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A Unified Formulation of the Central Limit Theorem for Small and Large Deviations from the Mean

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1. Introduction

Let μ^{n*} denote the n:th convolution of a positive measure μ on $\mathbb{R}$. Put

$$\phi(a)=\int e^{ax}\,\mu(dx) \tag{1.1}$$

and

$$s(x)=\inf_{a\in A}(\log\phi(a)-a\,x), \tag{1.2}$$

where A denotes the interior of the set of a for which $\phi(a)<\infty$. Let furthermore $m(a)$ and $v(a)$ stand for the meanvalue and variance of the probability measure μ_a, where

$$\mu_a(dx)=\frac{e^{ax}}{\phi(a)}\,\mu(dx). \tag{1.3}$$

We have $m'(a)=v(a)$ and hence the mapping $A\to a\to m(a)\in m(A)$ is one to one unless $v(a)=0$ i.e. μ is concentrated at a single point. Let us write $\hat{a}$ for the inverse mapping, $\hat{a}=m^{-1}$.

Cramér 1938 showed that if $0\in A$ (and $v(0)>0$) then

$$\frac{\mu^{n*}([x,\infty))}{1-\Phi(\tilde{x})}=\phi(0)^n\,e^{\frac{\tilde{x}^3}{\sqrt{n}}\lambda\left(\frac{\tilde{x}}{\sqrt{n}}\right)}\left[1+O\left(\frac{\tilde{x}}{\sqrt{n}}\right)\right] \tag{1.4}$$

for $1<\tilde{x}=o(\sqrt{n})$. Here $\tilde{x}=(x-nm(0))/\sqrt{nv(0)}$, $\Phi(x)=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)\,dy$, and λ is a certain power series. (Cramér's error term was slightly worse because the Berry-Esseen theorem was not known at that time.) He also showed that if, in addition, μ has a non-vanishing absolutely continous component with respect to the Lebesgue measure, then for fixed c satisfying

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$m(0) < c \in m(A)$

$$\mu^{n*}([n\,c, \infty)) = \frac{e^{ns(c)}}{\sqrt{2\pi\, n\, v(\hat{a}(c))}\,\hat{a}(c)^2}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{1.5}$$

Blackwell and Hodges 1959 showed that if the support of μ is contained in a coset of a discrete subgroup of $\mathbb{R}$, then (after a suitable normalization)

$$\mu^{n*}([n\,c, \infty)) = \frac{e^{ns(c)}}{\sqrt{2\pi\, n\, v(\hat{a}(c))}}\,\frac{1}{1 - e^{-\hat{a}(c)}}\left[1 + O\left(\frac{1}{n}\right)\right] \tag{1.6}$$

where again c is fixed and $m(0) < c \in m(A)$.

Bahadur and Ranga Rao 1960 pointed out that (1.5) holds (with $O(1/n)$ replaced by $o(1)$) in all cases not covered by (1.6). (We are still excluding the case $v(0) = 0$.) Petrov 1965 showed that the convergence in (1.5) and (1.6) is uniform when c stays away from $m(0)$ and the boundary of A.

The object of this paper is to give a unified formulation of these results, and to give conditions under which our approximation holds not only when x/n (or c) belongs to compact sets but also when x/n is close to ∞.

Related results can be found in the book [8] of Ibragimov and Linnik and in that of Petrov [10]. See for example Ch. 14 of the former. A k-dimensional result which is related to ours in the continuous case was given by Borovkov and Rogozin 1965.

