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Smoluchowski's Theory of Coagulation in Colloids Holds Rigorously in the Boltzmann-Grad-Limit*

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Summary. We consider Smoluchowski's model of coagulation in colloids: n particles move in three-dimensional euclidean space according to Brownian motions independently of each other as long as the particles are at a distance greater than R. When two particles come to within a distance R they stick together and form a "double particle", which itself is in Brownian motion – and so on. In the Boltzmann-Grad-limit $n \to \infty$, $nR = \text{constant}$, we prove "propagation of chaos" and derive the kinetic equations for the densities of the k-fold particles.

Contents

1. Introduction

In [19] Smoluchowski developed a theory of coagulation in colloids based on Brownian motion. For a survey of his ideas see the beautiful lectures [18], part III: Theorie der Koagulation (p. 593–599), cf. also [5]. The underlying model may be described as follows. Consider n particles in three-dimensional

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euclidean space, each particle being surrounded by a sphere of influence of diameter R. The particles move independently of each other, according to a Brownian motion, as long as their distance is greater than R. When two particles meet at a distance R they stick together and form a "double particle", which itself is again in Brownian motion - and so on. Smoluchowski found on a heuristic basis an explicit formula for the average concentration of single, double, triple etc. particles at time t.

Our aim is to put Smoluchowski's theory on a firm mathematical basis. It turns out that it holds rigorously in the limit with $n\to\infty$, $R\to 0$ in such a way that nR remains constant. We refer the reader to [13] for a heuristic derivation of the kinetic equations for the densities of the k-fold particles as well as for an introduction to the basic ideas of the proof. In this paper we give detailed proofs.

To explain the meaning of the limit $nR=\text{const}$ let us consider for a moment the case of particles moving in general d-dimensional space $\mathbb{R}^d$ with $d\geqq 2$ (after that we restrict ourselves in the rest of the paper to the case $d=3$). Suppose we are given n point-particles in a bounded set $\Lambda\subset\mathbb{R}^d$ with volume V, each particle being surrounded by a ball of diameter R. Every particle moves according to a Brownian motion (with the diffusion constant D) as long as it does not meet any other particle. What is the average time T in which a tagged particle does not undergo a collision? Let $(\omega(t))_{0\leqq t}$ be the path of a Brownian particle; denote by

(1.1) $$W_R(t)=\{x+\omega(s)\colon x\in\mathbb{R}^d,\ |x|\leqq R,\ 0\leqq s\leqq t\}$$

the Wiener sausage, swept out by a ball of radius R during the time interval $[0,t]$, having the volume $|W_R(t)|$ and

(1.2) $$C(R)=\text{capacity of the ball with radius } R.$$

Then the mean free time T should be such that

(1.3) $$n|W_R(T)|\approx V \quad \text{for small } R.$$

By a scale transformation we get

(1.4) $$\begin{aligned}|W_R(t)|&=|\{x+\omega(R^2 s)\colon |x|\leqq R,\ 0\leqq s\leqq t/R^2\}|\\&=R^d\left|\left\{\frac{x}{R}+\frac{\omega(R^2 s)}{R}\colon |x|\leqq R,\ 0\leqq s\leqq t/R^2\right\}\right|\\&=R^d|W_1(t/R^2)|\end{aligned}$$

and therefore the condition (1.3) reads

(1.5) $$nR^d|W_1(T/R^2)|\approx V \quad \text{for small } R.$$

The asymptotic behaviour of the volume of the Wiener sausage for $t\to\infty$ is given by (cf. [20] and [9]; for a use of the Wiener sausage in connection with the Feynman-Kac formula see [17] and [13])

(1.6) $$|W_R(t)|\sim\begin{cases}t\cdot C(R)\cdot D, & d\geqq 3\\ \dfrac{4\pi D\cdot t}{\log t}, & d=2\end{cases}\qquad (t\to\infty).$$

From (1.5) and (1.6) we get for $R \to 0$:

$$(1.7) \qquad \begin{aligned} &nR^d \cdot (T/R^2) \cdot C(1) \cdot D \approx V, && d \geqq 3 \\ &nR^2 \cdot \frac{4\pi D T/R^2}{\log(T/R^2)} = \frac{4\pi D T}{(1/n)\cdot(\log T - \log R^2)} \approx V, && d=2. \end{aligned}$$

Neglecting the term $(1/n) \cdot \log T$ we arrive at

$$(1.8) \qquad T \approx \begin{cases} |(n/V) \cdot R^{d-2} \cdot C(1) \cdot D|^{-1}, & d \geqq 3 \\ |(n/V) \cdot (2\pi/|\log R|) \cdot D|^{-1}, & d=2. \end{cases}$$

We can write (1.8) in the following form (using the right normalization of the capacity):

$$(1.9) \qquad T \approx |(n/V) \cdot C(R) \cdot D|^{-1} \quad (d \geqq 2)$$

(according to [16], Chap. 3.5, at the top of p. 81, the capacity of a set $B \subset \mathbb{R}^2$, which is contained in an open unit disc, is given by (Robin constant of $B)^{-1}$; for a disc B of radius R the capacity can be computed as $2\pi/|\log R|$.).

Back to the case $d=3$: Because of (1.8) and (1.9) the limit $nR = \text{const}$ is the limit of constant mean free time (the so called Boltzmann-Grad-limit) or equivalently the limit in which the number of balls in the (fixed) volume V times the capacity of a ball is constant.

The organization of the paper is as follows:

In Sect. 2 we formulate the problem and the result precisely, in Theorem 2.1 for the case of unlabeled particles (disregarding their multiplicities) and in Theorem 2.2 for the general case of particles with multiplicities. To describe the result in the former case, write for Borel sets $A \subset \mathbb{R}^3$ and times $t \geqq 0$

$$(1.10) \qquad N(t; A) = \text{number of unlabeled particles in } A \text{ at time } t.$$

Then if the initial distributions of the n particles ($n \in \mathbb{N}$) are such that $N(0; A)/n$ converges in probability for $n \to \infty$ ($nR = \text{const}$) for all Borel sets $A \subset \mathbb{R}^3$ (and if the correlation functions at time 0 satisfy a certain natural boundedness condition which we do not formulate here), the same holds for $N(t; A)/n$ for all $t \geqq 0$ (this is another formulation of what is called propagation of chaos), and in fact $N(t; A)/n \to \int_A dx\, p(t, x)$, where $p(t, x)$ is the solution of the kinetic equation

$$(1.11) \qquad \frac{\partial}{\partial t} p(t, x) = D \cdot \Delta p(t, x) - 4\pi R D n (p(t, x))^2.$$

In Sects. 3 and 4 the four main propositions 3.1–3.4 are stated and the proof of Theorem 2.1 is given by means of them. We give the proofs of Propositions 3.1–3.4 in Sects. 5–8. The basic idea of the proof of Theorems 2.1 and 2.2 is the same as used by O.E. Lanford III in his work on the derivation of the Boltzmann equation ([10–12], cf. also [21]), namely to develop the correlation functions in a perturbation series and to estimate the norm of every term of the series uniformly in R. Therefore, in order to prove the convergence of the rescaled correlation functions for $R \to 0$ it is sufficient to show the convergence of every term of the series. In substantiating this scheme in the present case of Brownian motions some stochastic flavour comes in related to the Wiener sausage as indicated above. In Sect. 9 we sketch some additional ingredients needed for the proof of Theorem 2.2. Auxiliary results of a technical character, which are needed in the proofs given in Sects. 5–8, are collected in the last section.

Finally we mention briefly other work on related stochastic models: After the pioneering work of M. Kac [7] propagation of chaos was proved for different stochastic models, e.g. in [14], [1]. In these cases however the collision rule is purely stochastic, which is not so in Smoluchowski's model. Problems of coalescing and annihilating random walks on the lattice $\mathbb{Z}^d$, where the passage to the Boltzmann-Grad-limit is not necessary, are treated in [2, 3].

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2. Result

We begin with some basic notation.

(i) Initial data.

Let two numbers $\lambda > 0$, $D > 0$ be given, fixed for the rest of the paper. To every $n \in \mathbb{N}$ there corresponds a positive number $R > 0$ by the relation

$$4\pi R D n = \lambda. \tag{2.1}$$

λ^{-1} means the mean free time, called by Smoluchowski "coagulation time" (cf. (1.9) with $V=1$). When we write $n \to \infty$ in the sequel this implies always $R \to 0$ according to (2.1) and vice versa. Suppose further that we are given a symmetric probability density $\pi_n^{(R)}(x_1, \ldots, x_n)$ ($n \in \mathbb{N}$, $x_i \in \mathbb{R}^3$) such that

$$\int_{|x_i - x_k| \leqq R} dx_1 \ldots dx_n \, \pi_n^{(R)}(x_1, \ldots, x_n) = 0 \quad \text{for all } i \neq k \quad (i, k \leqq n). \tag{2.2}$$

Expectation with respect to the initial distribution $\pi_n^{(R)}$ will be denoted by $E^{(R)}$.

(ii) Time evolution.

Starting with n particles, each of them moves according to a Brownian motion with diffusion constant D, independently of all other particles as long as it is at a distance $> R$ away from them. At time $t = 0$ each particle has multiplicity 1; at the first time t when the distance between a pair of particles, say particle i and particle k, is R, the following happens (notice that triple and higher collisions do not occur with probability 1): with probability 1/2 the particle k is annihilated, the multiplicity of the particle i is augmented by the multiplicity of particle k, and particle i continues to describe a Brownian motion with the same diffusion constant D as before. With probability 1/2 the analogous event happens, with the role of k and i interchanged. For simplicity we assume that D and R remain constant after collisions, an assumption which is not unreasonable according to [18, 5]. The notation $(x_1, \ldots, x_j;\ k_1, \ldots, k_j)$ means that the particle at x_i has multiplicity $k_i \in \mathbb{N}$ $(1 \leqq i \leqq j)$.

(iii) Correlation functions.

Analogously to (1.10) where the multiplicities of the particles are disregarded we define for labeled particles

(2.3) $$N(t; A, k) = \text{number of particles with multiplicity } k, \text{ which are at time } t \text{ in the Borel set } A \quad (k \in \mathbb{N}).$$

The j^{th} correlation function $\rho_j^{(R)}(t; x_1, \ldots, x_j; k_1, \ldots, k_j)$ for labeled particles $(x_1, \ldots, x_j; k_1, \ldots, k_j) \in (\mathbb{R}^3)^j \times \mathbb{N}^j$ can be defined by

(2.4) $$\sum_{k_1 \in I} \cdots \sum_{k_j \in I} \int_{A \times \ldots \times A} \int dx_1 \ldots dx_j \, \rho_j^{(R)}(t; x_1, \ldots, x_j; k_1, \ldots, k_j)$$
$$= E^{(R)}\Big[\Big(\sum_{k_1 \in I} N(t; A, k_1)\Big)\Big(\sum_{k_2 \in I} N(t; A, k_2) - 1\Big) \ldots \Big(\sum_{k_j \in I} N(t; A, k_j) - j + 1\Big)\Big]$$
$$\text{for all Borel sets } A \subset \mathbb{R}^3 \text{ and all subsets } I \subset \mathbb{N} \quad (1 \leqq j \leqq n).$$

This is sufficient for the definition of $\rho_j^{(R)}(t; x, k)$ for almost all $x \in \mathbb{R}^{3j}$ and all $k \in \mathbb{N}^j$ because of the symmetry of the moment measures under permutations of the coordinates (cf. [15], p. 299). The j^{th} correlation function $\rho_j^{(R)}(t; x_1, \ldots, x_j)$ for unlabeled particles $(x_1, \ldots, x_j)$ is defined similarly by the special choice of $I = \mathbb{N}$ in (2.4). In particular $\rho_j^{(R)}(t; x, k)$ and $\rho_j^{(R)}(t; x)$ are related by

(2.5) $$\rho_j^{(R)}(t; x) = \sum_{k \in \mathbb{N}^j} \rho_j^{(R)}(t; x, k).$$

Especially for $t = 0$ the correlation function $\rho_j^{(R)}(0; x)$ can be expressed easily in terms of the initial density $\pi_n^{(R)}$ by the relation

(2.6) $$\rho_j^{(R)}(0; x_1, \ldots, x_j)$$
$$= n(n-1) \ldots (n-j+1) \int dx_{j+1} \ldots dx_n \, \pi_n^{(R)}(x_1, \ldots, x_n) \quad (1 \leqq j \leqq n).$$

Since n tends to infinity it is reasonable to rescale the correlation functions:

(2.7) $$f_j^{(R)}(t; x) = \begin{cases} n^{-j} \rho_j^{(R)}(t; x), & j \leqq n \\ 0 & j > n \end{cases} \quad (x \in \mathbb{R}^{3j}),$$

(2.8) $$f_j^{(R)}(t; x, k) = \begin{cases} n^{-j} \rho_j^{(R)}(t; x, k), & j \leqq n \\ 0 & j > n \end{cases} \quad (x \in \mathbb{R}^{3j}).$$

(iv) Norms.

The L^∞-norm of a function $f_j\colon (\mathbb{R}^3)^j \to \mathbb{R}$ is given by

(2.9) $$\|f_j\| = \inf\{M\colon |f_j(x)| \leqq M \text{ for almost all } x \in \mathbb{R}^{3j}\} \quad (j \in \mathbb{N}).$$

Given a positive number z, define on the space

(2.10) $$Y = \{f = (f_1, f_2, \ldots)\colon f_j \text{ is a measurable function on the space } \mathbb{R}^{3j} \text{ into } \mathbb{R}_+\}$$

the norm

$$\|f\|_z = \sup_{j \geq 1} z^{-j} \|f_j\|. \tag{2.11}$$

Now we are ready to formulate the theorems. We assume two conditions on the correlation functions to be fulfilled at time $t=0$:

Assumptions

(C1) *there exists a number* $z>0$ *such that* $\sup_R \|f^{(R)}(0)\|_z < \infty$

(C2) *there exists a function* $p_0\colon \mathbb{R}^3 \to \mathbb{R}_+$ *such that*

$$\lim_{R\to 0} f_j^{(R)}(0; x_1, \ldots, x_j) = \prod_{i=1}^{j} p_0(x_i)$$

almost everywhere ($j \in \mathbb{N}$).

Theorem 2.1. *Under the conditions* (C1) *and* (C2) *the following holds for all* $t \geq 0$:

(a) $N(t; A)/n \to \int_A dx\, p(t,x)$ *in probability as* $R \to 0$ ($A \subset \mathbb{R}^3$ *bounded Borel set*), *where* $p(t,x)$ *is the unique solution in* L^∞ *of the kinetic equation*

$$\frac{\partial}{\partial t} p(t,x) = D \cdot \Delta p(t,x) - \lambda(p(t,x))^2 \quad \textit{with initial condition } p_0. \tag{2.12}$$

(b) *propagation of chaos, i.e.*

$$\lim_{R\to 0} f_j^{(R)}(t; x_1, \ldots, x_j) = \prod_{i=1}^{j} p(t, x_i) \quad \textit{almost everywhere } (j \in \mathbb{N}).$$

Theorem 2.2. *Under the conditions* (C1) *and* (C2) *the following holds for all* $t \geq 0$:

(a) $N(t; A, k)/n \to \int_A dx\, p(t; x, k)$ *in probability as* $R \to 0$ ($A \subset \mathbb{R}^3$ *bounded Borel set*, $k \in \mathbb{N}$), *where the functions* $p(t; x, k)$, $k \in \mathbb{N}$ *are the unique solution in* L^∞ *of the system of kinetic equations*

$$\begin{aligned} \frac{\partial}{\partial t} p(t; x, k) &= D \cdot \Delta p(t; x, k) \\ &\quad + \lambda\Big(\sum_{i+j=k} p(t; x, i) \cdot p(t; x, j) - 2 \sum_{i \geq 1} p(t; x, k) \cdot p(t; x, i)\Big) \end{aligned} \tag{2.13}$$

with initial condition $p(0; \cdot, k) = p_0$ *for* $k=1$ *and* $p(0; \cdot, k) = 0$ *for* $k>1$.

(b) *propagation of chaos, i.e.*

$$\lim_{R\to 0} f_j^{(R)}(t; x_1, \ldots, x_j; k_1, \ldots, k_j) = \prod_{i=1}^{j} p(t; x_i, k_i)$$

almost everywhere ($k_i \in \mathbb{N}$; $j \in \mathbb{N}$).

Remark 2.1. To realize condition (C2), let be given a bounded domain $\Lambda \subset \mathbb{R}^3$ with volume V which remains fixed. Distribute n particles in the domain Λ with $|x_i - x_k| \geqq R$ for $i \neq k$ in such a way that $N(0; A)/n \to \int_A dx\, p_0(x)$ stochastically as $n \to \infty$ ($A \subset \Lambda$ Borel set). If $V = 1$, then the number of particles per volume is $n/V = n$, thus λ^{-1} in (2.1) can really be interpreted as the mean free time (cf. (1.9)). However in order to get the spatially homogeneous case, i.e. to realize (C2) in such a way that the limit function p_0 is constant on the whole space $\mathbb{R}^3$, the volume V has to tend to infinity at the same time as n tends to infinity, in such a way that $(n/V) \cdot R$ remains constant. Regrettably the proof of the Theorems does not work in this case because of technical reasons. In the proof of the norm estimate given in Proposition 3.3 we have to use the inequality $R \leqq \text{const}\, n^{-1}$, which we cannot replace by $R \leqq \text{const}(n/V)^{-1}$ if V tends to infinity (see (7.31), where we use (7.30) and (7.21)). However the statement of the Theorems should remain true also in the spatially homogeneous case; in this case the functions $p(t)$ and $p(t;k)$ are independent of x and one can solve explicitly – this is the result given originally by Smoluchowski – the Eqs. (2.12) and (2.13) by

$$p(t) = \frac{p_0}{1 + \lambda p_0 t}, \qquad p(t;k) = p_0 \frac{(\lambda p_0 t)^{k-1}}{(1 + \lambda p_0 t)^{k+1}} \quad (k \geqq 1). \tag{2.14}$$

Remark 2.2. In this paper we restrict ourselves to the case of dimension $d = 3$. For the norm estimate in Proposition 3.3 we need an estimate of (cf. Lemma 7.1, (7.11) and (10.4))

$$\frac{d}{dt} \int_{|x| > R} dx\, f(t,x), \quad \text{where } f(t,x) \text{ solves the problem} \tag{2.15}$$

$$\begin{aligned} &\frac{\partial}{\partial t} f(t,x) = D \cdot \Delta f(t,x) && t > 0,\ |x| > R \\ &f(t,x) \to 1 && |x| \downarrow R \\ &f(0,x) = 0 && |x| > R \end{aligned} \tag{2.16}$$

(2.16) has a simple explicit solution in the case $d = 3$ (cf. Lemma 10.1), whereas the formulas do not look so pleasant for $d \neq 3$. Therefore we do not know whether the proof of the theorems goes through in the case $d \neq 3$ without essential change.

3. Notation and Formulation of the Main Propositions

In this section we state the Propositions 3.1–3.4 on which the proof of Theorem 2.1 is based. To avoid cumbersome notation we consider only the case of unlabeled particles in Sects. 3–8 and sketch in Sect. 9 the ingredients needed in addition for the proof of Theorem 2.2 in the case of labeled particles.

Continuation of the list of basic notations, begun in the last section:

(i) Sets.

(3.1) $D_j^{(R)}=\{x=(x_1,\dots,x_j)\in(\mathbb{R}^3)^j;\ |x_i-x_k|>R \text{ for } i\neq k;\ i,k\leq j\}$ $(j\in\mathbb{N})$,

(3.2) $B_j^{(R)}=\bigcup_{\substack{i,k\leq j\\ i\neq k}}\{x=(x_1,\dots,x_j)\in(\mathbb{R}^3)^j: |x_i-x_k|\leq R\}=\mathbb{R}^{3j}\setminus D_j^{(R)}$ $(j\in\mathbb{N})$.

If the context is clear we omit sometimes the indices R and j.

(ii) Brownian motion.

(3.3) $(\omega_i(t))_{0\leq t}$, $i\in\mathbb{N}$ are the paths of a sequence of independent Brownian motions on $\mathbb{R}^3$, each of them with diffusion constant D.

$\omega_{1,\dots,j}(t)=(\omega_1(t),\dots,\omega_j(t))$; if the context is clear we write sometimes only $\omega(t)$ instead of $\omega_{1,\dots,j}(t)$. The path ω shifted by $t\geq 0$ is denoted by ω_t^+, i.e. $\omega_t^+(s)=\omega(s+t)$ $(s\geq 0)$.

(3.4) $(\bar{\omega}(t))_{0\leq t}$ is the path of a Brownian motion on $\mathbb{R}^3$ with diffusion constant $2D$. For $x=(x_1,\dots,x_j)\in\mathbb{R}^{3j}$ denote by

(3.5) $P_x=P_{x_1,\dots,x_j}$ the probability measure corresponding to the Brownian motion $(\omega_{1,\dots,j}(t))_{0\leq t}$ starting at time $t=0$ at $(x_1,\dots,x_j)$; define $\bar{P}_x$ similarly for Brownian motions with diffusion constant $2D$.

(3.6) $p_j(t;x,y)=(4\pi D t)^{-3j/2}\exp(-|x-y|^2/4Dt)$ $(j\in\mathbb{N}, x\in\mathbb{R}^{3j}, y\in\mathbb{R}^{3j})$

is the transition density of $\omega_{1,\dots,j}(t)$, whereas

(3.7) $y\to q_j^{(R)}(t;x,y)$ $(x,y\in\mathbb{R}^{3j})$ denotes the (appropriately defined) density of the measure $K\to P_x(\omega_{1,\dots,j}(t)\in K,\ B_j^{(R)}$ is not hit during $[0,t])$. We will make use of the following four properties of $q_j^{(R)}$:

(3.8) $$q_j^{(R)}(t;x,y)=q_j^{(R)}(t;y,x),$$

(3.9) $q_j^{(R)}$ is continuous in every point $(t;x,y)\in(0,\infty)\times D_j^{(R)}\times D_j^{(R)}$,

(3.10) $q_j^{(R)}(t;x,y)\to 0$ if y converges towards a point $y_0\in B_j^{(R)}$,

(3.11) $$q_j^{(R)}(t;x,y)\to 0 \quad \text{if } |y|\to\infty$$

(for (3.8) and (3.9) see [16], Theorem 4.3; (3.10) follows because all points in $B_j^{(R)}$ are regular, see [16], Proposition 3.3).

