen, E.: On the speed of convergence in Strassen's law of the iterated logarithm. Ann. Probability **6**, 668–672 (1978)
3. Cameron, R.H., Martin, W.T.: Transformation of Wiener integrals under translation. Ann. of Math. **45**, 386–396 (1944)
4. Chung, K.L.: On the maximum partial sums of sequences of independent random variables. Trans. Amer. Math. Soc. **64**, 205–233 (1948)
5. Feller, W.: An introduction to probability theory and its applications. Vol. 2. New York: Wiley 1966
6. Skorokhod, A.V.: Random processes with independent increments. (Russian). Moscow: Izdat. Nauka 1964
7. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 211–226 (1964)

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