2. Results

We shall need two functions namely

$$\tau(\lambda) = e^{\lambda/2}(1 - \Phi(\sqrt{\lambda})) \tag{2.1}$$

and

$$\rho(\lambda, s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp[\lambda(e^{i\xi} - 1 - i\,\xi)]}{1 - s\,e^{i\xi}}\,d\xi. \tag{2.2}$$

τ is completely monotone ($(-1)^n\,\tau^{(n)}(\lambda) \geqq 0$) and satisfies $\tau(0) = 1/2$ and

$$1 - \frac{1}{\lambda} < \tau(\lambda)\sqrt{2\pi\lambda} < 1, \qquad \lambda > 0. \tag{2.3}$$

For ρ the following is true

$$\lim_{l \to 0} \rho\left(\frac{v}{l^2}, e^{-la}\right) = \tau(v\,a^2), \qquad (v > 0, a > 0) \tag{2.4}$$

and for fixed $0 \leqq s < 1$ as $\lambda \to \infty$

$$\rho(\lambda, s) = \frac{1}{(1 - s)\sqrt{2\pi\lambda}}\left[1 + O\left(\frac{1}{\lambda}\right)\right]. \tag{2.5}$$

Some of these statements will be proved in Sect. 5.

Let $G(\mu)$ stand for the smallest closed subgroup of $\mathbb{R}$ containing all the differences of the numbers in the support of μ. Then either $G(\mu)=\mathbb{R}$, $G(\mu)=l\mathbb{Z}$ for some $0<l<\infty$, or $G(\mu)=\{0\}$.

Theorem A. (i) *If* $G(\mu)=\mathbb{R}$, *then*

$$\mu^{n*}([x,\infty))=e^{ns(x/n)}\,\tau(n\,v(\hat{a}(x/n))\,\hat{a}(x/n)^2)\,[1+o(1)] \tag{2.6}$$

uniformly in x *when* $\hat{a}(x/n)$ *stays within compact subsets of* $A\cap[0,\infty)$. *The error* $o(1)$ *may be replaced by* $O(1/\sqrt{n})$ *if Condition 2.1 below is satisfied.*

(ii) *If* $G(\mu)=l\mathbb{Z}$, *then*

$$\mu^{n*}([x,\infty))=e^{ns(x/n)}\,\rho\left(\frac{n\,v(\hat{a}(x/n))}{l^2}, e^{-l|\hat{a}(x/n)|}\right)\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right] \tag{2.7}$$

uniformly in x *when* $\hat{a}(x/n)$ *stays within compact subsets of* $A\cap[0,\infty)$ *(and* x *belongs to that coset of* $G(\mu)$ *that contains* $\operatorname{supp}\mu^{n*}$*).*

The theorem remains true if we replace $\mu^{n*}([x,\infty))$ by $\mu^{n*}((-\infty,x])$ and $A\cap[0,\infty)$ by $A\cap(-\infty,0]$ (this is why we used the absolute value sign to the right in (2.7)). Note that we formally get (2.6) from (2.7) if we let $l\to 0$ and use (2.4). Note also that we have not assumed that μ has finite total mass (the left tail may be infinite). We may write $O(\min(n^{-1/2}, (n\,\hat{a}(x/n))^{-1}))$ instead of $O(1/\sqrt{n})$ in (2.7). The same remark applies to (2.6) under the additional Condition 2.1 below. Also, the error $o(1)$ in (2.6) can be replaced by the smallest of $o(1)$ and $O(\max(n^{-1/2}, x/n-m(0)))$.

Condition 2.1. $\limsup_{|\alpha|\to\infty}|\phi(a+i\alpha)|<\phi(a)$ *for some* $a\in A$.

It follows from Lemma 4 in [1] that the inequality in Condition 2.1 holds for all $a\in A$ if it holds for some. This was pointed out to me by R.N. Bhattacharya.

Corollary. *All the results described in the introduction follow from Theorem A, provided we replace the errors in* (1.4), (1.5) *and* (1.6) *by* $o(1)$. *They follow as they stand if we use the modified error terms mentioned above.*

We already known that the results of Section 1 hold so we will only sketch a proof.