(iii) Hitting times.

Denote by T_K the first hitting time of a set K. Special abbreviations:

(3.12) $T_{1,\dots,j}^{(R)}=T_{B_j^{(R)}}$ (sometimes simply $T_{1,\dots,j}$ or T),

(3.13) $T_{1,\dots,j/j+1}^{(R)}=T_K$ for $K=\bigcup_{i=1}^{j}\{|x_i-x_{j+1}|\leq R\}$,

(3.14) $$T_R=T_{\{x\in\mathbb{R}^3:|x|\leq R\}}.$$

(iv) Special measures.

Define for $j\in\mathbb{N}$, $t>0$, $x\in(\mathbb{R}^3)^{j+1}$, $h>0$, $R>0$ the following measures:

(3.15) $$\mu_{j+1}^{(h,R)}(t;x,dy) = (8\pi RD)^{-1} h^{-1} P_x(T_{1,\dots,j+1}^{(R)}\in(t,t+h], \omega_{1,\dots,j+1}(t+h)\in dy)$$

(measure on $(\mathbb{R}^3)^{j+1}$),

(3.16) $$e_{j+1}^{(h,R)}(t;x,dy) = (8\pi RD)^{-1} h^{-1} P_x(T_{1,\dots,j+1}^{(R)}\in(t,t+h], \omega_{1,\dots,j}(t+h)\in dy)$$

(measure on $\mathbb{R}^{3j}$),

(3.17) $\mu_{j+1}^{(R)}(t;x,dy)$, $e_{j+1}^{(R)}(t;x,dy)$ are the weak limits of $\mu_j^{(h,R)}(t;x,dy)$ respectively $e_{j+1}^{(h,R)}(t;x,dy)$ as $h\to 0$ (the existence of these limits will be shown in Sect. 5).

(v) Operators.

Given $k\in\mathbb{N}$, $0<t_1<t_2<\dots<t_k<t$, define the operator $A_k^{(R)}(t;t_1,\dots,t_k)$ on functions $f_{j+k}\in L^\infty((\mathbb{R}^3)^{j+k})$ by

(3.18) $$\begin{aligned}(A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k})(x) &= \int_{\mathbb{R}^{3(j+k)}} dx^{(k)} f_{j+k}(x^{(k)}) \int_{\mathbb{R}^{3(j+k-1)}} e_{j+k}^{(R)}(t_1;x^{(k)},dx^{(k-1)}) \\ &\quad\cdot \int_{\mathbb{R}^{3(j+k-2)}} e_{j+k-1}^{(R)}(t_2-t_1;x^{(k-1)},dx^{(k-2)}) \\ &\quad\dots \int_{\mathbb{R}^{3j}} e_{j+1}^{(R)}(t_k-t_{k-1};x^{(1)},dx^{(0)})\, q_j^{(R)}(t-t_k;x^{(0)},x).\end{aligned}$$

Furthermore define for $j\in\mathbb{N}$, $t>0$ the operators

(3.20) $$(S_j(t)f_j)(x) = \int dy\, f_j(y)\, p_j(t;y,x) \qquad (f_j\in L^\infty(\mathbb{R}^{3j}), x\in\mathbb{R}^{3j}),$$

(3.21) $$(S_j^{(R)}(t)f_j)(x) = \int dy\, f_j(y)\, q_j^{(R)}(t;x,y) \qquad (f_j\in L^\infty(\mathbb{R}^{3j}), x\in\mathbb{R}^{3j}).$$

C_{j+1}: $L^\infty(\mathbb{R}^{3(j+1)})\to L^\infty(\mathbb{R}^{3j})$ is defined by

(3.22) $$(C_{j+1}f_{j+1})(x) = \sum_{i=1}^{j} f_{j+1}(x_1,\dots,x_j,x_i)(f_{j+1}\in L^\infty(\mathbb{R}^{3(j+1)}), x=(x_1,\dots,x_j)\in\mathbb{R}^{3j}).$$

(3.18)–(3.22) give rise to operators $A_k^{(R)}(t;t_1,\dots,t_k)$, $S(t)$, $S^{(R)}(t)$ and C acting on the space of sequences Y into Y by the formulas

(3.23) $$\begin{gathered}(A_k^{(R)}(t;t_1,\dots,t_k)f)_j = A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k} \\ (S(t)f)_j = S_j(t)f_j, \qquad (S^{(R)}(t)f)_j = S_j^{(R)}(t)f_j, \\ (Cf)_j = C_{j+1}f_{j+1} \qquad (f\in Y, j\in\mathbb{N}).\end{gathered}$$

This ends the list of the notation basic for the rest of the paper and we can now state Propositions 3.1–3.4:

Proposition 3.1. *Let $t>0$, $R>0$, $x\in\mathbb{R}^{3(j+1)}$, $j\in\mathbb{N}$ be given. Then the weak limit*

$$\lim_{h\to 0} e_{j+1}^{(h,R)}(t;x,dy)=e_{j+1}^{(R)}(t;x,dy) \tag{3.24}$$

exists.

Proposition 3.2. *The rescaled correlation functions $f_j^{(R)}(t)$ $(j\in\mathbb{N}, t\geqq 0)$ have a series expansion of the following form:*

$$f_j^{(R)}(t;x)=(S_j^{(R)}(t)f_j^{(R)}(0))(x) + \sum_{k\geqq 1}(-\lambda)^k \int\limits_{0<t_1<\ldots<t_k<t}\int dt_1\ldots dt_k (A_k^{(R)}(t;t_1,\ldots,t_k)f^{(R)}(0))_j(x). \tag{3.25}$$

Proposition 3.3. *Let $z>0$ be given. There exists a universal constant $c<\infty$ and a positive number t_0, which only depends on D, λ and z, such that the following estimate holds uniformly in R:*

$$\begin{aligned}&\|\lambda^k \int\limits_{0<t_1<\ldots<t_k<t}\ldots\int dt_1\ldots dt_k A_k^{(R)}(t;t_1,\ldots,t_k)f^{(R)}(0)\|_{2z}\\ &\leqq c\cdot\left(\frac{t}{t_0}\right)^{k/2}\|f^{(R)}(0)\|_z \qquad (k\in\mathbb{N}; 0\leqq t\leqq 1).\end{aligned} \tag{3.26}$$

Proposition 3.4. *Assume condition* (C1) *and denote by $f^{(0)}(0)$ the function in Y which is given by*

$$f_j^{(0)}(0;x_1,\ldots,x_j)=\prod_{i=1}^{j}p_0(x_i) \qquad (j\in\mathbb{N}, x_i\in\mathbb{R}^3).$$

Then for all $k\in\mathbb{N}$ and $j\in\mathbb{N}$

$$\begin{aligned}&\lim_{R\to 0}\int\limits_{0<t_1<\ldots<t_k<t}\ldots\int dt_1\ldots dt_k(A_k^{(R)}(t;t_1,\ldots,t_k)f^{(R)}(0))_j(x)\\ &=\int\limits_{0<t_1<\ldots<t_k<t}\ldots\int dt_1\ldots dt_k(S(t-t_k)CS(t_k-t_{k-1})C\ldots S(t_2-t_1)CS(t_1)f^{(0)}(0))_j(x)\end{aligned} \tag{3.27}$$

almost everywhere.

4. Proof of Theorem 2.1 by Means of Propositions 3.1–3.4

Let the rescaled correlation functions $f^{(R)}(0)$ be given. Suppose that the conditions (C1) and (C2) are satisfied. Assuming furthermore that Propositions 3.1–3.4 hold, we prove Theorem 2.1 in this section. There are two reasons why it is possible to prove Theorem 2.1 for all times $t\geqq 0$ in contrast to the Boltzmann equation (for the latter case, cf. [10–12]). Firstly the kind of convergence to be proved for $t>0$ is the same as supposed in (C2) at time $t=0$, namely almost-sure convergence. Secondly the following Lemma 4.1 shows that also condition (C1) still holds for $t>0$, and in fact with the same parameter z. These two facts enable us to extend Theorem 2.1 successively to all positive t, after having it proved for small times t.

Lemma 4.1. *Let $R>0$ be given. Then*

$$\|\rho_j^{(R)}(t)\| \leqq \|\rho_j^{(R)}(0)\| \quad \text{for all } j\in\mathbb{N},\ t\geqq 0. \tag{4.1}$$

Proof. Compare the system of coalescing particles with a corresponding system of particles which start with the same initial distribution but move according to independent Brownian motions without coalescing. Then we get

$$\rho_j^{(R)}(t;x) \leqq (S_j(t)\,\rho_j^{(R)}(0))(x) \quad \text{a.e. } (t\geqq 0) \tag{4.2}$$

and therefore

$$\|\rho_j^{(R)}(t)\| \leqq \|S_j(t)\,\rho_j^{(R)}(0)\| \leqq \|\rho_j^{(R)}(0)\| \quad (t\geqq 0). \tag{4.3}$$

The proof of Theorem 2.1 consists of three small steps.

Step 1. It is sufficient to prove the statement (b) concerning the propagation of chaos: We repeat here the simple argument, given in [10], p. 108/109, because it shows the basic role of propagation of chaos. According to the definition of the rescaled correlation functions (cf. (2.5), (2.7)) we have for Borel sets $A\subset\mathbb{R}^3$

$$E^{(R)}[N(t;A)] = n\int_A dx_1\, f_1^{(R)}(t;x_1) \tag{4.4}$$

and

$$\begin{aligned} E^{(R)}[N(t;A)^2] &= E^{(R)}[N(t;A)\cdot(N(t;A)-1)] + E^{(R)}[N(t;A)] \\ &= n^2 \int_{A\times A} dx_1\,dx_2\, f_2^{(R)}(t;x_1,x_2) + n\int_A dx_1\, f_1^{(R)}(t;x_1) \end{aligned} \tag{4.5}$$

and therefore

$$\begin{aligned} E^{(R)}[N(t;A)/n] &= \int_A dx_1\, f_1^{(R)}(t;x_1) \\ E^{(R)}[(N(t;A)/n)^2] &= \int_{A\times A} dx_1\,dx_2\, f_2^{(R)}(t;x_1,x_2) + 1/n\int_A dx_1\, f_1^{(R)}(t;x_1). \end{aligned} \tag{4.6}$$

By (4.1) condition (C1) holds also for $f^{(R)}(t)$, so, if we assume the statement (b) about the convergence a.e. of the $f_j^{(R)}(t)$, we can apply Lebesgue's theorem and this leads to

$$\begin{aligned} \lim_{n\to\infty} E^{(R)}[N(t;A)/n] &= \int_A dx\, p(t,x) \\ \lim_{n\to\infty} E^{(R)}[(N(t;A)/n)^2] &= \int_{A\times A} dx_1\,dx_2\, p(t,x_1)\,p(t,x_2) = \Big(\int_A dx\, p(t,x)\Big)^2 \end{aligned} \tag{4.7}$$

which implies statement (a) of Theorem 2.1. By the way, a similar argument given in [10] shows, that conversely if we assume the convergence almost everywhere of the $f_j^{(R)}(t)$ as $R\to 0$ as well as $\sup_R \|f^{(R)}(t)\|_z < \infty$ $(t\geqq 0)$, then (a) implies (b).

Step 2. Proof of (b) for small t.

Let t_0 be as in Proposition 3.3 and choose $\bar{t}_0 < \min\{(\lambda z)^{-1}, t_0\}$.

(i) As suggested by (3.27) and (3.25) we define for $j \in \mathbb{N}$

$$(4.8)\quad f_j^{(0)}(t) = S_j(t) f_j^{(0)}(0) + \sum_{k \geqq 1} (-\lambda)^k \int_{0<t_1<\ldots<t_k<t} \ldots \int dt_1 \ldots dt_k (S(t-t_k)\, C \ldots S(t_2-t_1)\, C S(t_1) f^{(0)}(0))_j.$$

That this series is well defined at least for $t < (\lambda z)^{-1}$ is shown by

Lemma 4.2. *There exists a universal constant $c < \infty$ such that*

$$(4.9)\qquad \|\lambda^k \int_{0<t_1<\ldots<t_k<t} \ldots \int dt_1 \ldots dt_k S(t-t_k)\, C \ldots S(t_2-t_1)\, C S(t_1) f^{(0)}(0)\|_{2z} \leqq c \cdot (\lambda z)^k t^k \|f^{(0)}(0)\|_z \quad (t \geqq 0).$$

($\|f^{(0)}(0)\|_z < \infty$ by (C1) and (C2)). The proof is much simpler than that of the corresponding estimate (3.26) in Proposition 3.3, so we omit it.

(ii) Since the estimate (3.26) is uniform in R we get from (3.27) and (4.8), (4.9)

$$(4.10)\qquad \lim_{R \to 0} f_j^{(R)}(t) = f_j^{(0)}(t) \quad \text{a.e. } (j \in \mathbb{N}, t \leqq \bar{t}_0)$$

(iii) the series (4.8) factorizes, i.e.

$$(4.11)\qquad f_j^{(0)}(t; x_1, \ldots, x_j) = \prod_{i=1}^{j} f_1^{(0)}(t; x_i) \quad (t < (\lambda z)^{-1}; x_i \in \mathbb{R}^3, j \in \mathbb{N}).$$

To prove this multiply the series for $f_1^{(0)}(t; x_1)$ by that for $f_1^{(0)}(t; x_2)$ (for $t < (\lambda z)^{-1}$ there are no problems of convergence because of (4.9)) and get just the series for $f_2^{(0)}(t; x_1, x_2)$. The computation is based on the following three facts: factorization of $f_j^{(0)}(0)$ according to (C2);

$$S_j(t) \left(\bigotimes_{i=1}^{j} g_i \right) = \bigotimes_{i=1}^{j} (S_1(t)\, g_i)$$

for functions $g_i \in L^\infty(\mathbb{R}^3)$; the operator C_{j+1} is by Definition (3.22) the sum of i operators acting on pairs of coordinates.

(iv) $f_1^{(0)}(t)$ is the unique solution in L^∞ of the kinetic Eq. (2.12) on the interval $[0, \bar{t}_0]$: (4.8) and (4.11) imply

$$(4.12)\qquad f_1^{(0)}(t) = S_1(t) f_1^{(0)}(0) - \lambda \int_0^t dt_1\, S_1(t-t_1)\, C_2(f_1^{(0)}(t_1) \otimes f_1^{(0)}(t_1)) = S_1(t) f_1^{(0)}(0) - \lambda \int_0^t dt_1\, S_1(t-t_1) (f_1^{(0)}(t_1))^2.$$

Considerations from the regularity theory of partial differential equations, which are more or less standard and therefore omitted here, show that (4.12) has a unique solution which is moreover in the class $C^{1,2}((0,\infty) \times \mathbb{R}^3)$. This implies that $f_1^{(0)}(t)$ agrees on the time interval $[0, \bar{t}_0]$ with the unique solution in L^∞ of the kinetic Eq. (2.12) with initial condition p_0.

Step 3. Extension to all $t \geqq 0$.

Define for $t \in [0, \bar{t}_0]$

$$(4.13)\quad \bar{f}_j^{(0)}(t) = S_j(t) f_j^{(0)}(\bar{t}_0) + \sum_{k \geqq 1} (-\lambda)^k \int_{0 < t_1 < \ldots < t_k < t} \ldots \int dt_1 \ldots dt_k (S(t-t_k) C \ldots \ldots S(t_2 - t_1) C S(t_1) f^{(0)}(\bar{t}_0))_j.$$

From Lemma 4.1 it follows $\sup_R \|f^{(R)}(\bar{t}_0)\|_z < \infty$, hence condition (C1) is satisfied at time $\bar{t}_0$, and in fact with the same z as assumed at time $t=0$. Moreover, according to what is proved in step 2 we have $\lim_{R \to 0} f_j^{(R)}(\bar{t}_0) = \bar{f}_j^{(0)}(0)$ a.e. and $\bar{f}_j^{(0)}(0)$ factorizes. Therefore condition (C2) is also satisfied for $\bar{f}^{(0)}(0)$ and we can apply Theorem 2.1 proved so far for $t \in [0, \bar{t}_0]$ (Notice that the number of particles at time $\bar{t}_0$ is no longer fixed, i.e. the situation at time $\bar{t}_0$ is slightly different from that assumed at time 0, where we started with the initial distribution $\pi_n^{(R)}$ of n particles. However this circumstance does not cause any essential change in the proof.). Observing that $\bar{t}_0$ depends only on D, z, λ it follows that the conclusion of Theorem 2.1 when applied to the initial correlation functions $f^{(R)}(\bar{t}_0)$ holds in the same time interval $[0, \bar{t}_0]$.

This implies that the L^∞-function

$$(4.14)\quad g(t) = \begin{cases} f_1^{(0)}(t), & 0 \leqq t \leqq \bar{t}_0 \\ \bar{f}_1^{(0)}(t - \bar{t}_0), & \bar{t}_0 \leqq t \leqq 2\bar{t}_0 \end{cases}$$

solves (2.12) with initial condition p_0 in the time interval $[0, 2\bar{t}_0]$. But (2.12) has a unique L^∞-solution. Therefore Theorem 2.1 is proved for all $t \in [0, 2\bar{t}_0]$. In this way we can extend the result to the whole positive time axis. Hence Theorem 2.1 is proved.

5. Proof of Proposition 3.1: Existence of the Measures $e^{(R)}(t; x, dy)$

The aim of this section is to prove Proposition 3.1. Thereto it is sufficient to show that the weak limit of the measures $\mu_j^{(h,R)}(t; x, dy)$ as defined in (3.15) exists as h tends to zero. The proof of the following lemma will be given in Sect. 10 (here and in the following sections we state a series of computational or merely technical lemmas, the proofs of which are postponed until Sect. 10. They will be numbered by (10.i) with $1 \leqq i \leqq 8$ according to the order in which their proofs appear in the last section).

Lemma 10.2. *Let* $t > 0$, $x \in \mathbb{R}^{3j}$, $R > 0 (j \geqq 2)$ *be given. Then the measures* $\{\mu_j^{(h,R)}(t; x, dy): 0 < h < 1\}$ *have a uniformly bounded total mass and are tight.*

Because of this lemma we have to prove that all convergent subsequences of the measures $\mu_j^{(h,R)}$ have the same limit. The idea to attack this problem is similar to that of Itô-McKean [6], Chap. 7.7, where the convergence of the charges $dz\, h^{-1} P_z(T_R \leqq h,\ T_R(\omega_h^+) = \infty)$ to the equilibrium charge is shown (with

$z \in \mathbb{R}^3$ and T_R as defined in (3.14)). The Green function of the domain $\mathbb{R}^{3j}$ is denoted by

$$G(a,b) = \int_0^\infty ds\, p_j(s;a,b). \tag{5.1}$$

For the rest of this section, j and R are fixed, so they can be dropped ($B = B_j^{(R)}$, $\mu^{(h)} = \mu_j^{(h,R)}$). Assume for the moment the following two lemmas being proved:

Lemma 5.1. *For all $a, x \in \mathbb{R}^{3j}$ the limit*

$$\lim_{h \to 0} \int \mu^{(h)}(t;x,dy)\, G(a,y) \tag{5.2}$$

exists.

Lemma 5.2. *Let μ be a measure on $\mathbb{R}^{3j}$ and $h_k (k \in \mathbb{N})$ a sequence of positive numbers such that $h_k \to 0$ and $\mu^{(h_k)} \to \mu$. Then*

$$\lim_{k \to \infty} \int \mu^{(h_k)}(t;x,dy)\, G(a,y) = \int \mu(dy)\, G(a,y) \quad \textit{for all } a \notin \partial B. \tag{5.3}$$

Then the proof of Proposition 3.1 is finished because a finite measure on $\mathbb{R}^{3j}$ is uniquely determined by its Newtonian potential, more precisely: let μ and ν be finite measures on $\mathbb{R}^{3j}$ such that

$$\int \mu(dy)\, G(a,y) = \int \nu(dy)\, G(a,y) \tag{5.4}$$

almost everywhere, then $\mu = \nu$ (see [16], Chap. 3, Prop. 1.1). So it remains to prove Lemma 5.1 and 5.2.

Proof of Lemma 5.1. In the course of the proof we write e.g. $P_x(\omega(t+h) = y; T \in (t, t+h])$ instead of the correct but cumbersome

$$\int du\, q_j^{(R)}(t;x,u)\{p_j(h;u,y) - q_j^{(R)}(h;u,y)\}.$$

Using time reversal (cf. (3.8)) we get then

$$\begin{aligned}
(5.5)\quad & 8\pi R D \int \mu^{(h)}(t;x,dy)\, G(a,y) \\
&= h^{-1} \int dy\, P_x(\omega(t+h) = y; T \in (t, t+h]) \int_0^\infty ds\, P_a(\omega(s) = y) \\
&= h^{-1} \int_0^\infty ds \int dy\, P_a(\omega(s) = y)\, P_y(\omega(t+h) = x; T \leqq h, T(\omega_h^+) > t) \\
&= h^{-1} \int_0^\infty ds\, P_a\,(B \text{ is hit during } [s, s+h], \\
&\qquad \text{but not during } (s+h, s+h+t];\ \omega(s+t+h) = x).
\end{aligned}$$

To handle this expression more easily we define the "$t/2$-escape-times from B" in the following way (notice that the usual escape time from B corresponds to the case $t/2 = \infty$):

$$\begin{aligned}
(5.6)\quad & T^{(1)} = \inf\{s \geqq 0 : T_B(\omega_s^+) > t/2\} \\
& T^{(n+1)} = \inf\{s \geqq T^{(n)} + t/2 : T_B(\omega_s^+) > t/2\} \qquad (n \geqq 1).
\end{aligned}$$

Observe that $T^{(n)}+t/2$ is a stopping time (but not $T^{(n)}$ itself). To write (5.5) in a more appropriate manner we use

Lemma 10.4. *Let* $a\in\mathbb{R}^{3j}$, $x\in\mathbb{R}^{3j}$. *Then*

$$(5.7)\quad \int_0^\infty ds\, P_a\,(B \text{ is hit during } [s, s+h], \text{ but not during } (s+h, s+h+t];\, \omega(s+t+h)=x)$$
$$=\int_0^\infty ds \sum_{n\geqq 1} \int P_a(T^{(n)}\in[s,s+h];\, \omega(T^{(n)}+t/2)\in du)\, q_j^{(R)}(t/2;u,x)+o(h).$$

Proof. In Sect. 10.