Proof. Define λ in (1.4) by $z^3\,\lambda(z)=s(m(0)+z\sqrt{v(0)})+z^2/2$ and note that $s'(x)=-\hat{a}(x)$, $\hat{a}'(x)=1/v(\hat{a}(x))$, $x/n=m(0)+\tilde{x}\sqrt{v(0)/n}$, and hence that $n\,v(\hat{a}(x/n))\,\hat{a}(x/n)^2=(1+O(\tilde{x}/\sqrt{n}))$. When $G(\mu)=\mathbb{R}$ (1.4) now follows from the inequality $|\tau(\lambda(1+\varepsilon))-\tau(\lambda)|\leqq \text{Const.}\,|\varepsilon|\,\tau(\lambda)$ valid for $|\varepsilon|\leqq 1/2$ and $\lambda\geqq 0$. When $G(\mu)=l\mathbb{Z}$ we need in addition a refined form of (2.4) namely $|\rho(\lambda, e^{-b})-\tau(\lambda b^2)|\leqq \text{Const.}(b+\lambda^{-1/2})\,\tau(\lambda b^2)$ valid for $b>0$ and $\lambda>0$.

The remaining results of Sect. 1 follow via (2.3) and (2.5). Note that Condition 2.1 is satisfied if μ has a non-vanishing absolutely continous component with respect to the Lebesgue measure. The normalization mentioned just before (1.6) is $l=1$.

If $\mu(dx)=e^{-x^2}dx$ then (2.6) holds not only when x/n belongs to compacts but uniformly for $x/n \geqq m(0)$. It can also be verified that the same is true when μ is concentrated on $(0, \infty)$ and there defined by $\mu(dx)=e^{-x}dx$. A similar remark applies to (2.7). There are more measures with this property, namely measures satisfying the following condition.

Condition 2.2. The support of μ is bounded to the right, sup supp $\mu=r<\infty$, *and;*

(i) $G(\mu)=\mathbb{R}$.

For some $\sigma>0$ and some L that varies slowly at 0

$$\mu([x, \infty))=(r-x)^{\sigma} L(r-x), \quad \textit{for } x<r. \tag{2.8}$$

(ii) $G(\mu)=l\,\mathbb{Z}$.

$$\mu(r-l)>0. \tag{2.9}$$

Theorem B. *Suppose that Condition 2.2 is satisfied and that $0 \leqq b \in A$. Then the approximations* (2.6) *and* (2.7) *hold uniformly in x when $x/n \geqq m(b)$.*

Here again the error terms can be improved. We interprete the right hand side of (2.6) and (2.7) as 0 when $s(x/n)=-\infty$. The letter b occurs in the formulation of the theorem just to cover cases where $0 \notin A$.

3. The Discrete Case

We are now going to prove Theorem A and B when $G(\mu)=l\,\mathbb{Z}$, and shall assume that $l=1$ and that supp μ is a subset of $\mathbb{Z}$ itself. This is just a normalization.

In Theorem A we consider only values of x for which $x/n \in m(A)$, but in Theorem B it may happen that $x/n \geqq r$. Let us first point out that if Condition 2.2 holds, then $s(x)=-\infty$ for $x>r$, $s(r)=\log \mu(r)$ and

$$\rho(n\,v(\hat{a}(r)), e^{-\hat{a}(r)})=\rho(n\,v(\infty), e^{-\infty})=\rho(0,0)=1.$$

So the conclusion of Theorem B is true for $x/n \geqq r$. It therefore suffices to consider values of x for which $\hat{a}(x/n)$ is finite.

By Fourier inversion

$$\mu^{n*}(y)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(a+i\alpha)y}\,\phi(a+i\alpha)^n\,d\alpha. \tag{3.1}$$

Sum both sides of this identity over y from x to ∞, put $a=\hat{a}(x/n)$, note that

$$s(x)=\log \phi(\hat{a}(x))-\hat{a}(x)\,x \tag{3.2}$$

and thus conclude

$$\mu^{n*}([x, \infty))=e^{ns(x/n)}\,I_n(\hat{a}(x/n)). \tag{3.3}$$

Here

$$I_n(a)=\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\gamma_a(\alpha)^n}{1-e^{-a-i\alpha}}\,d\alpha \tag{3.4}$$

where

$$\gamma_a(\alpha)=\frac{\phi(a+i\alpha)}{\phi(a)}\,e^{-i\alpha m(a)}=\int e^{i\alpha(x-m(a))}\,\mu_a(dx). \tag{3.5}$$

The usual approximation here is to replace $\gamma_a(\alpha)$ by $\exp(-\alpha^2 v(a)/2)$. This approximation is a good one provided $v(a)$ stays away from 0. Theorem B covers, however, cases for which this is not true. We will instead approximate $\gamma_a(\alpha)$ by $\exp q(\alpha)$, where $q(\alpha)=v(a)(e^{-i\alpha}-1+i\alpha)$.