Furthermore, given $x\in\mathbb{R}^{3j}$ fixed, define the function $F\colon[0,\infty)\to[0,\infty)$ by

$$(5.8)\qquad F(s)=\sum_{n\geqq 1}\int P_a(s\leqq T^{(n)}<\infty;\, \omega(T^{(n)}+t/2)\in du)\, q_j^{(R)}(t/2;u,x).$$

Then we get from (5.5), (5.7) and (5.8)

$$(5.9)\quad 8\pi RD\int \mu^{(h)}(t;x,dy)\,G(a,y)=h^{-1}\int_0^\infty ds(F(s)-F(s+h))+O(h)\qquad (a,x\in\mathbb{R}^{3j}).$$

Since the function F is decreasing and vanishes at infinity we get from (5.9)

$$(5.10)\qquad \lim_{h\to 0} 8\pi RD\int\mu^{(h)}(t;x,dy)\,G(a,y)=\lim_{h\to 0} h^{-1}\int_0^h ds\,F(s)=F(0).$$

So Lemma 5.1 is proved.

Proof of Lemma 5.2. Given $h_k\to 0$, $\mu^{(h_k)}\to\mu$, define for $b\in\mathbb{R}^{3j}$

$$(5.11)\qquad g_k(b)=\int\mu^{(h_k)}(t;x,dc)\,G(b,c),$$

$$(5.12)\qquad g(b)=\lim_{k\to\infty} g_k(b)\qquad \text{(the limit exists by virtue of Lemma 5.1).}$$

Then we have to show

$$(5.13)\qquad g(a)=\int\mu(dc)\,G(a,c)\qquad \text{for all } a\notin\partial B.$$

For the proof we use the following

Lemma 10.5(a). *The function* g *defined by* (5.12) *is continuous in every point* $b\notin\partial B$.

Proof. Postponed until Sect. 10.

The function $c\to G(b,c)$ is not bounded, so we cannot pass immediately from (5.11) to (5.13). However for $\tau>0$ the smoothed function $c\to\int db\, p(\tau;a,b)\,G(b,c)$ is continuous and bounded. This suggests the following trick: let $\tau>0$ and $a\notin\partial B$ be given and consider

$$(5.14)\qquad \int db\, p(\tau;a,b)\,g(b)=\int db\, p(\tau;a,b)\lim_{k\to\infty} g_k(b).$$

Proceeding formally – the justification is given in Lemma 10.5(b) – we get

$$(5.15)\qquad \begin{aligned}\int d b\, p(\tau; a, b)\, g(b) &= \lim_{k\to\infty} \int d b\, p(\tau; a, b) \int \mu^{(h_k)}(t; x, d c)\, G(b, c)\\ &= \lim_{k\to\infty} \int \mu^{(h_k)}(t; x, d c) \int d b\, p(\tau; a, b)\, G(b, c)\\ &= \int \mu(d c) \int d b\, p(\tau; a, b)\, G(b, c)\\ &= \int d b\, p(\tau; a, b) \int \mu(d c)\, G(b, c).\end{aligned}$$

Here we have used the above remark and the weak convergence $\mu^{(h_k)} \to \mu$. By Lemma 10.2, μ is a finite measure, hence the right hand side of (5.15) is finite. If $\tau \to 0$ in (5.15), the left hand side converges to $g(a)$ by the continuity of g at $a \notin \partial B$ (Lemma 10.5), and the right hand side converges to $\int \mu(dc)\, G(a, c)$ because the function $b \to \int \mu(d c)\, G(b, c)$ is an excessive function (cf. [6], p. 244). So (5.13) is proved, and the proof of Proposition 3.1 is finished.

6. Proof of Proposition 2: Derivation of the Perturbation Series

a) Heuristic Derivation of the BBGKY-Hierarchy

We begin with a heuristic consideration. The change $\delta \rho_j^{(R)}(t, x)$ during δt is due to free Brownian flow on the one hand and to the collision of one of the particles at $x_1, \ldots, x_j$ with an additional particle on the other hand:

$$(6.1)\qquad \delta \rho_j^{(R)}(t; x) = D \cdot \Delta \rho_j^{(R)}(t; x) \cdot \delta t + \tfrac{1}{2} \delta_{\mathrm{coll}} \rho_j^{(R)}(t; x).$$

To find $\delta_{\mathrm{coll}} \rho_j^{(R)}(t; x)$ we write the diffusion equation $\frac{\partial}{\partial t} p(t, x) = D \Delta p(t, x)$ in the form of a continuity equation

$$(6.2)\qquad \frac{\partial}{\partial t} p(t, x) + \operatorname{div}(-D \operatorname{grad} p(t, x)) = 0$$

giving to the term $-D \operatorname{grad} p(t, x)$ the meaning of the density of the particle flux, i.e. the flux of particles which streams through a surface element $d\sigma$ during the time interval δt is given by

$$(6.3)\qquad -D \vec{n} \cdot \operatorname{grad} p(t, x)\, d\sigma \cdot \delta t \qquad (\vec{n} \text{ is the normal vector at the point } x \in d\sigma).$$

Apply this consideration to our system of j particles $x_1, \ldots, x_j$. The surface at which a collision with an additional particle x_{j+1} can happen is

$$(6.4)\qquad S(x_1, \ldots, x_j) = \bigcup_{i=1}^{j} \{x_{j+1} \in \mathbb{R}^3 : |x_{j+1} - x_k| \geqq R\ (1 \leqq k \leqq j),\ |x_{j+1} - x_i| = R\}.$$

Observing that both colliding particles move according to a Brownian motion with a diffusion constant D, we have to replace D by $2D$ in the formula (6.3) and get

(6.5) $\delta_{\text{coll}}\rho_j^{(R)}(t;x) = -2D \int\limits_{S(x_1,\dots,x_j)} \sigma(dx_{j+1})\vec{n}\cdot \text{grad}_{j+1}\,\rho_{j+1}^{(R)}(t;x_1,\dots,x_j,x_{j+1})\cdot\delta t$

where σ = surface measure on $S(x_1,\dots,x_j)$

$\vec{n}$ = normal vector at S

grad_{j+1} = gradient with respect to the variable x_{j+1}.

Inserting (6.5) into (6.1) we arrive at the so called BBGKY-hierarchy (this is the usual name for the corresponding hierarchy in the case of deterministic dynamics given by Newton's equations, see for example [10]):

$$(6.6)\quad \frac{\partial}{\partial t}\rho_j^{(R)}(t;x) = D\cdot\Delta\rho_j^{(R)}(t;x) - D\int\limits_{S(x_1,\dots,x_j)} \sigma(dx_{j+1})\vec{n}\cdot\text{grad}_{j+1}\,\rho_{j+1}^{(R)}(t;x_1,\dots,x_{j+1}) \qquad (j\in\mathbb{N}).$$

Having (6.6) we could pass to the corresponding integral equations for all $j\in\mathbb{N}$ which would give us a series similar to that desired in Proposition 3.2. However there are analytical difficulties to derive (6.6) rigorously, in particular one has to be sure that the normal derivative of $\rho_{j+1}^{(R)}(t)$ exists on the boundary $\partial D_{j+1}^{(R)}$. One can formulate this problem in terms of partial differential equations, but $\partial D_{j+1}^{(R)}$ is not a smooth manifold and it does not seem to be easy to overcome the difficulties caused by the corners.

So we prefer a purely probabilistic procedure, giving first a combinatorial derivation of a series for $\rho_j^{(R)}(t)$ involving first hitting times and then transforming this series into the desired one with the help of Proposition 3.1.

b) Derivation of the ê-Series

Let $R>0$ be given, fixed in this section (and therefore omitted sometimes). We use the abbreviation

$$(6.7)\quad \hat{e}_{j+1}(dt;x,dy) = P_x(T_{1,\dots,j} > T_{1,\dots,j/j+1};\, T_{1,\dots,j/j+1}\in dt,\, \omega_{1,\dots,j}(T_{1,\dots,j/j+1})\in dy)$$

($x\in\mathbb{R}^{3(j+1)}$; the $\hat{e}_{j+1}(\cdot;x,\cdot)$ are therefore measures on $[0,\infty)\times\mathbb{R}^{3j}$).

Proposition 6.1. *$\rho_j^{(R)}(t)$ can be developed in a series of the following form*

$$(6.8)\quad \rho_j^{(R)}(t;x) = (S_j^{(R)}(t)\,\rho_j^{(R)}(0))(x) + \sum_{k\geq 1}(-1)^k 2^{-k}\int\limits_{\mathbb{R}^{3(j+k)}} dx^{(k)}\,\rho_{j+k}^{(R)}(0;x^{(k)})\int\limits_{(0,t)}\int \hat{e}_{j+k}(dt_1;x^{(k)},dx^{(k-1)}) \cdot\int\limits_{(t_1,t)}\int \hat{e}_{j+k-1}(dt_2-t_1;x^{(k-1)},dx^{(k-2)}) \cdots\int\limits_{(t_{k-1},t)}\int \hat{e}_{j+1}(dt_k-t_{k-1};x^{(1)},dx^{(0)})\,q_j^{(R)}(t-t_k;x^{(0)},x).$$

Proof. In the following we write ρ_j simply for $\rho_j^{(R)}$, $P_y(T_{1,\dots,j}>t;\ \omega_{1,\dots,j}(t)=x)$ for $q_j^{(R)}(t;y,x)$ etc. and we use the explicit definition (2.6) of $\rho_j(0)$. Furthermore, in order to avoid a formal contradiction, we think of a particle which is annihilated as being "coloured" and use this new terminology. A particle, after being coloured, is allowed to move on freely without interruption, but ceases to interact with other particles. It is disregarded from this time on.

A subsystem of $\{1,\dots,n\}$ is regarded as if it were coloured if and only if one of its members is coloured.

For abbreviation we define the following:

$S_K=$ the first time at which the system K is coloured ($K\subset\{1,\dots,n\}$),
$S_{K/i}=$ the time at which K is coloured by the particle i ($i\notin K$),
$\hat{P}_y=$ the probability measure induced by the process of coalescing (colouring) Brownian motions, starting at $y\in\mathbb{R}^{3n}$.
$\hat{P}=\hat{P}_{\pi_n}=\int dy\,\pi_n(y)\,\hat{P}_y$;
$J=\{1,\dots,j\}$,
$F=\{T_{1,\dots,j}>t,\omega_{1,\dots,j}(t)=x\}$,
$(n)_k=n(n-1)\dots(n-k+1)$ for $k\leqq n$.

We work with the following definition of $\rho_j(t)$ which turns out to be equivalent to the definition (2.4) (with $I=\mathbb{N}$)

$$\begin{aligned}\rho_j(t;x)&=(n)_j\hat{P}(F;J\text{ is not coloured}\\&\qquad\text{by any other particle } i\notin J\text{ during }[0,t])\\&=(n)_j\hat{P}(F;S_{1,\dots,j}>t).\end{aligned}\tag{6.9}$$

The main idea of the proof is the decomposition (6.11) below, which is first applied to the last term of (6.9).

Let for $i=j+1,\dots,n$

$$A_i=\{F;t\geqq S_{1,\dots,j/i},S_{1,\dots,j,i}=S_{1,\dots,j/i}\}.\tag{6.10}$$

A_i is the event that in the stream of $\{1,\dots,n\}$ the system J is coloured for the first time by the (still living) particle i. It therefore depends on the other particles outside $J\cup i$, whereas, for instance, the event $\{t\geqq S_{1,\dots,j/i}\}$ alone, by our convention, does not depend on $k\notin J\cup i$. Then we have

$$\begin{aligned}\hat{P}(F;S_{1,\dots,j}>t)&=\hat{P}(F)-\hat{P}(F;J\text{ is coloured by a particle } i\notin J)\\&=\hat{P}(F)-\hat{P}\left(F\cap\bigcup_{i=j+1}^{n}A_i\right)\\&=\hat{P}(F)-\sum_{i=j+1}^{n}\hat{P}(F\cap A_i)\\&=\hat{P}(F)-(n-j)\,\hat{P}(F\cap A_{j+1})\qquad(\text{symmetry of }\pi_n).\end{aligned}\tag{6.11}$$

In our original language (6.11) means roughly speaking that the probability of surviving of a subsystem J is equal to the probability of the free motion of J minus the probability that J is destroyed by a particle not in J.

Now, we apply the same argument to the last term in (6.11) and then successively to all terms in which events of the form

$$\{S_{1,\dots,j+k} = S_{1,\dots,j+k-1/j+k}\}$$

are still present. We get for example

$$\begin{aligned}(6.12)\qquad &\hat{P}(F; t \geqq S_{1,\dots,j/j+1}, S_{1,\dots,j+1} = S_{1,\dots,j/j+1}) \\ &= \hat{P}(F; t \geqq S_{1,\dots,j/j+1}) - (n-j-1)\,\hat{P}(F; t \geqq S_{1,\dots,j/j+1} \geqq \\ &\quad S_{1,\dots,j+1/j+2}, S_{1,\dots,j+2} = S_{1,\dots,j+1/j+2}) \\ &= \hat{P}(F; t \geqq S_{1,\dots,j/j+1}) - (n-j-1)\,\hat{P}(F; t \geqq \\ &\quad S_{1,\dots,j/j+1} \geqq S_{1,\dots,j+1/j+2}) \\ &\quad + (n-j-1)(n-j-2)\,\hat{P}(F; t \geqq S_{1,\dots,j/j+1} \geqq S_{1,\dots,j+1/j+2} \geqq \\ &\quad S_{1,\dots,j+2/j+3}, S_{1,\dots,j+3} = S_{1,\dots,j+2/j+3}),\end{aligned}$$

etc. By inserting (6.11), (6.12) etc. into (6.10), we obtain

$$\begin{aligned}(6.13)\qquad \rho_j(t; x) &= (n)_j \cdot \hat{P}(F) - (n)_{j+1} \cdot \hat{P}(F; t \geqq S_{1,\dots,j/j+1}) \\ &\quad + (n)_{j+2} \cdot \hat{P}(F; t \geqq S_{1,\dots,j/j+1} \geqq S_{1,\dots,j+1/j+2}) \\ &\quad \mp \dots .\end{aligned}$$

In view of the fact that

$$\begin{aligned}(n)_j \hat{P}(F) &= (n)_j \cdot \int dy\, \pi_n(y)\, P_y(T_{1,\dots,j} > t;\ \omega_{1,\dots,j}(t) = x) \\ &= (S_j(t)\, \rho_j(0))(x)\end{aligned}$$

and

$$\begin{aligned}&(n)_{j+1} \cdot \hat{P}(F; t \geqq S_{1,\dots,j/j+1}) \\ &\quad = 2^{-1} (n)_{j+1} \cdot \int dy\, \pi_n(y)\, P_y(F; t \geqq T_{1,\dots,j/j+1}) \\ &\quad = 2^{-1} \int dy\, \rho_{j+1}(0; y)\, P_y(T_{1,\dots,j} > t;\ \omega_{1,\dots,j}(t) = x,\ t \geqq T_{1,\dots,j/j+1})\end{aligned}$$

etc., the identity (6.13) is exactly the desired series (6.8).

c) Passage from Proposition 6.1 to Proposition 3.2

Observing the Definitions (3.16), (3.18) and (2.1) and the rescaling given by (2.7) it is, in view of Proposition 6.1, sufficient for the proof of Proposition 3.2 to show

Proposition 6.2. *Let* $j, k \in \mathbb{N}$, $t > 0$, $u \in \mathbb{R}^{3j}$ *be given. The measures* $\hat{e}^{(R)}_{j+1}$ *are defined by* (6.7). *Then*

$$(6.13)\qquad \int dx^{(k)}\,\rho_{j+k}^{(R)}(0;x^{(k)})\int_{(0,t)}\int \hat{e}_{j+k}^{(R)}(dt_1;x^{(k)},dx^{(k-1)})$$
$$\cdot\int_{(t_1,t)}\int \hat{e}_{j+k-1}^{(R)}(dt_2-t_1;x^{(k-1)},dx^{(k-2)})$$
$$\ldots\int_{(t_{k-1},t)}\int \hat{e}_{j+1}^{(R)}(dt_k-t_{k-1};x^{(1)},dx^{(0)})\,q_j^{(R)}(t-t_k;x^{(0)},u)$$
$$=(8\pi RD)^k\int_0^t dt_1\int_{t_1}^t dt_2\ldots\int_{t_{k-1}}^t dt_k\int\int$$
$$\ldots\int dx^{(k)}\,\rho_{j+k}^{(R)}(0;x^{(k)})\,e_{j+k}^{(R)}(t_1;x^{(k)},dx^{(k-1)})$$
$$\ldots e_{j+1}^{(R)}(t_k-t_{k-1};x^{(1)},dx^{(0)})\,q_j^{(R)}(t-t_k;x^{(0)},u).$$

Proof. We proceed by induction, beginning with the integration with respect to the variable t_k. To this purpose we need the following two lemmas:

Lemma 6.1. *Let* $\varepsilon>0$, $\alpha>\varepsilon$, $j\in\mathbb{N}$. *We consider functions* $\varphi\colon[\varepsilon,\alpha]\times\mathbb{R}^{3j}\to\mathbb{R}_+$ *which have the following three properties*

$$(6.14)\qquad \varphi \text{ is uniformly continuous,}$$

$$(6.15)\qquad \varphi(t,y)=0 \quad \text{if } y\in B_j^{(R)} \text{ and } t\in[\varepsilon,\alpha],$$

$$(6.16)\qquad \lim_{|x|\to\infty}\max_{\varepsilon\le t\le\alpha}\varphi(t,x)=0.$$

Then for all $s\in[0,\alpha-\varepsilon]$

$$(6.17)\qquad \int_{(\varepsilon+s,\alpha)}\int \hat{e}_{j+1}(dt-s;x,dy)\,\varphi(t,y)$$
$$=8\pi RD\int_{\varepsilon+s}^{\alpha}dt\int e_{j+1}(t-s;x,dy)\,\varphi(t,y)\qquad (x\in\mathbb{R}^{3(j+1)}).$$

Proof. At the end of this section.

Lemma 10.6. *Let* $\varepsilon>0$, $\alpha>\varepsilon$, $j\in\mathbb{N}$, *given a function* $\varphi\colon[\varepsilon,\alpha]\times\mathbb{R}^{3j}\to\mathbb{R}_+$ *which satisfies the properties* (6.14)–(6.16) *in Lemma* 6.1. *Then the function* $\bar{\varphi}\colon[\varepsilon,\alpha-\varepsilon]\times\mathbb{R}^{3(j+1)}\to\mathbb{R}_+$, *defined by*

$$(6.18)\qquad \bar{\varphi}(t,x)=\int_{t+\varepsilon}^{\alpha}ds\int e_{j+1}(s-t;x,dy)\,\varphi(s,y)$$

has again the properties (6.14)–(6.16).

Proof. In Sect. 10.

To prove Proposition 6.2 with the help of Lemma 6.1 and Lemma 10.6, we first change slightly the domain of integration in (6.13). We replace $\{0<t_1<\ldots<t_k<t\}$ by the domain

$$(6.19)\qquad \{(t_1,\ldots,t_k)\colon t_{k-1}+\varepsilon<t_k<t-\varepsilon,\ t_{k-2}+\varepsilon<t_{k-1}<t-2\varepsilon,$$
$$\ldots,\varepsilon<t_1<t-k\varepsilon\}\qquad (\varepsilon<t/(k+1))$$

and prove (6.13) with this new integration domain on both sides. Then letting $\varepsilon\to 0$, we are finished.

To prove (6.13) in this modified form, we begin with the last integral

$$\int_{(t_{k-1}+\varepsilon,\, t-\varepsilon)} \int \hat{e}_{j+1}(dt_k - t_{k-1};\, x^{(1)}, dx^{(0)})\, q_j^{(R)}(t-t_k;\, x^{(0)}, u). \tag{6.20}$$

Setting

$$\varphi(\tau, z) = q_j^{(R)}(t-\tau;\, z, u), \qquad \alpha = t-\varepsilon,\ \ \tau \in [\varepsilon, \alpha], \tag{6.21}$$

we see that φ satisfies the properties (6.14)–(6.16) if we notice that $\tau \leqq \alpha$ implies $t-\tau \geqq \varepsilon$ and if we use the properties (3.9)–(3.11) of $q_j^{(R)}$. So we can apply Lemma 6.1 to φ and get for $s = t_{k-1}$ ($t_{k-1} \leqq t - 2\varepsilon = \alpha - \varepsilon$ according to (6.19)), that (6.20) is equal to

$$\int_{t_{k-1}+\varepsilon}^{t-\varepsilon} dt_k \int e_{j+1}(t_k - t_{k-1};\, x^{(1)}, dx^{(0)})\, q_j^{(R)}(t-t_k;\, x^{(0)}, u). \tag{6.22}$$

We continue inductively by setting

$$\bar{\varphi}(\tau, z) = \int_{\tau+\varepsilon}^{\alpha} ds \int e_{j+1}(s-\tau;\, z, dy)\, \varphi(s, y), \qquad \tau \in [\varepsilon, \alpha-\varepsilon]. \tag{6.23}$$

Because of Lemma 10.6 we can again apply Lemma 6.1 to this function $\bar{\varphi}$, and proceeding step by step in this manner we finally get the desired formula (6.13). So Proposition 6.2 is proved.

We end this section with the

Proof of Lemma 6.1. To save notation, we give the proof of formula (6.17) for $s=0$ only.