This will be done using the following standard estimates:

$$|\gamma^n-e^{nq}|\leqq n\max(|\gamma|,|e^q|)^{n-1}|\gamma-e^q|, \qquad |e^q-1-q|\leqq\tfrac{1}{2}|q|^2 e^{|q|},$$

and $|q|\leqq v(a)\,\alpha^2/2$. Hence (with obvious notation)

$$|\gamma^n-e^{nq}|\leqq n\max(|\gamma|,|e^q|)^{n-1}\left[|\gamma-1-q|+\frac{(v\alpha^2)^2\,e^{v\alpha^2/2}}{8}\right]. \tag{3.6}$$

But $1-\cos\alpha\geqq\frac{2}{\pi^2}\alpha^2$ for $|\alpha|\leqq\pi$, and hence

$$|e^q|\leqq e^{-\frac{2}{\pi^2}v(a)\alpha^2} \tag{3.7}$$

and

$$|1-s\,e^{-i\alpha}|^2\geqq(1-s)^2+s\left(\frac{2}{\pi}\right)^2\alpha^2. \tag{3.8}$$

The following proposition is a consequence of the definitions of I_n and ρ and the estimates (3.6), (3.7) and (3.8).

Proposition 3.1. *Define $I_n(a)$ and $\gamma_a(\alpha)$ as in* (3.4) *and* (3.5). *Let $0<\delta<2/\pi^2$. If*

$$|\gamma_a(\alpha)|\leqq e^{-\delta v(a)\alpha^2} \tag{3.9}$$

and

$$|\gamma_a(\alpha)-1-v(a)(e^{-i\alpha}-1+i\alpha)|\leqq B\,v(a)^{3/2}|\alpha|^3 \tag{3.10}$$

for $|\alpha|\leqq\pi$, then

$$|I_n(a)-\rho(n\,v(a),s)|\leqq C\left(\frac{B}{\sqrt{\delta^3}\sqrt{n}}+\frac{1}{\delta^2 n}\right)\min\left(\frac{1}{\sqrt{s}},\frac{1}{\sqrt{\delta\,n\,v(a)\,(1-s)^2}}\right) \tag{3.11}$$

for $n\geqq 2+1/\delta$. Here $s=e^{-a}$ and C is an absolute constant.

Proof of Theorem A. The moments $\int|x-m(a)|^k\,\mu_a(dx)$ are never 0 or ∞ and they are continuous functions of a and hence bounded away from 0 and ∞ when a is bounded away from the boundary of A. The estimate

$$\begin{aligned}|\gamma-1-q|&\leqq\left|\gamma-1+\frac{\alpha^2}{2}v\right|+v\left|e^{-i\alpha}-1+i\alpha+\frac{\alpha^2}{2}\right|\\&\leqq\frac{|\alpha|^3}{6}\int|x-m(a)|^3\,\mu_a(dx)+\frac{|\alpha|^3}{6}v(a)\end{aligned} \tag{3.12}$$

therefore shows that for each compact $K \subset A$ there is a constant $B = B(K)$ such that (3.10) holds for all $a \in K$. The minimum to the right in (3.11) is by Lemma 5.2 dominated by $\delta^{-1/2} \rho(n\, v(a), e^{-a}) \max_{a \in K} e^{a/2}$ and hence Theorem A will follow from the following lemma.

Lemma 3.2. *Let μ be a measure with $G(\mu) = \mathbb{Z}$. Then, for each compact $K \subset A$ there is a positive constant $\delta = \delta(K)$ such that*

$$|\hat{\mu}_a(\alpha)| \leqq e^{-\delta v(a) \alpha^2} \tag{3.13}$$

for $a \in K$ and $|\alpha| \leqq \pi$.