Step 1. For fixed $t \in [\varepsilon, \alpha]$ the following holds:

$$\begin{aligned} 8\pi R D \int e_{j+1}(t;\, x, dy)\, \varphi(t, y) &= \lim_{h\to 0} h^{-1} \int P_x(\omega_{1,\dots,j}(t+h) \in dy; \\ &\quad T_{1,\dots,j} > T_{1,\dots,j/j+1},\ T_{1,\dots,j/j+1} \in (t, t+h])\, \varphi(t, y). \end{aligned} \tag{6.24}$$

To prove (6.24) it is sufficient to assume $j=2$ and to show

$$\lim_{h\to 0} h^{-1} \int P_{x_1 x_2}(\omega(t+h) \in dy;\ T_{1,2} \in (t, t+h])\, \varphi(t, y) = 0. \tag{6.25}$$

Because of (6.14) φ is uniformly continuous, i.e. given $\eta > 0$ there exists $\delta > 0$ such that

$$|y_1 - y_2| \leqq R + \delta \ \text{ implies } \ \varphi(t, y_1, y_2) \leqq \eta \quad \text{for all } t \in [\varepsilon, \alpha] \ \ (\text{notice (6.15)}). \tag{6.26}$$

Furthermore

$$\begin{aligned} &\lim_{h\to 0} h^{-1} \int P_{x_1 x_2}(\omega(t+h) \in dy;\ T \in (t, t+h])\, \varphi(t, y) \\ &\quad = \lim_{h\to 0} h^{-1} \int_{|y_1 - y_2| \leqq R+\delta} P_{x_1 x_2}(\omega(t+h) \in dy;\ T \in (t, t+h])\, \varphi(t, y) \end{aligned} \tag{6.27}$$

(a detailed proof can be given along the lines of (10.16) (see the proof of Lemma 10.2)).

By (6.14) and (6.17), φ is bounded, and therefore we get from (6.26) and (6.27)

$$\begin{aligned}(6.28)\qquad &\lim_{h\to 0} h^{-1}\int P_{x_1x_2}(\omega(t+h)\in dy;\ T_{1,2}\in(t,t+h])\,\varphi(t,y)\\ &\leqq \eta\cdot \lim_{h\to 0} h^{-1} P_{x_1x_2}(T_{1,2}\in(t,t+h])\cdot\|\varphi\|\leqq \text{const}\cdot\eta;\end{aligned}$$

in the last inequality we have used the explicit computation done in Lemma 10.1.

Step 2. Passage in (6.24) from $\omega(t+h)$ to $\omega(T_{1,\ldots,j/j+1})$: Again let $t\in[\varepsilon,\alpha]$ be fixed. Then

$$\begin{aligned}(6.29)\qquad & h^{-1}\int P_{x_1,\ldots,x_{j+1}}(\omega_{1,\ldots,j}(t+h)\in dy;\\ &\quad T_{1,\ldots,j}>T_{1,\ldots,j/j+1}\in(t,t+h])\,\varphi(t,y)\\ &=h^{-1}\int\int_{(t,t+h]} P_x(T_{1,\ldots,j}>T_{1,\ldots,j/j+1}\in d\tau_j\\ &\quad \omega_{1,\ldots,j}(T_{1,\ldots,j/j+1})\in du)\cdot\int dy\, p_j(t+h-\tau;\,u,y)\,\varphi(t,y).\end{aligned}$$

By virtue of the assumptions (6.14)-(6.17) there exists to any given $\eta>0$ a $\delta>0$ such that for all $h<\delta$, all $\tau\in(t,t+h]$, all $u\in\mathbb{R}^{3j}$

$$(6.30)\qquad \left|\int dy\,\varphi(t,y)\,p_j(t+h-\tau;\,y,u)-\varphi(t,u)\right|<\varepsilon.$$

By (6.24), (6.29), (6.30) we obtain

$$\begin{aligned}(6.31)\qquad & 8\pi RD\int e_{j+1}(t;\,x,dy)\,\varphi(t,y)\\ &=\lim_{h\to 0} h^{-1}\int P_x(T_{1,\ldots,j}>T_{1,\ldots,j/j+1};\ T_{1,\ldots,j/j+1}\in(t,t+h],\\ &\quad \omega_{1,\ldots,j}(T_{1,\ldots,j/j+1})\in dy)\,\varphi(t,y).\end{aligned}$$

Step 3. Passage from $\int dt\, e(t)$ to $\int \hat{e}(dt)$.

Due to (6.16) and its uniform continuity, φ can be approximated in the supnorm by finite sums of the form

$$(6.32)\qquad \sum_{i,l}\alpha_{il}1_{I_i}(t)\,1_{K_l}(u)\qquad (I_i\subset[\varepsilon,\alpha],\ K_l\subset\mathbb{R}^{3j}\ \text{compact rectangles}).$$

In view of (6.31) and (6.32) it is therefore sufficient to show

$$\begin{aligned}(6.33)\qquad & \int_I dt\lim_{h\to 0} h^{-1}\int P_x(T_{1,\ldots,j}>T_{1,\ldots,j/j+1}\in(t,t+h];\,\omega_{1,\ldots,j}(T_{1,\ldots,j/j+1})\in K)\\ &=\int_I P_x(T_{1,\ldots,j}>T_{1,\ldots,j/j+1}\in dt;\ \omega_{1,\ldots,j}(T_{1,\ldots,j/j+1})\in K)\\ &\quad\text{for rectangles } I\subset[\varepsilon,\alpha] \text{ and } K\subset\mathbb{R}^{3j}.\end{aligned}$$

(6.33) follows from the fact that

$$t\to P_x(T_{1,\ldots,j}>T_{1,\ldots,j/j+1}\leqq t;\ \omega_{1,\ldots,j}(T_{1,\ldots,j/j+1})\in K)$$

is the distribution function of a measure which is absolutely continuous with respect to Lebesgue measure. The measure $I\to P_x(T_{1,\ldots,j}>T_{1,\ldots,j/j+1}\in I)$ is absolutely continuous because it is majorized by

$$I \to \sum_{i=1}^{j} P_{x_i x_{j+1}}(T_{i,j+1} \in I),$$

and each measure

$$I \to P_{x_i x_{j+1}}(T_{i,j+1} \in I)$$

is absolutely continuous with a density given by Lemma 10.1.

Hence Lemma 6.1 is proved.

7. Proof of Proposition 3.3: Estimate of the Norm, Uniformly in R

In this section Proposition 3.3 is proved. The operator $A_k^{(R)}(t;\, t_1, \ldots, t_k)$ as defined by (3.18) is essentially a product of k operators of the form $e_{j+i}^{(R)}(t_{k-i+1} - t_{k-i};\, x, dy)$ $(1 \leqq i \leqq k)$. In Lemma 7.1 we give the basic norm estimate for one such operator, and in Lemma 7.2 we iterate this estimate. Then we are ready to prove (3.26) by integrating over the set $\{0 < t_1 < \ldots < t_k < t\}$. During this section we use the notation

$$h(s) = 1/\sqrt{2\pi D s} \qquad (s > 0). \tag{7.1}$$

Lemma 7.1. *Given $j \in \mathbb{N}$ let $F\colon \mathbb{R}^{3j} \to \mathbb{R}_+$ be a function which satisfies the following four conditions:*

(7.2) *F is uniformly continuous,*

(7.3) *$F(y) = 0$ for $y \in B_j^{(R)}$ (we will also write $B = B_j^{(R)}$ for simplicity),*

(7.4) $\|F\| < \infty$,

(7.5) *$F \in L^1(\mathbb{R}^{3j}, dy)$, where dy denotes Lebesgue measure on $\mathbb{R}^{3j}$.*

Then

$$\int dx \int e_{j+1}^{(R)}(t;\, x, dy)\, F(y) \leqq j(1 + R \cdot h(t)) \int dy\, F(y) \qquad (t > 0). \tag{7.6}$$

Proof. Since F has the properties (7.2)–(7.4), the arguments given in the proof of step 1 in Lemma 6.1 ((6.24)–(6.28)) are applicable to F leading to

$$\begin{aligned} \int e_{j+1}(t;\, x, dy)\, F(y) = (8\pi R D)^{-1} \lim_{h \to 0} h^{-1} \int P_x(\omega(t+h) \in dy;\\ T_{1,\ldots,j} > T_{1,\ldots,j/j+1} \in (t, t+h])\, F(y) \end{aligned} \tag{7.7}$$

hence we can estimate (notation $\hat{y}_i$ means that the i^{th} coordinate is missing)

$$\begin{aligned} \int e_{j+1}(t;\, x, dy)\, F(y) \leqq (8\pi R D)^{-1} \sum_{i=1}^{j} \lim_{h \to 0} \int dy_1 \ldots \hat{d}y_i \\ \ldots dy_j\, p_{j-1}(t+h;\, x_1, \ldots, \hat{x}_i, \ldots, x_j;\, y_1 \ldots \hat{y}_i \ldots y_j) \\ \cdot h^{-1} \int dy_i \int dz\, P_{y_i, z}(\omega_{i,j+1}(t+h) = (x_i, x_{j+1});\\ T_{i,j+1} \leqq h,\, T_{i,j+1}(\omega_h^+) > t) \cdot F(y) \end{aligned} \tag{7.8}$$

(we have used time reversal, cf. Lemma 10.1 (10.4)).

Integrate both sides of (7.8) with respect to dx and interchange $\int dx$ with $\lim_{h\to 0}$ on the right hand side (we indicate at the end of the proof why this interchange is allowed) and get

$$\int dx \int e_{j+1}(t;x,dy)\,F(y) \leqq (8\pi RD)^{-1} \sum_{i=1}^{j} \lim_{h\to 0} h^{-1} \int dy_1 \ldots \widehat{dy_i} \ldots dy_j \cdot \int dy_i \int dz\, P_{y_i,z}(T_{i,j+1} \leqq h, T_{i,j+1}(\omega_h^+) > t) \cdot F(y). \tag{7.9}$$

The main effect of the integration over dx is that (recall the conventions (3.5), (3.4) and (3.14))

$$\begin{aligned} h^{-1} \int_{\mathbb{R}^3} dz\, P_{y_i,z}(T_{i,j+1} \leqq h, T_{i,j+1}(\omega_h^+) > t) &= h^{-1} \int_{\mathbb{R}^3} dz\, \bar{P}_{y_i - z}(T_R \leqq h, T_R(\bar{\omega}_h^+) > t) \\ &= h^{-1} \int_{\mathbb{R}^3} dz\, \bar{P}_z(T_R \leqq h, T_R(\bar{\omega}_h^+) > t) \end{aligned} \tag{7.10}$$

is independent of y_i. Moreover letting $h\to 0$ we can compute the limit of (7.10) explicitly with the help of Lemma 10.1:

$$\lim_{h\to 0} h^{-1} \int_{\mathbb{R}^3} dz\, \bar{P}_z(T_R \leqq h, T_R(\bar{\omega}_h^+) > t) = 8\pi RD \left(1 + \frac{R}{\sqrt{\pi 2Dt}}\right). \tag{7.11}$$

So we get from (7.9), (7.10) and (7.11)

$$\int dx \int e_{j+1}(t;x,dy)\,F(y) \leqq j \cdot (1 + R/\sqrt{2\pi Dt}) \int dy\, F(y). \tag{7.12}$$

Therefore Lemma 7.1 is proved. It only remains to say a few words about the above interchange of $\int dx$ and $\lim_{h\to 0}$. It is justified by Lebesgue's theorem, if we can show

$$\sup_{0<h\leqq 1} h^{-1} \int dy_1 \int dz\, P_{y_1,z}(\omega(t+h) = (x_1, x_{j+1}); T \leqq h, T(\omega_h^+) > t) \cdot \int dy_2 \ldots dy_j\, p_{j-1}(t+h; x_2, \ldots, x_j; y_2, \ldots, y_j)\, F(y) \in L^1(\mathbb{R}^{3(j+1)}, dx_1 \ldots dx_{j+1}). \tag{7.13}$$

(7.13) can be proved by the same arguments as given in the course of the proof of Lemma 10.6 (10.58), which we therefore omit here.

Lemma 7.2. *Let* $j, k \in \mathbb{N}$ *and* $f_{j+k} \in L^\infty(\mathbb{R}^{3(j+k)})$. *Then for all* $R > 0$ *and all times* $t_1, \ldots, t_k, t$ *such that* $0 < t_1 < \ldots < t_k < t$ *the following estimate holds:*

$$\begin{aligned} &\| A_k^{(R)}(t; t_1, \ldots, t_k) f_{j+k} \| \\ &\leqq j(j+1) \ldots (j+k-1)(1 + R \cdot h(t_1)) \prod_{i=1}^{k-1} (1 + R \cdot h(t_{i+1} - t_i)) \, \| f_{j+k} \|. \end{aligned} \tag{7.14}$$

Proof. For fixed $x \in \mathbb{R}^{3(j+k)}$ we can write with a certain function $F^{(k-1)}$

$$\begin{aligned} &(A_k^{(R)}(t; t_1, \ldots, t_k) f_{j+k})(x) \\ &= \int_{\mathbb{R}^{3(j+k)}} dx^{(k)} f_{j+k}(x^{(k)}) \int_{\mathbb{R}^{3(j+k-1)}} e_{j+k}^{(R)}(t_1; x^{(k)}, dx^{(k-1)})\, F^{(k-1)}(x^{(k-1)}) \\ &\leqq \| f_{j+k} \| \int dx^{(k)} \int e_{j+k}^{(R)}(t_1; x^{(k)}, dx^{(k-1)})\, F^{(k-1)}(x^{(k-1)}). \end{aligned} \tag{7.15}$$

By Lemma 10.7, $F^{(k-1)}$ satisfies the conditions (7.2)–(7.5), so we can apply Lemma 7.1 and get

$$(7.16)\qquad (A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k})(x) \leqq (j+k-1)(1+R\cdot h(t_1)) \int_{\mathbb{R}^{3(j+k-1)}} dx^{(k-1)} F^{(k-1)}(x^{(k-1)}).$$

Since $F^{(k-1)}$ can again be expressed in the form

$$(7.17)\qquad F^{(k-1)}(x^{(k-1)}) = \int e_{j+k-1}^{(R)}(t_2-t_1;x^{(k-1)},dx^{(k-2)})F^{(k-2)}(x^{(k-2)})$$

we can apply Lemma 7.1 successively and after $(k-1)$ steps the last factor is given by

$$(7.18)\qquad \begin{aligned} &\int_{\mathbb{R}^{3(j+1)}} dx^{(1)} \int_{\mathbb{R}^{3j}} e_{j+1}^{(R)}(t_k-t_{k-1};x^{(1)},dx^{(0)})\, q_j^{(R)}(t-t_k;x^{(0)},x)\\ &\quad\leqq j(1+R\cdot h(t_k-t_{k-1}))\int dx^{(0)}\, q_j^{(R)}(t-t_k;x^{(0)},x)\\ &\quad\leqq j(1+R\cdot h(t_k-t_{k-1}))\end{aligned}$$

(we have applied Lemma 7.1 to the function $x^{(0)}\to q_j^{(R)}(t-t_k;x^{(0)},x)$. This is possible by (3.9)–(3.11)).

Altogether we have shown

$$(7.19)\qquad \begin{aligned} &A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k}(x)\\ &\quad\leqq \|f_{j+k}\|(j+k-1)\cdot(j+k-2)\dots j\cdot(1+R\cdot h(t_1))\prod_{i=1}^{k-1}(1+R\cdot h(t_{i+1}-t_i))\end{aligned}$$

for all $x\in\mathbb{R}^{3(j+k)}$. Therefore Lemma 7.2 is proved.

Proof of Proposition 3.3. By the definition (2.11) of the norm $\|\ \|_{2z}$ we have

$$(7.20)\qquad \begin{aligned} &\Big\|\lambda^k \int\limits_{0<t_1<\dots<t_k<t}\!\!\dots\!\int dt_1\dots dt_k\, A_k^{(R)}(t;t_1,\dots,t_k)f^{(R)}(0)\Big\|_{2z}\\ &\quad=\sup_{j\geqq 1}\Big\{(2z)^{-j}\Big\|\lambda^k \int\limits_{0<t_1<\dots<t_k<t}\!\!\dots\!\int dt_1\dots dt_k\, A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k}^{(R)}(0)\Big\|\Big\}.\end{aligned}$$

In the case $k>n$ we have $f_{j+k}^{(R)}(0)=0$ for all $j\in\mathbb{N}$ (cf. the convention (2.7)), we can therefore assume without loss of generality

$$(7.21)\qquad k\leqq n.$$

Because of (7.14) in Lemma 7.2 we can further estimate (7.20) by

$$(7.22)\qquad \begin{aligned} &\sup_{j\geqq 1}\{(2z)^{-j}\lambda^k j(j+1)\dots(j+k-1)\|f_{j+k}^{(R)}(0)\|\\ &\quad\cdot\int\limits_{0<t_1<\dots<t_k<t}\!\!\dots\!\int dt_1\dots dt_k(1+R\cdot h(t_1))\prod_{i=1}^{k-1}(1+R\cdot h(t_{i+1}-t_i))\}.\end{aligned}$$

Given $k \in \mathbb{N}$, consider the function $j \to F_k(j) = 2^{-j} j(j+1)\dots(j+k-1)$. F_k has its maximum at $j = k+1$, and Stirling's formula shows

(7.23)
$$\sup_{k \geq 1} \sup_{j \geq 1} \frac{2^{-j} j(j+1)\dots(j+k-1)}{k^k} \leq \sup_{k \geq 1} k^{-k} \sup_{j \geq 1} F_k(j) = \sup_{k \geq 1} k^{-k} 2^{-k-1} \frac{(2k)!}{k!} = c < \infty .$$

Inserting (7.23) in (7.22) and using furthermore $\|f_{j+k}^{(R)}(0)\| \leq z^{j+k} \|f^{(R)}(0)\|_z$ we can estimate (7.20) by

(7.24)
$$c \cdot (\lambda z)^k \|f^{(R)}(0)\|_z \cdot k^k \int_{0<t_1<\dots<t_k<t} \dots \int dt_1 \dots dt_k (1 + R \cdot h(t_1)) \prod_{i=1}^{k-1} (1 + R \cdot h(t_{i+1} - t_i)).$$

By (7.21) and (7.24) it is sufficient for the proof of (3.26) to show: there exists a constant $c_3 < \infty$, which depends only on D, λ and z, such that

(7.25)
$$k^k \int_{0<t_1<\dots<t_k<t} \dots \int dt_1 \dots dt_k (1 + R \cdot h(t_1)) \cdot \prod_{i=1}^{k-1} (1 + R \cdot h(t_{i+1} - t_i)) \leq (c_3 t)^{k/2} \qquad (R > 0, k \leq n, 0 \leq t \leq 1).$$

To prove (7.25) multiply out the product

$$(1 + R \cdot h(t_1)) \cdot \prod_{i=1}^{k-1} (1 + R \cdot h(t_{i+1} - t_i)).$$

To a given i $(0 \leq i \leq k)$ there exist exactly $\binom{k}{i}$ terms containing i times the factor 1, and $(k-i)$ times a factor of the form $R \cdot h(\cdot)$. Each of these $\binom{k}{i}$ terms can be estimated by

(7.26)
$$\left(R \cdot \int_0^t ds\, h(s) \right)^{k-i} \cdot \int_0^t dt_1 1 \cdot \int_{t_1}^t dt_2 1 \dots \int_{t_{i-1}}^t dt_i 1$$

[for the proof of (7.26) use estimates of the type

(7.27)
$$\int_{t_p}^t dt_{p+1} h(t_{p+1} - t_p) \int_{t_{p+1}}^t dt_{p+2} 1 \leq \int_0^t ds\, h(s) \int_{t_p}^t dt_{p+2} 1$$

$(0 \leq t_p \leq t)$].

The computation of (7.26) gives

(7.28)
$$R^{k-i} \left(\frac{2\sqrt{t}}{\sqrt{2\pi D}} \right)^{k-i} \frac{t^i}{i!} \qquad \text{(notice } t \leq \sqrt{t} \leq 1)$$
$$\leq R^{k-i} (\sqrt{2/\pi D})^{k-i} \cdot t^{k/2} \cdot \frac{1}{i!} \qquad \text{(setting } c_1 = \max\{1, \sqrt{2/\pi D}\})$$
$$\leq c_1^k t^{k/2} R^{k-i} \frac{1}{i!}.$$

By (7.26) and (7.28) we can therefore estimate the left hand side of (7.25) by

$$k^k \sum_{i=0}^{k} \binom{k}{i} c_1^k t^{k/2} R^{k-i} \frac{1}{i!}. \tag{7.29}$$

Because of (2.1) there exists a constant c_2 depending only on D and λ such that

$$R = \frac{c_2}{n} \leqq \frac{c_2}{k} \quad \text{for } k \leqq n. \tag{7.30}$$

Taking into account the assumption (7.21) we can insert (7.30) into (7.29) and get the upper bound

$$\begin{aligned} &c_1^k k^k t^{k/2} \sum_{i=0}^{k} \binom{k}{i} c_2^{k-i} \left(\frac{1}{k}\right)^{k-i} \frac{1}{i!} \\ &\leqq (c_1 c_2)^k t^{k/2} \sum_{i=0}^{k} \binom{k}{i} \frac{k^i}{i!} \quad (\text{use } k^i/i! \leqq e^k \text{ for all } i \in \mathbb{N}) \\ &\leqq (c_1 c_2 e)^k t^{k/2} \cdot 2^k. \end{aligned} \tag{7.31}$$

Setting $c_3 = (2 e c_1 c_2)^2$ we get the inequality (7.25), and therefore the proof of Proposition 3.3 is finished.