Proof. By a standard estiamte

$$|\hat{\mu}_a(\alpha)| \leqq e^{-\alpha^2 v(a)/3} \tag{3.14}$$

for $|\alpha| \leqq c(a) = v(a) / \int |x - m(a)|^3 \mu_a(dx)$ (see Ch. XVI of [5]). So (3.14) holds for $|\alpha| \leqq c = \min_{a \in K} c(a) > 0$. The function $(a, \alpha) \to \hat{\mu}_a(\alpha)$ is continuous, the set $C = \{(a, \alpha) \mid a \in K,\ c \leqq |\alpha| \leqq \pi\}$ is compact, and $|\hat{\mu}_a(\alpha)| < 1$ for $(a, \alpha) \in C$ (see [5], Lemma 4, p. 501). Therefore $\max_{(a,\alpha) \in C} |\hat{\mu}_a(\alpha)| = q < 1$, and hence (3.13) holds with

$$\delta = \min \left[\tfrac{1}{3}, \left(\log \frac{1}{q}\right) \Big/ \left(\pi^2 \max_{a \in K} v(a)\right)\right]. \tag{3.15}$$

Proof of Theorem B. Let us start with a lemma that can be proved in a similar way as Lemma 1 of [7].

Lemma 3.3. *Let μ be a measure on $\mathbb{Z}$ such that $\mu(r) > 0$ and $\mu(r-t) > 0$ $(t \geqq 1)$ but $\mu(x) = 0$ for $r \neq x > r - t$. Then for each $b \in A$ there is a constant $C = C(b)$ such that*

$$\begin{aligned} &\left| m(a) - r + t \frac{\mu(r-t)}{\mu(r)} e^{-at} \right| \leqq C e^{-a(t+1)} \\ &\left| \int (x - m(a))^k \mu_a(dx) - (-t)^k \frac{\mu(r-t)}{\mu(r)} e^{-at} \right| \leqq C^k e^{-a(t+1)} \end{aligned} \tag{3.16}$$

for all $a \geqq b$. Also

$$\begin{aligned} {}^0\mu_a(0) &= 1 - 2 \frac{\mu(r-t)}{\mu(r)} e^{-at} + O(e^{-a(t+1)}) \\ {}^0\mu_a(\pm t) &= \frac{\mu(r-t)}{\mu(r)} e^{-at} + O(e^{-a(t+1)}) \end{aligned} \tag{3.17}$$

for $a \geqq b$. Here

$${}^0\mu_a(x) = \sum_{x_1 - x_2 = x} \mu_a(x_1)\, \mu_a(x_2). \tag{3.18}$$

Lemma 3.4. *Let μ be a measure on $\mathbb{Z}$ such that* $\max \operatorname{supp} \mu = r < \infty$, *and let $b \in A$. Then there is a constant $c = c(b) > 0$ such that*

$$|\hat{\mu}_a(\alpha)| \leqq e^{-cv(a)\alpha^2} \tag{3.19}$$

for all $a \geqq b$ and $|\alpha| \leqq \pi$ if and only if $\mu(r-1)>0$.

Proof. We have

$$\begin{aligned} 1-|\hat{\mu}_a(\alpha)|^2 &= \int (1-\cos\alpha x)\, {}^0\mu_a(dx) \\ &\geqq (1-\cos\alpha)\, 2\, {}^0\mu_a(1) \geqq (1-\cos\alpha)\, 2\, \mu_a(r)\, \mu_a(r-1). \end{aligned} \tag{3.20}$$

But

$$\mu_a(r)\, \mu_a(r-1) = \mu(r)\, \mu(r-1)\, e^{-a}(e^{ar}/\phi(a))^2 \tag{3.21}$$

and $e^{ar}/\phi(a)$ increases with a. Also, $1-\cos\alpha \geqq 2\alpha^2/\pi^2$ for $|\alpha| \leqq \pi$. Hence

$$1-|\hat{\mu}_a(\alpha)|^2 \geqq \left(\frac{2}{\pi}\right)^2 \alpha^2\, \mu(r)\, \mu(r-1)\, (e^{br}/\phi(b))^2\, e^{-a} \tag{3.22}$$

for $a \geqq b$. A glance at (3.16) now shows that $1-|\hat{\mu}_a(\alpha)|^2 \geqq 2c\alpha^2 v(a)$, and hence that (3.19) holds for some constant $c>0$, provided $\mu(r-1)>0$.