8. Proof of Proposition 3.4: Almost Sure Convergence

The core of the proof of the almost sure convergence (3.27) is the following

Proposition 8.1. *Let $F^{(R)}: \mathbb{R}^3 \to \mathbb{R}_+$ $(R = \lambda/4\pi D n;\ n = 1, 2, \ldots)$ be a sequence of functions with the following properties*

(8.1) *$F^{(R)}$ is continuous for all R,*

(8.2) *$\lim_{R\to 0} F^{(R)}(x) =: F(x)$ exists almost everywhere,*

(8.3) *there exists a function $\bar{F} \in L^1(\mathbb{R}^3)$ such that $F^{(R)}(x) \leqq \bar{F}(x)$ a.e. for all R*

$$\sup_R \|F^{(R)}\| < \infty. \tag{8.4}$$

Then it follows for all $t > 0$:

$$\lim_{R\to 0} \int_{\mathbb{R}^3} e_2^{(R)}(t; x, dy) F^{(R)}(y) = \int_{\mathbb{R}^3} dy\, p_2(t; x, (y, y)) F(y) \tag{8.5}$$

for all $x = (x_1, x_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $x_1 \neq x_2$.

To show Proposition 3.4 we proceed in the following way: in Sect. a) we prove Proposition 8.1, in Sect. b) we explain how the general problem (3.27) is reduced to Proposition 8.1 and in the final Sect. c) we give some error estimates needed for the reduction.

a) Proof of Proposition 8.1

(i) Setup of the proof:

The idea of the proof is to split up

$$e_2^{(R)}(t;x,dy)=\int q_2^{(R)}(t-\tau;x,du)\,e_2^{(R)}(\tau;u,dy),\qquad \tau \text{ small} \tag{8.6}$$

and to approximate the right hand side of (8.6) by $q_2^{(R)}(t-\tau;x,dy)$. We make precise this consideration by firstly letting $R\to 0$ and after that letting $\tau\to 0$: For *all* $\tau\in(0,t)$ we can write (using occasionally the notation $e_2^{(h,R)}(t;x,y)\,dy$ instead of $e_2^{(h,R)}(t;x,dy)$):

$$\begin{aligned}\int e_2^{(R)}(t;x,dy)\,F^{(R)}(y)&=\lim_{h\to 0}\int e_2^{(h,R)}(t;x,dy)\,F^{(R)}(y)\\ &=\lim_{h\to 0}\int du\,q_2^{(R)}(t-\tau;x,u)\int e_2^{(h,R)}(\tau;u,dy)\,F^{(R)}(y)\\ &=\lim_{h\to 0}\int dy\,F^{(R)}(y)\int_{\substack{|u_1-y|\leqq\eta\\ \wedge\,|u_2-y|\leqq\eta}} du\,e_2^{(h,R)}(\tau;u,y)\,q_2^{(R)}(t-\tau;x,u)\\ &\quad+\lim_{h\to 0}\int dy\,F^{(R)}(y)\int_{\substack{|u_1-y|\geqq\eta\\ \vee\,|u_2-y|\geqq\eta}} du\,e_2^{(h,R)}(\tau;u,y)\,q_2^{(R)}(t-\tau;x,u),\end{aligned}\tag{8.7}$$

where η is a positive number.

For the proof of (8.5) it is sufficient to show the following three estimates: For all $\eta>0$

$$\overline{\lim_{\tau\to 0}}\;\overline{\lim_{R\to 0}}\;\overline{\lim_{h\to 0}}\int dy\,F^{(R)}(y)\int_{\substack{|u_1-y|\geqq\eta\\ \vee\,|u_2-y|\geqq\eta}} du\,e_2^{(h,R)}(\tau;u,y)\,q_2^{(R)}(t-\tau;x,u)=0. \tag{8.8}$$

To every $\varepsilon>0$ there exists $\eta>0$ such that

$$\begin{aligned}&\overline{\lim_{\tau\to 0}}\;\overline{\lim_{R\to 0}}\;\overline{\lim_{h\to 0}}\int dy\,F^{(R)}(y)\int_{\substack{|u_1-y|\leqq\eta\\ \wedge\,|u_2-y|\leqq\eta}} du\,e_2^{(h,R)}(\tau;u,y)\,q_2^{(R)}(t-\tau;x,u)\\ &\quad\leqq\int dy\,F(y)\,p_2(t;x,(y,y))+\varepsilon.\end{aligned}\tag{8.9}$$

And finally: to every $\varepsilon>0$ there exists $\eta>0$ such that

$$\begin{aligned}&\underline{\lim}_{\tau\to 0}\;\underline{\lim}_{R\to 0}\;\underline{\lim}_{h\to 0}\int dy\,F^{(R)}(y)\int_{\substack{|u_1-y|\leqq\eta\\ \wedge\,|u_2-y|\leqq\eta}} du\,e_2^{(h,R)}(\tau;u,y)\,q_2^{(R)}(t-\tau;x,u)\\ &\quad\geqq\int dy\,F(y)\,p_2(t;x,(y,y))-\varepsilon.\end{aligned}\tag{8.10}$$

(ii) Proof of (8.8).

Since $q_2^{(R)}(t-\tau;x,u)$ is bounded uniformly in x, u, R and $\tau\in(0,t/2)$ we have to show

$$\overline{\lim_{\tau\to 0}}\;\overline{\lim_{R\to 0}}\;\overline{\lim_{h\to 0}}\int dx\,\bar{F}(y)\int_{\substack{|u_1-y|\geqq\eta\\ \vee\,|u_2-y|\geqq\eta}} du\,e_2^{(h,R)}(\tau;u,y)=0 \tag{8.11}$$

($\bar{F}$ as given by (8.3)).

However by means of spatial homogenity the term

$$(8.12)\qquad \int_{\substack{|u_1-y|\geqq\eta\\ \vee\,|u_2-y|\geqq\eta}} du\, e_2^{(h,R)}(\tau;u,y)$$

$$= \int_{\substack{|u_1-y|\geqq\eta\\ \vee\,|u_2-y|\geqq\eta}} du\, e_2^{(h,R)}(\tau;(u_1-y,u_2-y),0) = \int_{\substack{|v_1|\geqq\eta\\ \vee\,|v_2|\geqq\eta}} dv\, e_2^{(h,R)}(\tau;v,0)$$

is independent of y. Hence (8.11) follows from Lemma 10.8.

(iii) Proof of (8.9).

Let $\varepsilon>0$ be given. The function $(s,v)\to p_2(s;0,v)$ being uniformly continuous, say in $[t/2,\infty)\times\mathbb{R}^6$, there exist $\tau_0>0$ and $\eta>0$ such that

$$(8.13)\qquad \max_{\tau\leqq\tau_0}\max\{p_2(t-\tau;x,u): |u_1-y|\leqq\eta, |u_2-y|\leqq\eta\}$$

$$\leqq p_2(t;x,(y,y))+\varepsilon \quad \text{for all } y.$$

With this η and τ_0 we get for all $\tau\leqq\tau_0$

$$(8.14)\qquad \overline{\lim_{h\to 0}}\int dy\, F^{(R)}(y) \int_{\substack{|u_1-y|\leqq\eta\\ \wedge\,|u_2-y|\leqq\eta}} du\, e_2^{(h,R)}(\tau;u,y)\, q_2^{(R)}(t-\tau;x,u)$$

$$\leqq \overline{\lim_{h\to 0}}\int dy\, F^{(R)}(y)(p_2(t;x,(y,y))+\varepsilon)\cdot\int du\, e_2^{(h,R)}(\tau;u,0)$$

$$\leqq \left\{\int dy\, F^{(R)}(y)\, p_2(t;x,(y,y))+\varepsilon\cdot\int dy\, \bar{F}(y)\right\}\cdot(1+R/\sqrt{2\pi D\tau})$$

by Lemma 10.1.

Letting first $R\to 0$ and then $\tau\to 0$ we get (8.9).

(iv) Proof of (8.10).

Let $\tau\in(0,t)$ be given. By Lemma 10.3

$$\sup_R \sup_{0<h<1} \sup_u \sup_y e_2^{(h,R)}(\tau;u,y)=c<\infty.$$

So we can estimate

$$(8.15)\qquad \overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} \int dy\, F^{(R)}(y)\int du\, e_2^{(h,R)}(\tau;u,y)\{p_2(t-\tau;x,u)-q_2^{(R)}(t-\tau;x,u)\}$$

$$\leqq c\cdot\int dy\, \bar{F}(y)\, \overline{\lim_{R\to 0}}\, P_{x_1,x_2}(T_{1,2}\leqq t-\tau)$$

$$= c\cdot\int dy\, \bar{F}(y)\, \overline{\lim_{R\to 0}}\, \bar{P}_{x_1-x_2}(T_R\leqq t-\tau)=0$$

because, by assumption, $x_1\neq x_2$ and $\lim_{R\to 0}\bar{P}_a(T_R\leqq t-\tau)=0$ for any $a\neq 0$, $a\in\mathbb{R}^3$.

As a result of (8.15) we can derive (8.10) if we can show

$$(8.16)\qquad \underline{\lim_{\tau\to 0}}\ \underline{\lim_{R\to 0}}\ \underline{\lim_{h\to 0}}\int dy\, F^{(R)}(y) \int_{\substack{|u_1-y|\leqq\eta\\ \wedge\,|u_2-y|\leqq\eta}} du\, e_2^{(h,R)}(\tau;u,y)\, p_2(t-\tau;x,u)$$

$$\geqq \int dy\, F(y)\, p_2(t;x,(y,y))-\varepsilon.$$

Analogously to (8.14) one finds $\eta>0$ and $\tau_0>0$ such that for all $\tau\leqq\tau_0$

$$(8.17)\quad \varliminf_{\tau\to 0}\ \varliminf_{R\to 0}\ \varliminf_{h\to 0}\int dy\,F^{(R)}(y)\int_{\substack{|u_1-y|\leqq\eta\\ \wedge|u_2-y|\leqq\eta}} du\,e_2^{(h,R)}(\tau;u,y)\,p_2(t-\tau;x,u)$$

$$\geqq \varliminf_{\tau\to 0}\ \varliminf_{R\to 0}\ \varliminf_{h\to 0}\int dy\,F^{(R)}(y)(p_2(t-\tau;x,(y,y))-\varepsilon)$$

$$\cdot\int_{\substack{|u_1|\leqq\eta\\ \wedge|u_2|\leqq\eta}} du\,e_2^{(h,R)}(\tau;u,0)$$

$$=\int dy\,F(y)(p_2(t;x,(y,y))-\varepsilon)\cdot\varliminf_{\tau\to 0}\ \varliminf_{R\to 0}\ \varliminf_{h\to 0}\int_{\substack{|u_1|\leqq\eta\\ \wedge|u_2|\leqq\eta}} du\,e_2^{(h,R)}(\tau;u,0).$$

Again by Lemma 10.8 the second factor is equal to 1 and hence (8.10) follows, and the proof of Proposition 8.1 is complete.

b) Reduction of Proposition 3.4 to Proposition 8.1

Definition. Let $j\in\mathbb{N}$. Denote by $\mathfrak{F}_j$ the set of sequences $\{F_j^{(R)}\}$ $(R=\lambda/4\pi Dn;\ n=1,2,\ldots)$ of functions $F_j^{(R)}\colon \mathbb{R}^{3j}\to\mathbb{R}_+$ satisfying the following five properties (8.18)–(8.22):

(8.18) every $F_j^{(R)}$ is uniformly continuous

(8.19) $F_j^{(R)}(x)=0$ for $x\in B_j^{(R)}$ (for all R)

(8.20) $\lim_{R\to 0} F_j^{(R)}(x)=F_j(x)$ exists a.e.

(8.21) there exists a function $\bar{F}\in L^1(\mathbb{R}^{3j})$ such that $F_j^{(R)}\leqq\bar{F}$ for all R

(8.22) $\sup_R \|F_j^{(R)}\|<\infty$.

We now reduce Proposition 3.4 to Proposition 8.1 in three steps.

Step 1. Reduction to pairs of particles.

We would like to use Proposition 8.1 in a more general form as follows: if $\{F_j^{(R)}\}\in\mathfrak{F}_j$ $(j\in\mathbb{N})$ then

$$(8.23)\quad \lim_{R\to 0}\int_{\mathbb{R}^{3j}} e_{j+1}^{(R)}(t;x,dy)\,F_j^{(R)}(y)$$

$$=\sum_{i=1}^{j}\int_{\mathbb{R}^{3j}} dy\,p_{j+1}(t;x,(y,y_i))\,F_j(y)\qquad \text{for } x=(x_1,\ldots,x_{j+1})\in\mathbb{R}^{3(j+1)}$$

such that $x_i\neq x_k$ for $i\neq k$.

To prove (8.23) we note that the same arguments as given in (6.24)–(6.28) are applicable to $F_j^{(R)}$ (observe (8.18), (8.19), (8.22)), and we get

(8.24)
$$\int e^{(R)}_{j+1}(t;x,dy)\,F^{(R)}_j(y) = (8\pi RD)^{-1}\lim_{h\to 0} h^{-1}\int dy\,F^{(R)}_j(y)\,P_x(\omega_{1,\ldots,j}(t+h)=y, T_{1,\ldots,j} > T_{1,\ldots,j/j+1}\in(t,t+h]).$$

We would like to replace the right hand side of (8.24) by the following upper bound

(8.25)
$$(8\pi RD)^{-1}\sum_{i=1}^{j}\lim_{h\to 0} h^{-1}\int dy\,F^{(R)}_j(y)\,P_x(\omega_{1,\ldots,j}(t+h)=y; T_{i,j+1}\in(t,t+h])$$
$$=(8\pi RD)^{-1}\sum_{i=1}^{j}\lim_{h\to 0} h^{-1}\int dy\,F^{(R)}_j(y) \cdot\int_{\mathbb{R}^3} dz\,P_{y,z}(\omega(t+h)=x; T_{i,j+1}\leqq h, T_{i,j+1}(\omega_h^+)>t)$$

(use time reversal in the last equality).

The error caused by the replacement of (8.24) by (8.25) actually tends to zero in the limit $R\to 0$ in view of the following

Proposition 8.2. *Let* $x=(x_1,\ldots,x_{j+1})\in\mathbb{R}^{3(j+1)}$ *such that* $x_i\neq x_k$ *for* $i\neq k$. *Then*

(8.26)
$$\overline{\lim_{R\to 0}}(8\pi RD)^{-1}\overline{\lim_{h\to 0}}h^{-1}\int dy\,F^{(R)}_j(y) \cdot\int dz\,P_{yz}(\omega(t+h)=x; T_{i,j+1}\leqq h, T_{i,j+1}(\omega_h^+)>t, T_{k,l}(\omega_h^+)\leqq t)=0$$

for all $i\leqq j$ *and all* k, l *such that* $1\leqq k<l\leqq j+1$.

Furthermore

(8.27)
$$\overline{\lim_{R\to 0}}(8\pi RD)^{-1}\overline{\lim_{h\to 0}}h^{-1}\int dy\,F^{(R)}_j(y) \cdot\int dz\,P_{yz}(\omega(t+h)=x; T_{i,j+1}\leqq h, T_{i,j+1}(\omega_h^+)>t, T_{k,j+1}\leqq h)=0$$

for all $i\leqq j$ *and all* $k\neq i$ *such that* $k\neq j+1$.

Proof. Postponed until Sect. c).

To compute (8.25) in the limit $R\to 0$ we can apply Lebesgue's theorem using (8.21) and Lemma 10.3. By Proposition 8.1 we get just (8.23).

Step 2. Convergence of $A^{(R)}_k(t;t_1,\ldots,t_k)f^{(R)}_{j+k}(0)$.

Let $0<t_1<\ldots<t_k<t$ be fixed and consider $A^{(R)}_k(t;t_1,\ldots,t_k)f^{(R)}_{j+k}(0)$ as defined by (3.18).

Begin with the last integration with respect to $dx^{(0)}$ and apply (8.23) with

$$F^{(R)}_j(y)=q^{(R)}_j(t-t_k;y,x)\qquad (x\in\mathbb{R}^{3j}\text{ fixed}).$$

Proceed inductively by considering in the next step

(8.28)
$$F^{(R)}_{j+1}(x)=\int e^{(R)}_{j+1}(t_k-t_{k-1};x,dy)\,F^{(R)}_j(y).$$

Lemma 10.7. $\{F_{j+1}^{(R)}\}$, *defined by* (8.28), *belongs to the class* $\mathfrak{F}_{j+1}$, *if* $\{F_j^{(R)}\}\in\mathfrak{F}_j$.

Proof. (8.20) is just given by (8.23) in step 1. The remaining properties (8.18), (8.19), (8.21) and (8.22) are verified in Sect. 10.

Back to the proof of the convergence of $A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k}^{(R)}(0)$: Applying Lemma 10.7 step by step, we end with

$$(8.29)\qquad \int dx^{(k)} f_{j+k}^{(R)}(0;x^{(k)})\,F_{j+k}^{(R)}(x^{(k)}),\qquad \{F_{j+k}^{(R)}\}\in\mathfrak{F}_{j+k}.$$

Because of (8.21) and condition (C1) Lebesgue's theorem is applicable, and we obtain by (8.20) and condition (C2)

$$(8.30)\qquad \lim_{R\to 0}\int dx^{(k)} f_{j+k}^{(R)}(0;x^{(k)})\,F_{j+k}^{(R)}(x^{(k)})=\int dx^{(k)} f_{j+k}^{(0)}(0;x^{(k)})\,F_{j+k}(x^{(k)}).$$

Thus we have proved the existence of the limit

$$\lim_{R\to 0}(A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k}^{(R)}(0))(x)\qquad \text{for all } x\in\mathbb{R}^{3j}.$$

Proceeding inductively, this time beginning with (8.29), we can apply the explicit formula for $F_{j+k}=\lim_{R\to 0}F_{j+k}^{(R)}$ as given by (8.23). Using notation (3.20) and (3.22) we identify in this way the limit as

$$(8.31)\qquad (S_j(t-t_k)\,C_{j+1}\,S_{j+1}(t_k-t_{k-1})\,C_{j+2}\dots C_{j+k}\,S_{j+k}(t_1)f_{j+k}^{(0)}(0))(x).$$

Step 3. Application of Lebesgue's theorem.

By (7.14) and assumption (C1) we have

$$(8.32)\qquad \begin{aligned}\|A_k^{(R)}(t;t_1,\dots,t_k)f_{j+k}^{(R)}(0)\| &\leqq (1+h(t_1))\prod_{i=1}^{k-1}(1+h(t_{i+1}-t_i))\,\|f_{j+k}^{(R)}(0)\|\\ &\leqq \text{const}(1+h(t_1))\prod_{i=1}^{k-1}(1+h(t_{i+1}-t_i))\qquad \text{for all } R.\end{aligned}$$

The right hand side of (8.32) is integrable over the set $\{0<t_1<\dots<t_k<t\}$, therefore we can use Lebesgue's theorem to conclude (3.27) in Proposition 3.4. What remains to do is only to give the

c) Proof of Proposition 8.2.

(i) Reformulation of the problem.

Because of (8.21) there exists $\bar F_j\in L^1(\mathbb{R}^{3j})$ such that $F_j^{(R)}\leqq \bar F_j$ for all R. So replacing $F_j^{(R)}$ by $\bar F_j$ in (8.26) and (8.27), and using Lemma 10.3 we are sure that we can apply Fatou's lemma to (8.26) and (8.27). To prove (8.26) and (8.27) we have therefore to show

$$(8.33)\qquad \overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}}\,R^{-1}h^{-1}\int_{\mathbb{R}^3} dz\,P_{y,z}(\omega(t+h)=x;\dots)=0$$

where $y=(y_1,\dots,y_j)\in\mathbb{R}^{3j}$ is a given point such that $y_i\neq y_k$ for $i\neq k$.

Moreover using the technique of decomposition of the time interval $[0, t+h]$ by appropriate intermediate time points (it is demonstrated for example in the proof of Lemma 10.3, so we omit the details here) we can get rid of $\{\omega(t+h)=x\}$ in (8.33). Therefore it is enough to prove that, given a point $y=(y_1, y_2)\in(\mathbb{R}^3)^2$, $y_1 \neq y_2$, (8.34)–(8.36) hold:

$$\overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1}\int dz\, P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>t;\ T_{1,2}(\omega_h^+)\leqq t)=0, \tag{8.34}$$

$$\overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1}\int dz\, P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>t;\ T_{2,3}(\omega_h^+)\leqq t)=0, \tag{8.35}$$

$$\overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1}\int dz\, P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>t;\ T_{2,3}\leqq h)=0. \tag{8.36}$$

(8.36) can be shown quickly: let $\eta=|y_1-y_2|/3$ and consider only R such that $R<\eta$. Then

$$\begin{aligned}
&\overline{\lim_{h\to 0}}\, h^{-1}\int dz\, P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>t;\ T_{2,3}\leqq h) \\
&=\overline{\lim_{h\to 0}}\, h^{-1}\int_{|z-y_1|\leqq\eta} dz\, P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>t;\ T_{2,3}\leqq h) \\
&\leqq\overline{\lim_{h\to 0}}\, h^{-1}\int_{|z-y_1|\leqq\eta} dz\, P_{y_2,z}(T_{2,3}\leqq h) \\
&=\overline{\lim_{h\to 0}}\, h^{-1}\int_{|z-y_1|\leqq\eta} dz\, \bar{P}_{y_2-z}(\min_{0\leqq s\leqq h}|\bar{\omega}(s)|\leqq R)=0
\end{aligned} \tag{8.37}$$

because $|z-y_1|\leqq\eta$ implies $|y_2-z|\geqq R+\eta$ and $\bar{P}_0(\max_{0\leqq s\leqq h}|\bar{\omega}(s)|\geqq\eta)=o(h)$.

Therefore it remains to show (8.34) and (8.35). The proof of (8.35) is quite similar to that of (8.34), so we omit it and give the details only for (8.34).

(ii) Proof of (8.34).

For all $\tau\in(0, t)$ an upper bound of (8.34) is given by $I_1(\tau)+I_2(\tau)$, where

$$I_1(\tau)=\overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1} \cdot\int dz\, P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau;\ T_{1,2}(\omega_h^+)\leqq\tau) \tag{8.38}$$

$$\begin{aligned}
I_2(\tau)=\overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1} &\cdot\int dz\, P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau; \\
&T_{1,2}(\omega_h^+)>\tau,\ T_{1,2}(\omega_{\tau+h}^+)\leqq t-\tau).
\end{aligned} \tag{8.39}$$

We have to show $\overline{\lim_{\tau\to 0}} I_1(\tau)=\overline{\lim_{\tau\to 0}} I_2(\tau)=0$. The proof is based on the following

Lemma 10.8. *For all* $\eta>0$

(a) $$\overline{\lim_{\tau\to 0}}\ \overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1}\int_{\mathbb{R}^3} dz\, P_z(\max_{h\leqq s\leqq\tau+h}|\omega(s)|\geqq\eta;\ T_R\leqq h,\ T_R(\omega_h^+)>\tau)=0.$$

(b) $$\overline{\lim_{\tau\to 0}}\ \overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1}\int_{\mathbb{R}^3} dz\, P_{y_1,z}(\max_{h\leqq s\leqq\tau+h}|\omega_1(s)-y_1|\geqq\eta;\ T_{1,2}\leqq h,\ T_{1,2}(\omega_h^+)>\tau)=0 \quad (y_1\in\mathbb{R}^3).$$

To apply this lemma, choose $\eta=|y_1-y_2|/3$. We begin with $I_1(\tau)$ and decompose

$$\begin{aligned}(8.40)\quad & R^{-1}h^{-1}\int dz\,P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau;\ T_{1,2}(\omega_h^+)\geqq\tau)\\ &=R^{-1}h^{-1}\int dz\,P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau;\\ &\quad T_{1,2}(\omega_h^+)\leqq\tau;\ \max_{h\leqq s\leqq\tau+h}|\omega_1(s)-y_1|>\eta)\\ &\quad +R^{-1}h^{-1}\int dz\,P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau;\\ &\quad T_{1,2}(\omega_h^+)\leqq\tau;\ \operatorname{may}_{h\leqq s\leqq\tau+h}|\omega_1(s)-y_1|\leqq\eta).\end{aligned}$$

The first summand on the right hand side of (8.40) is handled by Lemma 10.8(b). Since

$$\begin{aligned}(8.41)\quad & \{T_{1,2}(\omega_h^+)\leqq\tau,\ \max_{h\leqq s\leqq\tau+h}|\omega_1(s)-y_1|\leqq\eta\}\subset\{\max_{h\leqq s\leqq\tau+h}|\omega_2(s)-y_2|>\eta\}\\ & (R<\eta;\ \text{recall } |y_1-y_2|=3\eta),\end{aligned}$$

the second summand in (8.40) is estimated by

$$\begin{aligned}(8.42)\quad & R^{-1}h^{-1}\int dz\,P_{y_1,y_2,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau;\ \max_{h\leqq s\leqq\tau+h}|\omega_2(s)-y_2|>\eta)\\ &=R^{-1}h^{-1}\int dz\,P_{y_1,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau)\\ &\quad\cdot P_{y_2}(\max_{h\leqq s\leqq\tau+h}|\omega_2(s)-y_2|>\eta)\end{aligned}$$

because ω_2 is independent of ω_1 and ω_3. Letting $h\to 0$, $R\to 0$ the first factor tends to a finite constant by virtue of Lemma 10.1 and becomes in the limit independent of τ. After that, letting $\tau\to 0$ we see that the second factor tends to zero. So $\overline{\lim}_{\tau\to 0} I_1(\tau)=0$ is proved.

In order to handle $I_2(\tau)$ similarly, decompose

$$\begin{aligned}(8.43)\quad I_2(\tau)\leqq & \overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}}\,R^{-1}h^{-1}\int dz\int_{|v_1-y_1|\geqq\eta}dv_1\,dv_2\,P_{y_1,y_2,z}(\omega_{1,2}(\tau+h)=(v_1,v_2);\\ & T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau)\\ & +\overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}}\,R^{-1}h^{-1}\int dz\int_{|v_2-y_2|\geqq\eta}dv_1\,dv_2\,P_{y_1,y_2,z}(\omega_{1,2}(\tau+h)=(v_1,v_2);\\ & T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau)\\ & +\overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}}\,R^{-1}h^{-1}\int dz\int_{\{|v_1-y_1|\leqq\eta,\ |v_2-y_2|\leqq\eta\}}dv_1\,dv_2\,P_{y_1,y_2,z}(\omega_{1,2}(\tau+h)\\ & =(v_1,v_2);\quad T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau)\cdot P_{v_1,v_2}(T_{1,2}\leqq t-\tau).\end{aligned}$$

The first term on the right side of (8.43) can again be treated by Lemma 10.8(b), the second is already handled by (8.42). To estimate the third term observe that $|y_1-y_2|=3\eta$, $|v_1-y_1|\leqq\eta$, $|v_2-y_2|\leqq\eta$ imply $|v_1-v_2|\geqq\eta$, and hence, if we choose a fixed point $a\in\mathbb{R}^3$ such that $|a|=\eta$, we have

(8.44) $P_{v_1,v_2}(T_{1,2}\leqq t-\tau)\leqq \bar{P}_a(T_R\leqq t-\tau)$ for all (v_1, v_2) such that $|v_i-y_i|\leqq\eta$ for $i=1,2$.

Therefore the last term on the right hand side of (8.43) is bounded by

$$(8.45)\quad \overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}} R^{-1}h^{-1}\int dz\, P_{y_1,z}(T_{1,3}\leqq h,\ T_{1,3}(\omega_h^+)>\tau)\cdot \bar{P}_a(T_R\leqq t-\tau)$$
$$=\overline{\lim_{R\to 0}} R^{-1}\,8\pi RD(1+R/\sqrt{2\pi D\tau})\cdot \bar{P}_a(T_R\leqq t-\tau)\quad \text{(by Lemma 10.1)}$$
$$\leqq 8\pi D(\overline{\lim_{R\to 0}}\ \bar{P}_a(T_R\leqq t-\tau)+\overline{\lim_{R\to 0}}\ R/\sqrt{2\pi D\tau})=0\quad \text{for all } \tau>0,$$

whence $\overline{\lim_{\tau\to 0}} I_2(\tau)=0$.

This ends the proof of (8.34).

9. Additional Ingredients for the Proof of Theorem 2.2

a) Heuristics and Statement of Proposition 9.1

In order to give a heuristic derivation of the BBGKY-hierarchy in the case of labeled particles, we proceed much as in Sect. 6a. Now the change $\delta_{\text{coll}}\rho_j^{(R)}(t;x,l)$ consists of two terms: a positive one representing the increase of $\rho_j^{(R)}(t;x,l)$ due to formation of l-fold particles, and a negative one representing the decrease of $\rho_j^{(R)}(t;x,l)$ due to destroying l-fold particles. Hence the following hierarchy seems to be plausible:

$$(9.1)\quad \frac{\partial}{\partial t}\rho_j^{(R)}(t;x,l)$$
$$=D\cdot\Delta\rho_j^{(R)}(t;x,l)+2D\sum_{i=1}^{j}\Big\{\tfrac{1}{2}\sum_{p+q=l_i}\int_{S_i(x_1,\ldots,x_j)}\sigma(dx_{j+1})$$
$$\vec{n}\cdot\operatorname{grad}_{j+1}\rho_{j+1}^{(R)}(t;x,x_{j+1};l_1,\ldots,l_{i-1},p,l_{i+1},\ldots,l_j,q)$$
$$-\sum_{p\geqq 1}\int_{S_i(x_1,\ldots,x_j)}\sigma(dx_{j+1})\,\vec{n}\cdot\operatorname{grad}_{j+1}\rho_{j+1}^{(R)}(t;x,x_{j+1};l,p)\Big\}$$

where

$$S_i(x_1,\ldots,x_j)=\{x_{j+1}\in\mathbb{R}^3\colon |x_{j+1}-x_k|\geqq R \text{ for } 1\leqq k\leqq j,\ |x_{j+1}-x_i|=R\}$$
$\sigma=$ surface measure of $S_i(x_1,\ldots,x_j)$
$\vec{n}=$ normal vector at $S_i(x_1,\ldots,x_j)$.

In view of the result for unlabeled particles we expect that the rescaled correlation functions $f_j^{(R)}$ tend in the limit $R\to 0$ to functions $f_j^{(0)}$ satisfying the following hierarchy (recall that $f_j^{(R)}$ is defined by (2.8) and λ by (2.1))

$$(9.2)\quad \frac{\partial}{\partial t}f_j^{(0)}(t)=D\cdot\Delta f_j^{(0)}(t)$$
$$+2\lambda\sum_{i=1}^{j}\{\tfrac{1}{2}C_{j+1,i}^{+}f_{j+1}^{(0)}(t)-C_{j+1,i}^{-}f_{j+1}^{(0)}(t)\}\quad (j\in\mathbb{N}),$$

where the operators $C^{\pm}_{j+1,i}$ act on functions $g: \mathbb{R}^{3(j+1)} \times \mathbb{N}^{j+1} \to \mathbb{R}$ in the following way:

$$(9.3)\quad (C^{+}_{j+1,i}g)(x_1, \ldots, x_j; l_1, \ldots, l_j) = \sum_{p+q=l_i} g(x_1, \ldots, x_j, x_i; l_1, \ldots, l_{i-1}, p, l_{i+1}, \ldots, l_j, q),$$

$$(9.4)\quad (C^{-}_{j+1,i}g)(x_1, \ldots, x_j; l_1, \ldots, l_j) = \sum_{p \geqq 1} g(x_1, \ldots, x_j, x_i; l_1, \ldots, l_j, p) \quad (l_i \in \mathbb{N},\ 1 \leqq i \leqq j).$$

Furthermore, the form of the limit hierarchy (9.2) suggests that propagation of chaos holds since one can check that product functions

$$g_j(t; x, l) = \prod_{i=1}^{j} p(t; x_i, l_i) \quad (j \in \mathbb{N})$$

satisfy the hierarchy (9.2) if and only if $p(t)$ satisfies Eq. (9.2) with $j=1$.

However, for the same reasons as in the case of unlabeled particles we do not derive (9.2) via the hierarchy (9.1). Instead of that we work again with an appropriate series for $f_j^{(R)}(t; x, l)$. Our aim is to prove

Proposition 9.1. *Let t_0 be as in Proposition 3.3. Then the limit*

$$(9.5)\quad \lim_{R \to 0} f_j^{(R)}(t; x, l) = f_j^{(0)}(t; x, l)$$

exists for almost all $x \in \mathbb{R}^{3j}$, all $l \in \mathbb{N}^j$, all $t \in [0, t_0/16)$ and satisfies

$$(9.6)\quad f_j^{(0)}(t) = S_j f_j^{(0)}(0) - 2\lambda \int_0^t dt_1\, S_j(t-t_1) \cdot \sum_{i=1}^{j} \{\tfrac{1}{2} C^{+}_{j+1,i} f^{(0)}_{j+1}(t_1) - C^{-}_{j+1,i} f^{(0)}_{j+1}(t_1)\} \quad (j \in \mathbb{N}).$$

The proof is sketched in the next section b.

As before in the case of Theorem 2.1, Proposition 9.1 implies Theorem 2.2 first for $t \in [0, t_0/16)$: Iterating (9.6) we can express $f_j^{(0)}(t)$ in the form of a series; multiplying out two such series we can derive propagation of chaos similarly as before. Furthermore, observing that

$$(9.7)\quad \sum_{l \geqq 1} f_1^{(0)}(t; x, l) = p(t; x),$$

where $p(t)$ is the unique L^∞-solution of the kinetic Eq. (2.12), one can show that the solution of the Eq. (9.6) for $j=1$ is the unique L^∞-solution of the system of Eqs. (2.13). Finally, one can proceed quite analogously as before to extend these results to all times $t \geqq 0$.

b) Sketch of the Proof of Proposition 9.1

The basic ideas of the proof of Proposition 9.1 are the same as were used for the proof of Proposition 3.1–3.4. We only need some additional ingredients in order to handle the complications caused by the correct counting of the multiplicities. Therefore, we only sketch the proofs and omit the details.

First we have to develop $\rho_j^{(R)}(t; x, l)$ in a perturbation series similarly to the way it was done in Proposition 3.2 for $\rho_j^{(R)}(t; x)$. For this purpose we again work with the following definition which can be seen to be equivalent to that given in (2.4)

(9.8) $\rho_j^{(R)}(t; x_1, \ldots, x_j; l)\, dx_1 \ldots dx_j = n(n-1)\ldots(n-j+1) \int dy\, \pi_n(y)$
$\hat{P}_y$ (the particles $1, \ldots, j$ survive in the time interval $[0, t]$;
$\omega_{1,\ldots,j}(t) \in dx_1 \ldots dx_j$; $\omega_i(t)$ has multiplicity l_i for $1 \leqq i \leqq j$),

where, as previously defined, $\hat{P}_y$ denotes the probability measure induced by the coalescing Brownian motions starting in $y \in \mathbb{R}^{3n}$ (in particular, $\hat{P}_y$ respects the collision rule that after a collision between two particles p, q, the particle p resp. q disappears with probability 1/2).

Let $j \in \mathbb{N}$, $l \in \mathbb{N}^j$ and $x \in \mathbb{R}^{3j}$ be given. We use the notation

(9.9) $$|l| = \sum^{j} l_i,$$

(9.10) $$m = |l| - j,$$

(9.11) $$J = \{1, \ldots, j\}, \qquad S = \{j+1, \ldots, j+m\},$$
$$S_k = \{j+m+1, \ldots, j+m+k\} \qquad (k \geqq 0).$$

In order to compute $\rho_j^{(R)}(t; x, l)$, we first define all the streams leading to the right multiplicity l at time t, then we compute the probability of their surviving in the time interval $[0, t]$. We notice that such a stream survives as long as it does not meet any other particle. A special case of this is the following stream, which is defined by the particles $1, \ldots, j+m$ as follows:

(9.12) $F = \{$First particle $j+m$ collides with another particle $i < j+m$ and disappears, then $j+m-1$ collides with another one and disappears, ..., and lastly $j+1$ collides with a particle $i \in J$ and disappears; $T_{1,\ldots,j} > t$, $\omega_{1,\ldots,j}(t) = x$ and $\omega_i(t)$ has multiplicity l_i for $1 \leqq i \leqq j\}$.

Because of the symmetry of π_n we obtain

(9.13) $$\rho_j^{(R)}(t; x, l) = (n)_j (n-j) \ldots (n-j-m+1) \int \pi_n^{(R)}(dy)\, \hat{P}_y(F; S_F > t),$$

where

$$S_F = \sup\{s \geqq 0\text{: the stream } F \text{ survives in the time interval } [0, s]\}.$$

We need some further definitions for the formulation of Proposition 9.2 as well as for the norm estimate.

(9.14) Γ_k = set of those permutations γ of $S \cup S_k$ which satisfy:

if $\gamma_p, \gamma_q \in S$ resp. $\gamma_p, \gamma_q \in S_k$ then $p<q$ imply $\gamma_p < \gamma_q$.

(9.15) B_γ^i ($\gamma \in \Gamma_k$, $0 \leqq i \leqq k$) are the events described by the following four conditions (during the time interval $[0, t]$):

(i) The particles collide in the order $\gamma_{j+m+k}, \ldots, \gamma_{j+1}$ (i.e. first particle γ_{j+m+k} collides with one of the other particles $i \leqq j+m+k$ and disappears, then $\gamma_{j+m+k-1}$, and so on).

(ii) Any particle $\gamma_p \in S$ is restricted to collide only with particles of the set $J \cup S$.

(iii) The particles inside the system $J \cup S$ form the right stream F (see (9.12)).

(iv) There are exactly i collisions between particles p and q with $p \in J \cup S$ and $q \in S_k$, with other words there are exactly i collisions with the stream F.

After these preparations we can formulate the analogue to Proposition 6.1.

Proposition 9.2. *Let* $j \in \mathbb{N}$, $l \in \mathbb{N}^j$, $m = |l| - j$. *Then*

$$\rho_j^{(R)}(t; x, l) = \sum_{k \geqq 0} (-1)^k \int dy \, \rho_{j+m+k}^{(R)}(0; y) \sum_{\gamma \in \Gamma_k} \sum_{i=0}^{k} 2^i \hat{P}_y(B_\gamma^i). \tag{9.16}$$

Proof. We proceed as in the proof of Proposition 6.1 by using the same argument given there. In the first step the stream F is destroyed by the particle $j+m+1$. This happens at the collision time $T_{F/j+m+1}$ of the particle $j+m+1$ with one of the yet living particles in the stream F. In the second step the system $(F, j+m+1)$ is destroyed by the particle $j+m+2$, at the collision time $T_{F, j+m+1/j+m+2}$, with

$$T_{F, j+m+1/j+m+2} \leqq T_{F/j+m+1},$$

and so on.

In this procedure two points are to be observed. On the one hand, the times of collisions caused by the particles $j+m+1, \ldots, j+m+k$ are well ordered as just described. However, they are not ordered with respect to the times of collisions between the particles inside the stream F. All arrangements are possible. These possibilities are classified by the elements $\gamma \in \Gamma_k$. On the other hand the system $(F, j+m+1, \ldots, j+m+k-1)$ is destroyed with probability 1 if a collision happens between $j+m+k$ and F, and it is destroyed with probability 1/2 as in the case of coalescence if the collision happens between $j+m+k$ with one of the particles $j+m+1, \ldots, j+m+k-1$. Therefore, in view of the definition of $\hat{P}$, we have to multiply by a factor 2 at every collision with F. If $k \geqq 1$, there exists at least one collision with F, i.e. $i \geqq 1$. This explains the weight 2^i of $\hat{P}_y(B_\gamma^i)$ in the Proposition. Hence Proposition 9.2 is proved.

We formulate the perturbation series (9.16) in terms of the rescaled correlation functions $f_j^{(R)}$.

Corollary 9.3. *We denote by*

$$I_{j,k}^{(R)}(t; x, l) = (1/4\pi R D)^{m+k} \int dy \, f_{j+m+k}^{(R)}(0; y) \sum_{\gamma \in \Gamma_k} \sum_{i=0}^{k} 2^i \hat{P}_y(B_\gamma^i) \tag{9.17}$$

$$(j \in \mathbb{N}, \ k \geqq 0).$$

Then $f_j^{(R)}(t)$ *can be developed into the following series*

(9.18) $$f_j^{(R)}(t;x,l)=\sum_{k\geqq 0}(-1)^k\lambda^{|l|-j+k}I_{j,k}^{(R)}(t;x,l).$$

Proposition 9.4. *Let* t_0 *be as in Proposition* 3.3. *There exists a universal constant* $\bar{c}<\infty$ *such that the following estimate holds uniformly in* R:

(9.19) $$\sum_{l\in\mathbb{N}^j}\lambda^{|l|-j+k}\|I_{j,k}^{(R)}(t;\cdot,l)\|\leqq\bar{c}\cdot\left(\frac{t}{t_0/16}\right)^{k/2}\|f^{(R)}(0)\|_z\cdot(2z)^j$$

for all $j\in\mathbb{N}$, $k\in\mathbb{N}$ *and* $t\leqq t_0/16$.

Proof. Denote by

(9.20) $B=\{$the particle $j+m+k$ is the first to collide in the time interval $[0,t]$ with another particle in $J\cup S\cup S_k$ and disappears; then the particle $j+m+k-1,\ldots$; lastly the particle $j+1$; the particles $1,\ldots,j$ are still alive at time t and $\omega_{1,\ldots,j}(t)=x\}$.

From the symmetry of the initial distribution π_n we get for all $\gamma\in\Gamma_k$ $(k\geqq 0)$

(9.21) $$\int dy\, f_{j+m+k}^{(R)}(0;y)\sum_{i=0}^{k}2^i\hat{P}_y(B_\gamma^i)\leqq 2^k\int dy\, f_{j+m+k}^{(R)}(0;y)\hat{P}_y(B),$$

and by using the notation (3.18), we can write the right hand side of (9.21) in the form

(9.22) $$2^k\cdot(4\pi RD)^{m+k}\int_0^t dt_1\int_{t_1}^t dt_2\ldots\int_{t_{m+k-1}}^t dt_{m+k}$$
$$(A_{m+k}^{(R)}(t;t_1,\ldots,t_{m+k})f_{j+m+k}^{(R)}(0))(x).$$

(9.21), (9.22) and (3.26) imply

(9.23) $$\lambda^{|l|-j+k}\|I_{j,k}^{(R)}(t;\cdot,l)\|\leqq 2^k\cdot|\Gamma_k|\cdot c\cdot\left(\frac{t}{t_0}\right)^{(|l|-j+k)/2}\|f^{(R)}(0)\|_z(2z)^j$$

for all R,j,k,l and $t\leqq 1$.

The cardinality $|\Gamma_k|$ of Γ_k is just the number of the different possibilities for placing k indistinguishable balls in $m+1$ cells (namely the $m+1$ "cells" which are built from the time intervals between successive collisions in the system $J\cup S$: $(0,t_1)$, (t_1,t_2), $\ldots$, (t_m,t)) so that

(9.24) $$|\Gamma_k|=\binom{m+1+k-1}{k}=\frac{(m+k)!}{m!\,k!},$$

which can be estimated as in (7.23) by

(9.25) $$|\Gamma_k|\leqq\text{const}\,2^{m+k}\qquad(m\in\mathbb{N},\ k\in\mathbb{N}).$$

By inserting (9.25) into (9.23) we get with a universal constant $\bar{c}<\infty$:

$$(9.26)\quad \lambda^{|l|-j+k}\,\|I^{(R)}_{j,k}(t;\cdot,l)\| \leqq \bar{c}\cdot 4^k\cdot 2^m \left(\frac{t}{t_0}\right)^{(m+k)/2} \|f^{(R)}(0)\|_z\cdot(2z)^j$$

$$=\bar{c}\left(\frac{t}{t_0/4}\right)^{m/2}\cdot\left(\frac{t}{t_0/16}\right)^{k/2}\|f^{(R)}(0)\|_z\cdot(2z)^j.$$

From the summation of (9.26) over $m\geqq 0$ together with the fact that $\left(\frac{t}{t_0/4}\right)^{1/2}\leqq\frac{1}{2}$ for $t\leqq t_0/16$ we get (9.19).