Conversely, with t as in Lemma 3.3

$$\begin{aligned} 1-|\hat{\mu}_a(\alpha)|^2 &= (1-\cos\alpha t)\, 2\, {}^0\mu_a(t) + O(1-{}^0\mu_a(0)-{}^0\mu_a(t)-{}^0\mu_a(-t)) \\ &= \frac{1-\cos\alpha t}{t^2}\, v(a) + O(v(a)^{1+\frac{1}{t}}) \end{aligned} \tag{3.23}$$

contradicting (3.19) unless $t=1$ (put $\alpha = 2\pi/t$ and let $a \to \infty$). The lemma follows.

The relation (3.16) with $t=1$ implies in exactly the same way as in [7] p. 176 that

$$|\gamma_a(\alpha) - 1 + v(a)(e^{-i\alpha} - 1 + i\alpha)| \leqq \text{Const.}\, |\alpha|^3\, v(a)^2 \tag{3.24}$$

for $a \geqq b$ and $|\alpha| \leqq \pi$. Also $v(a)\, e^a \leqq$ Const. for $a \geqq b$. Proposition 3.1 is thus applicable with $B = \text{Const.}\sqrt{v(a)}$ and $\delta =$ the c of Lemma 3.4. Hence (note that $s = e^{-a} \leqq 1$)

$$|I_n(a) - \rho(n\,v(a), s)| \leqq \frac{\text{Const.}}{\sqrt{n}} \left[1 + \frac{1}{\sqrt{n\,v(a)}}\right] \min\left(1, \frac{1}{\sqrt{n\,v(a)\,(1-s)^2}}\right) \tag{3.25}$$

for $a \geqq b$. So Theorem B follows from Lemma 5.2 and the following remark: From (3.16) (with $t=1$)

$$n(r - m(a)) = n\,v(a) + O(n\,v(a)^2). \tag{3.26}$$

But

$$n(r - m(\hat{a}(x/n))) = nt - x \in \mathbb{Z}_+ \tag{3.27}$$

and hence $n\,v(\hat{a}(x/n)) \geqq c > 0$ for $b \leqq \hat{a}(x/n) < \infty$ and n large. Also, if $n\,v(\hat{a}(x/n)) \leqq \lambda_0$ then $n\,v(\hat{a}(x/n))$ differs from one of the integers $1, 2, \ldots, \lambda_0$ by at most $O(1/n)$ and hence (see (5.5))

$$\begin{aligned}\rho(n\,v(\hat{a}(x/n)), s) &\geqq \min_{k=1,\dots,\lambda_0} \rho(k,s) + O\left(\frac{1}{n}\right) \\ &\geqq \min_{k=1,\dots,\lambda_0} \rho(k,0) + O\left(\frac{1}{n}\right) \geqq c > 0\end{aligned} \tag{3.28}$$

for large n.