Now we are ready to give the

Proof of Proposition 9.1.

Step 1. $\lim\limits_{R\to 0} f^{(R)}_j(t)=f^{(0)}_j(t)$ exists a.e. $(t<t_0/16)$.

By virtue of (9.18) and Proposition 9.4 it is enough to show that every term of (9.18) converges a.e. in the limit $R\to 0$: When we have $j,k\in\mathbb{N}$, $l\in\mathbb{N}^j$, $t<t_0/16$, the limit

$$(9.27)\qquad \lim_{R\to 0} I^{(R)}_{j,k}(t;x,l)=I_{j,k}(t;x,l)$$

exists for all $x\in\mathbb{R}^{3j}$ with $x_p\neq x_q$ for $p\neq q$.

The proof of (9.27) is the same as already given for Proposition 3.4. We only have to notice the following point: Because of (ii) in the definition (9.15) of B^i_γ – for the sake of simplicity, we only consider the special case that γ is the identical permutation – the particles $j+1,\dots,j+m$ cannot be killed by one of the particles $j+m+1,\dots,j+m+k$. Therefore in the case that a particle $j+1,\dots,j+m$ collides with another we have slightly to change the definition of the measures $e^{(R)}_{j+1}(t;x,dy)$. The existence of these new measures is proved in a way similar to the proof of Proposition 3.1. We only have to work with the Greenian domain

$$\{x\in\mathbb{R}^{3(j+m+k)}: |x_p-x_q|>R \text{ for } p=j+1,\dots,j+m;\; q=j+m+1,\dots,j+m+k\}$$

instead of the whole space $\mathbb{R}^{3(j+m+k)}$ as used before.

Step 2. Transformation of the series $\sum\limits_{k\geqq 0}(-1)^k\lambda^{m+k}I_{j,k}(t;x,l)$ into the desired form (9.6).

The idea is to split up every $I_{j,k}$ $(k\geqq 0)$ into two parts according to what happens at the last collision time.

$$(9.28)\qquad I_{j,k}(t;x,l)=I^+_{j,k}(t;x,l)+I^-_{j,k}(t;x,l),$$

where $I^+_{j,k}$ indicates that the multiplicity of the particles $1,\dots,j$ inside the stream

F becomes the right multiplicity l not before the last collision of F with other particles, whereas $I_{j,k}^-$ indicates that the multiplicity of the particles $1, \ldots, j$ inside F becomes the right multiplicity l already before the last collision of F with other particles.

Then one can check the following identities

$$\sum_{k \geq 0} (-1)^k \lambda^{m+k} I_{j,k}^+(t; x, l) = (S_j(t) f_j^{(0)}(0))(x, l) + \lambda \int_0^t dt_1 \left(S_j(t-t_1) \sum_{i=1}^{j} C_{j+1,i}^+ f_{j+1}^{(0)}(t_1) \right)(x, l), \tag{9.29}$$

$$\sum_{k \geq 1} (-1)^k \lambda^{m+k} I_{j,k}^-(t; x, l) = -2\lambda \int_0^t dt_1 \left(S_j(t-t_1) \sum_{i=1}^{j} C_{j+1,i}^- f_{j+1}^{(0)}(t_1) \right)(x, l). \tag{9.30}$$

For the proof we have to write down the explicit series for $f_{j+1}^{(0)}(t_1)$ and to compare the series of the left hand side with that of the right hand side. The case (9.29) is straightforward, whereas in the case (9.30) we have carefully to compare the terms with the same number of collisions during $[0, t]$. The details are somewhat involved and are omitted. Thus Proposition 9.1 is shown and hence the proof of Theorem 2.2 finished.

10. Technical Lemmas

In Sect. a) we collect three computational lemmas. Lemma 10.1 gives explicit formulas e.g. for $P_x(T_R \leq t)$ with $x \in \mathbb{R}^3$, in particular it shows, how such probabilities depend on R. Furthermore we give detailed proofs for Lemma 10.2 and 10.3 in order to demonstrate two techniques which sometimes are used at other opportunities without any more comment: The first is the use of the fact that the processes $\omega_1 - \omega_2$, $\omega_1 + \omega_2$ are independent (this is a characterizing property of normal distribution) and the second is a decomposition of the time interval $[0, t]$ by appropriate intermediate times.

Sections b), c), d) contain the remaining technical lemmas, in the order in which they are needed in Sects. 5, 7 and 8.

Survey of Sect. 10:

a) Basic Computational Lemmas

Lemma 10.1. *Let $P_x(T_R \leqq t)$ be the probability, that a Brownian motion with diffusion constant D, starting from $x \in \mathbb{R}^3$, hits the ball $\{y \in \mathbb{R}^3 : |y| \leqq R\}$ during the time interval $[0, t]$. The following formulas hold:*

$$(10.1) \qquad P_x(T_R \leqq t) = \begin{cases} \dfrac{R}{|x|} \cdot \dfrac{2}{\sqrt{\pi}} \displaystyle\int_{(|x|-R)/2\sqrt{Dt}}^{\infty} du\, e^{-u^2}, & |x| \geqq R \\ 1, & |x| \leqq R \end{cases} \qquad (t>0)$$

$$(10.2) \qquad \frac{\partial}{\partial t} P_x(T_R \leqq t) = 4\pi R D \cdot \frac{(|x|-R)^+}{|x|} \cdot (4\pi D t)^{-3/2} \exp(-(|x|-R)^2/4Dt)$$

$$(10.3) \qquad \int_{|x|>R} dx\, P_x(T_R \leqq t) = 2Dt \cdot C(R) + 4 \cdot (2\pi)^{-3/2} (C(R))^2 (2Dt)^{1/2}$$
$$= 4\pi R D \cdot t + 8(\pi D)^{1/2} R^2 \cdot t^{1/2}$$

(*where* $C(R) = 2\pi R$ *is the capacity of a ball with radius* R),

$$(10.4) \qquad \lim_{h \to 0} h^{-1} \int dx\, P_x(T_R \in (t, t+h]) = \lim_{h \to 0} h^{-1} \int dx\, P_x(T_R \leqq h, T_R(\omega_h^+) > t)$$
$$= \frac{d}{dt} \int_{|x|>R} dx\, P_x(T_R \leqq t) = 4\pi R D (1 + R \cdot (\pi D t)^{-1/2}).$$

Proof. $P_x(T_R \leqq t)$ ($|x| > R$, $t > 0$) is the solution of the heat conduction problem

$$(10.5) \qquad \frac{\partial}{\partial t} f(t, x) = D \cdot \Delta f(t, x) \qquad (t>0, |x|>R)$$
$$f(t, x) \to 1 \quad \text{if } |x| \downarrow R \quad (t>0)$$
$$f(0, x) = 0 \quad (|x|>R).$$

Therefore, [4], Chap. 9.10, p. 247, formula (2) gives us (10.1), which in turn implies (10.2)–(10.4). To prove the first equality in (10.4), use time reversal (cf. (3.8)) and the convention on notation as used before (see for example the beginning of the proof of Lemma 5.1):

$$(10.6) \qquad \int dx\, P_x(T_R \leqq h, T_R(\omega_h^+) > t)$$
$$= \int\int\int dx\, du\, dv\, P_x(\omega(h) = u, T_R \leqq h) \cdot P_u(\omega(t) = v, T_R > t)$$
$$= \int\int\int dx\, du\, dv\, P_u(\omega(h) = x, T_R \leqq h) \cdot P_v(\omega(t) = u, T_R > t)$$
$$= \int dv\, P_v(T_R \in (t, t+h]) = \int dx (P_x(T_R \leqq t+h) - P_x(T_R \leqq t))$$
$$= \int_{|x|>R} dx\, P_x(T_R \leqq t+h) - \int_{|x|>R} dx\, P_x(T_R \leqq t).$$

Lemma 10.2. *Let $j \geqq 2$, $R > 0$, $t > 0$, $x \in \mathbb{R}^{3j}$ be fixed. The measures*

$$(10.7) \qquad \mu^{(h,R)}(t; x, dy), \qquad 0 < h \leqq 1$$

are tight and have a uniformly bounded total mass.

Proof. We first show boundedness. Since

$$(10.8)\quad 8\pi RD\mu^{(h,R)}(t;x,dy) \leqq h^{-1}\sum_{1\leqq i<k\leqq j} P_{x_i,x_k}(\omega_{i,k}(t+h)\in dy_i\,dy_k;\ T_{i,k}\in(t,t+h]) \cdot P_{x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_k,\ldots,x_j}(\omega(t+h)\in dy_1\ldots d\hat{y}_i\ldots d\hat{y}_k\ldots dy_j)$$

it is enough to consider the measures

$$(10.9)\quad \mu^{(h)}(dy)=h^{-1}P_x(\omega(t+h)\in dy;\ T\in(t,t+h]) \quad \text{where}\quad x=(x_1,x_2)\in\mathbb{R}^3\times\mathbb{R}^3,\quad T=T_{1,2}.$$

By (10.2) the total mass

$$(10.10)\quad \int_{\mathbb{R}^6}\mu^{(h)}(dy)=h^{-1}P_x(T\in(t,t+h])=h^{-1}\bar{P}_{x_1-x_2}(T_R\in(t,t+h])$$

is bounded uniformly in h.

To prove the tightness of the $\mu^{(h)}$, $0<h\leqq 1$, we pass to new coordinates in $\mathbb{R}^3\times\mathbb{R}^3$ defined by the following bijective map σ: $\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3\times\mathbb{R}^3$

$$(10.11)\quad \sigma(x_1,x_2)=(x_1-x_2,x_1+x_2).$$

Using a basic property of the normal distribution we get that the transformed process

$$(10.12)\quad (\sigma(\omega_1(t),\omega_2(t)))_{0\leqq t}=(\omega_1(t)-\omega_2(t),\omega_1(t)+\omega_2(t))_{0\leqq t}$$

is again a pair of *independent* Brownian motions, this time with diffusion constant $2D$.

Noticing that the Jacobian of σ is 8 we therefore have the following decomposition

$$(10.13)\quad h^{-1}P_x(\omega(t+h)=y;\ T\leqq h,\ T(\omega_h^+)>t) =8\cdot h^{-1}\cdot\bar{P}_{x_1-x_2}(\bar{\omega}(t+h)=y_1-y_2;\ T_R\leqq h,\ T_R(\omega_h^+)>t) \cdot\bar{P}_{x_1+x_2}(\bar{\omega}(t+h)=y_1+y_2).$$

The second factor on the right hand side of (10.13) makes no problem, so it is enough to show the following: given $x\in\mathbb{R}^3$, the measures

$$(10.14)\quad h^{-1}P_x(\omega(t+h)\in dy;\ T_R\in(t,t+h]),\qquad 0<h\leqq 1$$

are tight.

Given $\varepsilon>0$ we can find a $\delta>0$ such that

$$(10.15)\quad h^{-1}\int_{|y|\geqq R+1} dy\,P_x(\omega(t+h)=y;\ T_R\in(t,t+h])<\varepsilon \text{ for all } h\in(0,\delta].$$

This can be done because

$$(10.16)\qquad \begin{aligned} &h^{-1}\int_{|y|\geqq R+1} dy\, P_x(\omega(t+h)=y;\ T_R\in(t,t+h])\\ &=h^{-1}\int_{|y|\geqq R+1} dy \int P_y(\omega(h)\in du;\ T_R\leqq h)\cdot q_1^{(R)}(t;u,x)\\ &\leqq \text{const}\, h^{-1}\int_{|y|\geqq R+1} dy\, P_y(T_R\leqq h)\end{aligned}$$

(use time reversal and $\sup_u q_1^{(R)}(t;u,x)=\text{const}<\infty$).

Using the explicit formula given by (10.1) one can check that the last expression tends to zero in the limit $h\to 0$. Hence (10.15) is proved.

Furthermore there exists a compact set $K\supset\{y\colon |y|\leqq R+1\}$ such that

$$(10.17)\qquad \sup_{\delta\leqq h\leqq 1}\delta^{-1}\int_{K^c} dy\, P_x(\omega(t+h)=y)\leqq\varepsilon.$$

Together with (10.15) we get

$$(10.18)\qquad h^{-1}\int_{K^c} dy\, P_x(\omega(t+h)=y;\ T_R\in(t,t+h])\leqq\varepsilon \text{ for all } h\in(0,1].$$

Hence (10.14) follows, and therefore Lemma 10.2 is proved.

Lemma 10.3. *Let $t>0$ be fixed. Then*

$$(10.19)\qquad \begin{aligned}&\sup_R\ \sup_{h\in(0,1]}\ \sup_{x\in\mathbb{R}^6}\ \sup_{y_1\in\mathbb{R}^3} R^{-1}h^{-1}\\ &\cdot\int_{\mathbb{R}^3} dy_2\, P_{y_1,y_2}(\omega_{1,2}(t+h)=x;\ T_{1,2}^{(R)}\leqq h,\ T_{1,2}^{(R)}(\omega_h^+)>t)<\infty.\end{aligned}$$

Proof.

$$(10.20)\qquad \begin{aligned} &R^{-1}h^{-1}\int_{\mathbb{R}^3} dy_2\, P_{y_1,y_2}(\omega(t+h)=x;\ T_{1,2}^{(R)}\leqq h,\ T_{1,2}^{(R)}(\omega_h^+)>t)\\ &=R^{-1}h^{-1}\int dy_2\int P_{y_1,y_2}\left(\omega\left(\frac{t}{2}+h\right)\in du;\ T_{1,2}^{(R)}\leqq h,\ T_{1,2}^{(R)}(\omega_h^+)>t/2\right)\\ &\quad\cdot q_2^{(R)}(t/2;u,x)\qquad \left(\text{use }\sup_{u\in\mathbb{R}^6}\sup_{x\in\mathbb{R}^6} q_2^{(R)}(t/2;u,x)=c<\infty\right)\\ &\leqq c\cdot R^{-1}h^{-1}\int dy_2\, P_{y_1,y_2}(T_{1,2}^{(R)}\leqq h,\ T_{1,2}^{(R)}(\omega_h^+)>t/2)\\ &=c\cdot R^{-1}h^{-1}\int dy_2\, \bar P_{y_1-y_2}(T_R\leqq h,\ T_R(\bar\omega_h^+)>t/2)\\ &=c\cdot R^{-1}h^{-1}\int_{\mathbb{R}^3} dz\, \bar P_z(T_R\leqq h,\ T_R(\bar\omega_h^+)>t/2)\quad \text{independent of } x \text{ and } y_1.\end{aligned}$$

By (10.4) this last expression is bounded in $h\in(0,1]$ and R, hence Lemma 10.3 is proved.

b) Lemmas Needed for Section 5

Lemma 10.4. *Let $j\geqq 2$, $B=B_j^{(R)}$, $a\in\mathbb{R}^{3j}$, $x\in\mathbb{R}^{3j}$. Then*

(10.21)
$$\int_0^\infty ds\, P_a(B \text{ is hit during } [s, s+h],$$
$$\text{but not during } (s+h, s+h+t];\ \omega(s+t+h)=x)$$
$$=\int_0^\infty ds \sum_{n\geq 1} \int P_a(T^{(n)}\in[s, s+h];$$
$$\omega(T^{(n)}+t/2)\in du)\, q_j^{(R)}(t/2; u, x)+o(h).$$

Proof. In the case $x\in B$, both sides of (10.21) are equal to zero. Hence we assume $x\notin B$. The left side of (10.21) can be defined precisely in the following way (assume $h<t/2$; notice that $T^{(n)}+t/2$ is a stopping time; write q instead of $q_j^{(R)}$):

(10.22)
$$\int_0^\infty ds \sum_{n\geq 1} \int_{[s,s+h]} \int P_a(T^{(n)}\in d\tau;\ \omega(T^{(n)}+t/2)\in du)$$
$$\int dv\, q(t/2; u, v)\, q(s+h-\tau; v, x).$$

Using time reversal we get

(10.23)
$$|\int dv\, q(t/2; u, v)\, p(s+h-\tau; v, x)-\int dv\, q(t/2; u, v)\, q(s+h-\tau; v, x)|$$
$$=P_u(T\in(t/2, t/2+(s+h-\tau)];\ \omega(t/2+(s+h-\tau))=x)$$
$$=\int P_x(T\leqq s+h-\tau;\ \omega(s+h-\tau)\in dv)\, q(t/2; v, u)$$
$$\leqq \text{const}\, P_x(T\leqq s+h-\tau)\leqq \text{const}\, P_x(T\leqq h)$$

uniformly in $u\in\mathbb{R}^{3j}$, $\tau\in[s, s+h]$. But the assumption $x\notin B$ implies $P_x(T\leqq h)=o(h)$. Hence we can replace $q(s+h-\tau; v, x)$ in (10.22) by $p(s+h-\tau; v, x)$ and it is enough for the proof of (10.21) to show that the following difference tends to zero in the limit $h\to 0$:

(10.24)
$$|h^{-1}\int_0^\infty ds \sum_{n\geq 1} \int_{[s,s+h]} \int P_a(T^{(n)}\in d\tau;\ \omega(T^{(n)}+t/2)\in du)\, q(t/2; u, x)$$
$$-h^{-1}\int_0^\infty ds \sum_{n\geq 1} \int_{[s,s+h]} \int P_a(T^{(n)}\in d\tau; \omega(T^{(n)}+t/2)\in du)$$
$$\int dv\, q(t/2; u, v)\, p(s+h-\tau; v, x)\Big|.$$

But given $\varepsilon>0$ we can find $h_0>0$ such that (use (3.9)-(3.11))

(10.25)
$$|q(t/2; u, x)-\int dv\, p(s+h-\tau; x, v)\, q(t/2; u, v)|<\varepsilon$$
$$\text{for all } h\leqq h_0, \text{ all } \tau\in[s, s+h] \text{ and all } u\in\mathbb{R}^{3j}.$$

Therefore (10.24) is bounded for all $h\leqq h_0$ by

(10.26)
$$\varepsilon\cdot h^{-1}\int_0^\infty ds \sum_{n\geq 1} P_a(T^{(n)}\in[s, s+h]).$$

With the notation

(10.27)
$$G(s)=\sum_{n\geq 1} P_a(\infty>T^{(n)}\geqq s)$$

(10.26) can be written in the form (notice: G is decreasing and $G(0)<\infty$)

$$\varepsilon \cdot h^{-1} \int_0^\infty ds(G(s)-G(s+h)) = \varepsilon \cdot h^{-1} \int_0^h ds\, G(s) \leqq \varepsilon \cdot G(0). \tag{10.28}$$

Hence Lemma 10.4 is proved.

Lemma 10.5. (a) *The function* $g\colon \mathbb{R}^{3j} \to \mathbb{R}$, *defined by* (5.2) *in Lemma* 5.1, *or explicitly by*

$$g(a) = \lim_{h\to\infty} h^{-1} \int_0^\infty ds \int dy\, P_a(\omega(s)=y)\, P_y(T \leqq h,\, T(\omega_h^+) > t;\, \omega(t+h)=x) \tag{10.29}$$

$$(x \in \mathbb{R}^{3j} \text{ fixed})$$

is continuous in every point $a \notin \partial B$;

(b) (5.15) *can be justified.*

Proof. We introduce the notation

$$h(y) = P_y(T \leqq h,\, T(\omega_h^+) > t;\, \omega(t+h)=x). \tag{10.30}$$

In the proof of (a) let a point $a \notin \partial B$ be given and fixed. Proceed in two steps:

Step 1. Given $\tau > 0$, the function

$$b \to \lim_{h\to 0} h^{-1} \int_\tau^\infty ds \int dy\, p(s; b, y)\, h(y)$$

is continuous (this limit exists, cf. the proof of the similar fact for (10.29)).

Proof. Given $\varepsilon > 0$ there exists a compact $K \subset \mathbb{R}^{3j}$ such that

$$p(\tau; b, u) \leqq \varepsilon \exp(-|u|) \tag{10.31}$$

for all $u \notin K$ and all b such that $|b-a| \leqq 1$.

We decompose the integral

$$\begin{aligned} h^{-1} \int_\tau^\infty ds \int dy\, p(s; b, y)\, h(y) &= h^{-1} \int_0^\infty ds \int_K du\, p(\tau; b, u) \int dy\, p(s; u, y)\, h(y) \\ &\quad + h^{-1} \int_0^\infty ds \int_{K^c} du\, p(\tau; b, u) \int dy\, p(s; u, y)\, h(y). \end{aligned} \tag{10.32}$$

By (10.31) we can estimate the second term by

$$
\begin{aligned}
(10.33)\qquad & \varepsilon\cdot h^{-1}\int_0^\infty ds\int du\exp(-|u|)\int dy\, p(s;u,y)\,h(y)\\
&=\varepsilon\cdot h^{-1}\iint dy\,du\exp(-|u|)\,G(u,y)\,h(y)\\
&\leqq\varepsilon\cdot h^{-1}\int dy\,P_x(T\in(t,t+h],\ \omega(t+h)=y)\\
&\quad\cdot\sup_y\int du\exp(-|u|)\,G(u,y)\\
&=\varepsilon\cdot h^{-1}P_x(T\in(t,t+h])\cdot\text{const}\\
&\quad(\text{because }\sup_y\int du\exp(-|u|)\,G(u,y)=\text{const}<\infty).
\end{aligned}
$$

By (10.2) there is no problem with the first factor. Therefore it is enough to consider the first term on the right-hand side of (10.32). To given $\varepsilon>0$ choose $\delta>0$ such that

$$
(10.34)\qquad |p(\tau;a,u)-p(\tau;b,u)|<\varepsilon \quad \text{for all } u\in K \text{ and all } b \text{ with } |b-a|<\delta.
$$

Then we get for all b such that $|b-a|<\delta$:

$$
\begin{aligned}
(10.35)\qquad & |h^{-1}\int_0^\infty ds\int_K du\, p(\tau;a,u)\int dy\, p(s;u,y)\,h(y)\\
&\quad -h^{-1}\int_0^\infty ds\int_K du\, p(\tau;b,u)\int dy\, p(s;u,y)\,h(y)|\\
&\leqq\varepsilon\cdot h^{-1}\int_K du\int dy\, G(u,y)\,P_y(T\leqq h,\ T(\omega_h^+)>t;\ \omega(t+h)=x)\\
&\leqq\varepsilon\cdot(\sup_y\int_K du\, G(u,y))\cdot h^{-1}P_x(T\in(t,t+h]).
\end{aligned}
$$

Because of $\sup_y\int_K du\, G(u,y)<\infty$, step 1 is finished.