4. The Continuous Case

Our starting point is again (3.3), but we shall here use the representation

$$I_n(a) = \int_{nm(a)}^{\infty} e^{-a(x-nm(a))}\, \mu_a^{n*}(dx) \tag{4.1}$$

which is a direct consequence of the definition of μ_a. Make the substitution $x \to n\,m(a) + x\sqrt{n\,v(a)}$ in (4.1) and define $D_n(a,x)$ by

$$\mu_a^{n*}((-\infty, n\,m(a) + x\sqrt{n\,v(a)})) = \Phi(x) + \gamma\,\Phi^{(3)}(x) + D_n(a,x) \tag{4.2}$$

where

$$\gamma = -\frac{\int (x-m(a))^3\, \mu_a(dx)}{6\sqrt{n}\sqrt{v(a)^3}}. \tag{4.3}$$

Then $I_n(a) = I_n'(a) + I_n''(a)$, where

$$I_n'(a) = \int_0^{\infty} e^{-tx}\, \Phi(dx) + \gamma \int_0^{\infty} e^{-tx}\, \Phi^{(3)}(dx) \tag{4.4}$$

$$I_n''(a) = \int_0^{\infty} e^{-tx}\, D_n(a, dx).$$

Here we used the abbreviation

$$t = a\sqrt{n\,v(a)}. \tag{4.5}$$

Some calculations show that

$$I_n'(a) = \tau(t^2)\,(1 + \gamma\,\theta(t)), \tag{4.6}$$

where

$$\theta(t) = t^3 + \frac{1-t^2}{\tau(t^2)\sqrt{2\pi}}. \tag{4.7}$$

It is easy to see that $0 \leqq \theta(t) \leqq 4$ for $0 \leqq t \leqq 1$, and the inequalities $1 - t^{-2} \leqq \tau(t^2)\sqrt{2\pi t} \leqq 1 - t^{-2} + 3t^{-4} \geqq 11/12$ (see [4] p. 179) applied to (4.7) yields

$0 \leqq \theta(t) \leqq 3t^{-1}(1-t^2+3t^{-4})^{-1} \leqq \frac{36}{11} t^{-1}$ for $t \geqq 1$. Hence

$$0 \leqq \theta(t) \leqq 4 \min(1, 1/t) \tag{4.8}$$

for $t \geqq 0$, and thus in particular $|\theta(t)| \leqq 4$. We conclude:

$$I_n'(a) = \tau(n\, v(a)\, a^2) \left[1 + \frac{v \int |x-m(a)|^3 \mu_a(dx)}{\sqrt{n}\sqrt{v(a)^3}}\right], \tag{4.9}$$

where $|v| \leqq 2/3$.

By a partial integration

$$I_n''(a) = -D_n(a,0) + t \int_0^\infty e^{-tx} D_n(a,x)\, dx \tag{4.10}$$

and hence

$$I_n''(a) \leqq 2 \sup_x |D_n(a,x)|. \tag{4.11}$$

Lemma 4.1

$$|D_n(a,x)| \leqq C \frac{k_4(a)}{T\sqrt{n}} + \theta_a(T)^n \log T \tag{4.12}$$

for $1 \leqq T \leqq \sqrt{n}$ *and all* x. *Here* C *is an absolute constant,*

$$k_v(a) = \int |x-m(a)|^v \mu_a(dx)\, v(a)^{-v/2},$$

and

$$\theta_a(T) = \max_{1 \leqq |\alpha| k_3(a)\sqrt{v(a)} \leqq T} |\hat{\mu}_a(\alpha)|.$$

Proof. The letter C denotes absolute constants in this proof. By Essen's lemma (see [5] p. 538)

$$|D_n(a,x)| \leqq \frac{1}{\pi} \int_{-S}^{S} \frac{|\hat{D}_n(a,\alpha)|}{|\alpha|} d\alpha + \frac{24}{\pi} \frac{M(a)}{S} \tag{4.14}$$

for $S>0$ and all x. Here

$$M(a) = \max_x |\Phi'(x) + \gamma \Phi^{(4)}(x)| \leqq C \left[1 + \frac{k_3(a)}{\sqrt{n}}\right]. \tag{4.15}$$