Step 2. Since $a\notin\partial B$, there exists $\delta>0$ such that $a\in D_\delta$, where

$$
(10.36)\qquad D_\delta=\bigcap_{i,k\leqq j}\{x\in\mathbb{R}^{3j}\colon |x_i-x_k|\leqq R-\delta \text{ or } |x_i-x_k|\geqq R+\delta\}.
$$

Part (a) is proved if we can show

$$
(10.37)\qquad \overline{\lim_{\tau\to 0}}\ \sup_{b\in D_\delta}\ \lim_{h\to 0} h^{-1}\int_0^\tau ds\int dy\, p(s;b,y)\,h(y)=0.
$$

Choose $\eta>0$ such that

$$
(10.38)\qquad \bigcup_{b\in D_\delta} K_b\subset D_{\delta/2},\qquad \text{where } K_b=\{y\in\mathbb{R}^{3j}\colon |y-b|\leqq\eta\}.
$$

Decompose $\int dy=\int_{K_b}dy+\int_{K_b^c}dy$ and consider first $\int_{K_b^c}dy$:

$$(10.39)\qquad h^{-1}\int_0^{\tau} ds \int_{K_b^c} dy\, p(s; b, y)\, h(y) \leqq \tau\,\varepsilon\, h^{-1} \int dy\, P_y(T \leqq h,\ T(\omega_h^+) > t;\ \omega(t+h) = x)$$

for all $\tau \leqq \tau_0$, if we choose τ_0 such that $p(s; 0, y) \leqq \varepsilon$ for all $|y| \geqq \eta$ and all $s \leqq \tau_0$. Because (10.39) holds uniformly in b, it remains to estimate the term $\int_{K_b} dy$. Since $b \in D_\delta$ implies $K_b \subset D_{\delta/2}$ because of (10.38), we get uniformly in $b \in D_\delta$:

$$(10.40)\qquad \begin{aligned} & h^{-1}\int_0^{\tau} ds \int_{K_b} dy\, p(s; b, y)\, h(y) \\ & \leqq h^{-1}\int_0^{\tau} ds \int dy\, p(s; b, y) \cdot \sup_{y \in D_{\delta/2}} h(y) \\ & = \tau \cdot h^{-1} \sup_{y \in D_{\delta/2}} P_y(T \leqq h,\ T(\omega_h^+) > t;\ \omega(t+h) = x). \end{aligned}$$

But $\bar{P}_z(T_R \leqq h,\ T_R(\bar{\omega}_h^+) > t) = o(h)$ uniformly on the set

$$\{z \in \mathbb{R}^3\colon\ |z| \leqq R - \delta \ \text{ or } \ |z| \geqq R + \delta\},$$

hence (10.37) is proved.

Finally we sketch the proof of (b) omitting the details which are quite similar to the arguments just used for the part (a). Consider the sets D_δ, $\delta > 0$ defined by (10.36). Then there exist functions $f_\delta\colon \mathbb{R}^{3j} \to [0, 1]$ $(\delta > 0)$ such that

(i) f_δ is continuous,

(ii) $\operatorname{supp} f_\delta \subset D_\delta$,

(iii) $f_\delta(x) \uparrow 1$ as $\delta \downarrow 0$, for all $x \in \bigcup_{\delta > 0} D_\delta$.

One can show (the proof is omitted) that for every $\delta > 0$

$$(10.41)\qquad \sup_{0 < h \leqq 1}\ \sup_{b \in D_\delta} h^{-1} \int_0^{\infty} ds \int dy\, p(s; b, y)\, h(y) < \infty,$$

which implies

$$(10.42)\qquad \sup_{k \geqq 1}\ \sup_{b \in \mathbb{R}^{3j}} f_\delta(b)\, g_k(b) < \infty \qquad (\delta > 0).$$

Given $\delta > 0$ we can apply Lebesgue's theorem because of (10.42) and so justify (5.15), where we replaced $g(b)$ by $f_\delta(b)\, g(b)$:

$$(10.43)\qquad \begin{aligned} \int db\, p(\tau; a, b)\, f_\delta(b)\, g(b) &= \int db\, p(\tau; a, b)\, f_\delta(b) \lim_{k \to \infty} g_k(b) \\ &= \lim_{k \to \infty} \int db\, p(\tau; a, b)\, f_\delta(b)\, g_k(b) \\ &= \int db\, p(\tau; a, b)\, f_\delta(b) \int \mu(dc)\, G(b, c) \qquad (\delta > 0). \end{aligned}$$

Letting $\delta \downarrow 0$ we get from (10.43)

$$(10.44)\qquad \int db\, p(\tau; a, b)\, g(b) = \int db\, p(\tau; a, b) \int \mu(dc)\, G(b, c) \qquad (a \in \mathbb{R}^{3j},\ \tau > 0).$$

So part (b) of Lemma 10.5 is proved.

c) Lemmas, Needed for Induction Steps

Lemma 10.6. *Let $\varepsilon>0$, $\alpha>\varepsilon$, $j\in\mathbb{N}$ and a function $\varphi\colon[\varepsilon,\alpha]\times\mathbb{R}^{3j}\to\mathbb{R}_+$ be given such that φ satisfies the following three properties*

$$(10.45)\qquad \varphi \text{ is uniformly continuous,}$$

$$(10.46)\qquad \varphi(t,y)=0 \quad \text{for all } t\in[\varepsilon,\alpha] \text{ and } y\in B_j^{(R)},$$

$$(10.47)\qquad \lim_{|x|\to\infty}\max_{\varepsilon\leqq t\leqq\alpha}\varphi(t,x)=0.$$

Then the function $\bar\varphi\colon[\varepsilon,\alpha-\varepsilon]\times\mathbb{R}^{3(j+1)}\to\mathbb{R}_+$, defined by

$$(10.48)\qquad \bar\varphi(t,x)=\int_{t+\varepsilon}^{\alpha}ds\int e_{j+1}(s-t;x,dy)\,\varphi(s,y)$$

has again the properties (10.45)-(10.47).

Proof. As sometimes before (cf. (6.11)-(6.15)) we can replace the measure $e_{j+1}(s-t;x,dy)$ in (10.48) by

$$(10.49)\qquad \bar e_{j+1}(s-t;x,dy) =\lim_{h\to 0}h^{-1}P_x(\omega_{1,\dots,j}(t+h)\in dy;\,T_{1,\dots,j}>T_{1,\dots,j/j+1}\in(t,t+h]).$$

Clearly $\bar\varphi$ satisfies (10.46). To prove the continuity of $\bar\varphi$, decompose

$$(10.50)\qquad \bar\varphi(t,x)=\int_{\varepsilon/2}^{\alpha-t-\varepsilon/2}ds\int du\,q_{j+1}(\varepsilon/2;x,u)\int e_{j+1}(s;u,dy)\,\varphi(s+t+\varepsilon/2,y) =\int du\,q_{j+1}(\varepsilon/2;x,u)\,h(t,u),$$

where

$$(10.51)\qquad h(t,u)=\int_{\varepsilon/2}^{\alpha-t-\varepsilon/2}ds\int e_{j+1}(s;u,dy)\,\varphi(s+t+\varepsilon/2,y).$$

One can check that $\|h\|<\infty$ and that given $\eta>0$ there exists $\delta>0$ such that

$$(10.52)\qquad |h(t,u)-h(t',u)|<\eta \quad \text{for all } u\in\mathbb{R}^{3(j+1)} \text{ and } |t-t'|<\delta$$

(use the same properties of φ and the explicit formula (10.2) and notice that the integration in (10.51) is over $s\geqq\varepsilon/2$). Hence it follows for $|t-t'|<\delta$

$$\begin{aligned}(10.53)\qquad &|\bar\varphi(t,x)-\bar\varphi(t',x')|\\ &\leqq\left|\int du\,q(\varepsilon/2;x,u)\,h(t,u)-\int du\,q(\varepsilon/2;x,u)\,h(t',u)\right|\\ &\quad+\left|\int du\,q(\varepsilon/2;x,u)\,h(t',u)-\int du\,q(\varepsilon/2;x',u)\,h(t',u)\right|\\ &\leqq\eta\cdot\int du\,q(\varepsilon/2;x,u)+\|h\|\int du\,|q(\varepsilon/2;x,u)-q(\varepsilon/2;x',u)|.\end{aligned}$$

Applying Lebesgue's theorem to the last term when $x'\to x$, we conclude from (10.53) that $\bar\varphi$ is continuous in every point. Therefore the proof of Lemma 10.6 is finished if we show that $\bar\varphi$ satisfies (10.47). Let $\eta>0$ be given. By (10.47) there exists a compact set $K \subset \mathbb{R}^{3j}$ such that

(10.55) $\varphi(t, y) \leqq \eta \quad$ for all $t \in [\varepsilon, \alpha]$, $y \in K^c$.

Hence

$$(10.56)\quad \begin{aligned}\bar{\varphi}(t, x) &\leqq \int_{\varepsilon/2}^{\alpha-\varepsilon/2} ds \int_{K^c} \bar{e}_{j+1}(s+\varepsilon/2; x, dy) \cdot \eta \\ &\quad + \int_{\varepsilon/2}^{\alpha-\varepsilon/2} ds \int_{K} \bar{e}_{j+1}(s+\varepsilon/2; x, dy) \|\varphi\| \\ &\leqq \text{const}\, \eta + \|\varphi\| \int_{\varepsilon}^{\alpha} ds \int_{K} \bar{e}_{j+1}(s; x, dy) \\ &\quad (t \in [\varepsilon, \alpha-\varepsilon],\ x \in \mathbb{R}^{3(j+1)}).\end{aligned}$$

In the last step we have used (10.2) and the fact that we integrate over $s \geqq \varepsilon/2$. In view of (10.56) it remains to show

$$(10.57)\qquad \lim_{|x| \to \infty} \int_{\varepsilon}^{\alpha} ds \int_{K} \bar{e}_{j+1}(s; x, dy) = 0 \qquad (K \text{ compact}).$$

For the proof we assume $j=1$ without loss of generality. Using the independence of the processes $\omega_1 - \omega_2$, $\omega_1 + \omega_2$ (cf. (10.11)–(10.13)) and (10.16), we get for any $\delta > R$

$$(10.58)\quad \begin{aligned} &\lim_{h \to 0} h^{-1} \int_K dy \int dz\, P_{y,z}(\omega(s+h) = (x_1, x_2);\ T \leqq h,\ T(\omega_h^+) > s) \\ &= \lim_{h \to 0} h^{-1} \int_K dy \int_{|z-y| \leqq \delta} dz\, P_{y,z}(\omega(s+h) = (x_1, x_2);\ T \leqq h,\ T(\omega_h^+) > s) \\ &= \lim_{h \to 0} h^{-1} \int_K dy\, 8 \cdot \int_{|z-y| \leqq \delta} dz\, \bar{P}_{y-z}(\bar{\omega}(s+h) = x_1 - x_2;\ T_R \leqq h,\ T_R(\omega_h^+) > s) \\ &\quad \cdot \bar{P}_{y+z}(\bar{\omega}(s+h) = x_1 + x_2) \\ &\leqq \overline{\lim_{h \to 0}}\, 8 \cdot h^{-1} \int_K dy \int_{|z-y| \leqq \delta} dz\, \bar{P}_{y-z}(\bar{\omega}(s+h) = x_1 - x_2;\ T_R \leqq h,\ T_R(\omega_h^+) > s) \\ &\quad \cdot \sup_{|a| \leqq \delta} \bar{P}_{2y+a}(\bar{\omega}(s+h) = x_1 + x_2) \\ &= \lim_{h \to 0} h^{-1} \bar{P}_{x_1 - x_2}(T_R \in (s, s+h]) \cdot \int_K dy \sup_{|a| \leqq \delta} \bar{P}_{2y+a}(\bar{\omega}(s) = x_1 + x_2). \end{aligned}$$

Using again (10.2) and the fact that we have to integrate in (10.57) only over $s \in [\varepsilon, \alpha]$, we get (10.57). Hence Lemma 10.6 is proved.

Lemma 10.7. *Let $\mathfrak{F}_j$ ($j \in \mathbb{N}$) be the class of sequences $\{F_j^{(R)}\}$ satisfying* (8.18)–(8.22) *Then for any sequence $\{F_j^{(R)}\}$ in $\mathfrak{F}_j$ the sequence of functions $F_{j+1}^{(R)}$ ($R = \lambda/4\pi D n$), defined by*

$$(10.59)\qquad F_{j+1}^{(R)}(x) = \int e_{j+1}^{(R)}(t; x, dy)\, F_j^{(R)}(y) \qquad (t > 0,\ \textit{fixed}),$$

belongs to $\mathfrak{F}_{j+1}$.

Proof. (8.19) is obvious, (8.20) is already proved in Sect. 8b (see (8.23)). It remains to prove the following three properties:

(10.60) there exists $\bar{F}_{j+1} \in L^1(\mathbb{R}^{3(j+1)})$ such that $F_{j+1}^{(R)} \leqq \bar{F}_{j+1}$ for all R

(10.61) $$\lim_{|x| \to \infty} F_{j+1}^{(R)}(x) = 0 \quad \text{for all } R,$$

(10.62) $$\sup_R \|F_{j+1}^{(R)}\| < \infty,$$

because (10.60) and (10.61) already imply the uniform continuity of every $F_{j+1}^{(R)}$: by (10.61) it is enough to prove that $F_{j+1}^{(R)}$ is continuous. To this purpose choose $\tau \in (0, t)$ and write

(10.63) $$F_{j+1}^{(R)}(x) = \int d u\, q_{j+1}^{(R)}(t-\tau; x, u) \int e_{j+1}^{(R)}(\tau; u, d y)\, F_j^{(R)}(y).$$

But (10.60) holds for all parameters $t > 0$, so we can apply it in particular to $\int e_{j+1}^{(R)}(\tau; u, d y)\, F_j^{(R)}(y)$. Therefore we can use Lebesgue's theorem to prove the continuity of $F_{j+1}^{(R)}$.

For the proof of (10.62) we again assume without loss of generality $j = 1$. Because of $\sup_R \|F_j^{(R)}\| < \infty$ we have therefore to show

(10.64) $$\sup_R \sup_x \int e_2^{(R)}(t; x, d y) < \infty,$$

but this fact follows from the explicit formula (10.2). In order to prove (10.61) and (10.60) we proceed in the same way as in the proof of (10.57) above: first reduce the problem to a one particle problem with the help of the independence of $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$ (cf. (10.58)). Then apply (10.2) which gives also the estimate uniformly in R, as needed for (10.60). We omit the details which are just the same as before.

d) Proof of Lemma 10.8

Lemma 10.8. *Let* $\eta > 0$. *Then*

(a) $$\overline{\lim_{\tau \to 0}}\ \overline{\lim_{R \to 0}}\ \overline{\lim_{h \to 0}}\ R^{-1} h^{-1} \int_{\mathbb{R}^3} d z\, P_z\big(\max_{h \leqq s \leqq \tau + h} |\omega(s)| \geqq \eta;\ T_R \leqq h,\ T_R(\omega_h^+) > \tau\big) = 0.$$

(b) $$\overline{\lim_{\tau \to 0}}\ \overline{\lim_{R \to 0}}\ \overline{\lim_{h \to 0}}\ R^{-1} h^{-1} \int_{\mathbb{R}^3} d z\, P_{y_1, z}\big(\max_{h \leqq s \leqq \tau + h} |\omega_1(s) - y_1| \geqq \eta;\ T_{1,2} \leqq h,\ T_{1,2}(\omega_h^+) > \tau\big) = 0 \qquad (y_1 \in \mathbb{R}^3).$$

Proof. Using the independence of $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$ one can deduce (b) from (a) in the following way:

(10.65) $$\{\max_{h \leqq s \leqq \tau + h} |\omega_1(s) - y_1| \geqq \eta\} \subset \{\max_{h \leqq s \leqq \tau + h} |(\omega_1(s) - y_1) - (\omega_2(s) - z)| \geqq \eta\}$$
$$\cup \{\max_{h \leqq s \leqq \tau + h} |(\omega_1(s) - y_1) + (\omega_2(s) - z)| \geqq \eta\}$$

implies

(10.66) $$\int dz P_{y_1,z}(\max_{h\leqq s\leqq \tau+h} |\omega_1(s)-y_1|\geqq \eta; T_{1,2}\leqq h, T_{1,2}(\omega_h^+)>\tau)$$
$$\leqq \int dz \bar{P}_{y_1-z}(\max_{h\leqq s\leqq \tau+h} |\bar{\omega}(s)-(y_1-z)|\geqq \eta; T_{1,2}\leqq h, T_{1,2}(\omega_h^+)>\tau)$$
$$+\int dz \bar{P}_{y_1+z}(T_R\leqq h, T_R(\bar{\omega}_h^+)>\tau)\cdot \bar{P}_{y_1+z}(\max_{h\leqq s\leqq \tau+h} |\bar{\omega}(s)-(y_1+z)|\geqq \eta).$$

There is no problem with the last summand. It consists of two factors the first of which tends to a constant in the limit $h\to 0$, $R\to 0$ (independent of τ; use (10.4)), whereas the second factor becomes of the form $\bar{P}_0(\max_{0\leqq s\leqq \tau} |\bar{\omega}(s)|\geqq \eta)$ which tends to zero for $\tau\to 0$.

To handle the first summand on the right hand side of (10.66) we assume $R<\eta/2$, and get

(10.67) $$\overline{\lim_{h\to 0}}\, h^{-1}\int dz \bar{P}_z(\max_{h\leqq s\leqq \tau+h} |\bar{\omega}(s)-z|\geqq \eta; T_R\leqq h, T_R(\bar{\omega}_h^+)>\tau)$$
$$=\overline{\lim_{h\to 0}}\, h^{-1}\int_{|z|\leqq \eta/2} dz \bar{P}_z(\max_{h\leqq s\leqq \tau+h} |\bar{\omega}(s)-z|\geqq \eta; T_R\leqq h, T_R(\bar{\omega}_h^+)>\tau)$$
$$\leqq \overline{\lim_{h\to 0}}\, h^{-1}\int dz \bar{P}_z(\max_{h\leqq s\leqq \tau+h} |\bar{\omega}(s)|\geqq \eta/2; T_R\leqq h, T_R(\bar{\omega}_h^+)>\tau).$$

Now we can apply (a) to this last expression, hence (b) is proved.

Turning to the proof of (a), assume $R<\eta$ and use the notations

(10.68) $$K=\{x\in\mathbb{R}^3: |x|<\eta\}, \qquad L=K^c=\{x\in\mathbb{R}^3: |x|\geqq \eta\}.$$

By time reversal (a) is equivalent to

(10.69) $$\overline{\lim_{\tau\to 0}}\ \overline{\lim_{R\to 0}}\ \overline{\lim_{h\to 0}}\, R^{-1}h^{-1}\int dz P_z(T_L\leqq \tau; T_R\in(\tau,\tau+h])=0.$$

In order to prove (10.69) consider first

(10.70) $$\lim_{h\to 0} h^{-1}\int_L dz P_z(T_L\leqq \tau; T_R\in(\tau,\tau+h])=\frac{d}{d\tau}\int_L dz P_z(T_R\leqq \tau).$$

By means of (10.2) an interchange of differentiation and integration is allowed and

(10.71) $$\overline{\lim_{\tau\to 0}}\ \overline{\lim_{R\to 0}}\, R^{-1}\int_L dz \frac{d}{d\tau} P_z(T_R\leqq \tau)=0.$$

Therefore it remains to consider

(10.72) $$\int_K dz P_z(T_L\leqq \tau; T_R\in(\tau,\tau+h])$$
$$\leqq \int_K dz \int_{[0,\tau]}\int P_z(T_L\in dt, \omega(T_L)\in du) P_u(T_R\in(\tau-t,\tau-t+h]).$$

By rotation symmetry we can choose any point y on the boundary of L, i.e.

(10.73) $$|y| = \eta$$

and write the last expression of (10.72) in the form

(10.74) $$\int_{[0,\tau]} \int_K dz\, P_z(T_L \in dt) \cdot P_y(T_R \in (\tau - t, \tau - t + h]).$$

Using (10.2) once more as well as (10.73) we get

(10.75) $$\begin{aligned} &\overline{\lim_{R\to 0}}\, R^{-1} \lim_{h\to 0} \int_{[0,\tau]} \int_K dz\, P_z(T_L \in dt) \cdot h^{-1} P_y(T_R \in (\tau - t, \tau - t + h]) \\ &= \overline{\lim_{R\to 0}}\, R^{-1} \int_{[0,\tau]} \int_K dz\, P_z(T_L \in dt) \frac{d}{d\tau} P_y(T_R \leqq \tau - t) \\ &\leqq \overline{\lim_{R\to 0}} \int_{[0,\tau]} \int_K dz\, P_z(T_L \in dt) \cdot (4\pi D(\tau - t))^{-3/2} \\ &\quad \cdot \exp(-(\eta - R)^2/4D(\tau - t)) \\ &= \int_{[0,\tau]} \int_K dz\, P_z(T_L \in dt) \cdot (4\pi D(\tau - t))^{-3/2} \\ &\quad \cdot \exp(-|y|^2/4D(\tau - t)) \\ &= \int_{[0,\tau]} \int_K dz\, P_z(T_L \in dt, \omega(T_L) \in du) \cdot P_u(\omega(\tau - t) = 0) \\ &= \int_K dz\, P_z(\omega(\tau) = 0; T_L \leqq \tau) \\ &\leqq P_0(T_L \leqq \tau). \end{aligned}$$

But $\lim_{\tau \to 0} P_0(T_L \leqq \tau) = 0$, thus Lemma 10.8(a) is proved.

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