By a variant of the estimates in Ch. XVI.5 of [5]

$$|\hat{D}_n(a,\alpha)| \leqq C \frac{k_4(a)}{n} e^{-\alpha^2/3} \alpha^4 (1+\alpha^2) \tag{4.16}$$

for $|\alpha| \leqq \sqrt{n}/k_3(a)$. It is clear that

$$|\hat{D}_n(a,\alpha)| \leqq \theta_a(T)^n + e^{-\alpha^2/2}\left(1 + \frac{k_3(a)}{\sqrt{n}} |\alpha|^3\right) \leqq \theta_a(T)^n + C\left(\frac{1}{\alpha^2} + \frac{k_3(a)}{|\alpha|\sqrt{n}}\right) \tag{4.17}$$

for $\sqrt{n}/k_3(a) \leqq |\alpha| \leqq T\sqrt{n}/k_3(a)$. Put $S = T\sqrt{n}/k_3(a)$ in (4.14) and split the integral to the right into two parts: $|\alpha| \leqq \sqrt{n}/k_3(a)$ and $\sqrt{n}/k_3(a) \leqq |\alpha| \leqq S$. The lemma now follows from the estimates above and the moment inequalities $1 \leqq k_3(a)^2 \leqq k_4(a)$.

Let us sum up (note that $\tau(\lambda) \geqq c/\sqrt{\lambda+1}$).

Proposition 4.2. *Define $I_n(a)$ as in* (4.1) *and $\theta_a(T)$ and $k_\nu(a)$ as in Lemma* 4.1. *Then*

$$|I_n(a) - \tau(n\,v(a)\,a^2)| \leqq C\,\tau(n\,v(a)\,a^2)\left[\frac{k_3(a)}{\sqrt{n}} + \left(\frac{k_4(a)}{T\sqrt{n}} + \theta_a(T)^n \log T\right)\sqrt{n\,v(a)\,a^2+1}\right] \tag{4.18}$$

for $1 \leqq T \leqq \sqrt{n}$, and $a \in A$. Here C is an absolute constant.

Proof of Theorem A. Choose the compact $K \subset A \cap [0, \infty)$. In the same way as in the proof of Theorem A in the discrete case one finds that $k_\nu(a)$ and $v(a)$ are bounded away from 0 and ∞ when $a \in K$. Also, the function $(a, \alpha) \to \hat{\mu}_a(\alpha)$ is uniformly continuous on $K \times \mathbb{R}$ and $|\hat{\mu}_a(\alpha)| < 1$ for $\alpha \neq 0$ (see [5], Lemma 4, p. 501). Hence $\sup_{a \in K} \theta_a(T) < 1$ for each $T < \infty$, and if Condition 2.1 is satisfied then also $\sup_{a \in K} \theta_a(\infty) < 1$. Theorem A follows.

Proof of Theorem B. Theorem B is a consequence of Proposition 4.2 and the following two lemmas. Note that it suffices to consider large a, Theorem A takes care of the remaining values.

Lemma 4.3. *Suppose that μ satisfies Condition 2.2* (i). *Then*

$$\int (x - m(a))^k\, \mu_a(dx) \sim \frac{s_k(\sigma)}{a^k}, \qquad k = 2, 3, \ldots \tag{4.19}$$

as $a \to \infty$, where

$$s_k(\sigma) = (-1)^k \sum_{j=0}^{k} \binom{k}{j} (-\sigma)^{k-j} \frac{\Gamma(\sigma+j)}{\Gamma(\sigma)}. \tag{4.20}$$

Proof. Put $v(y) = y^\sigma L(y)$, $y > 0$. Then (Theorem 2, p. 283 of [5])

$$\int_0^x y^k\, v(dy) \sim \frac{\sigma}{\sigma+k}\, x^{k+\sigma} L(x) \tag{4.21}$$

and hence (Theorem 2, p. 445 of [5])

$$\int_0^\infty e^{-ay} y^k\, v(dy) \sim \frac{\sigma\Gamma(k+^\sigma)\, L\left(\frac{1}{a}\right)}{a^{k+\sigma}} \tag{4.22}$$

as $a \to \infty$. The lemma now follows from the identity

$$\begin{aligned}\int (x - m(a))^k\, \mu_a(dx) &= (-1)^k \int \left(y - \int \eta\, v_a(d\eta)\right)^k v_a(dy) \\ &= (-1)^k \sum_{j=0}^{k} \binom{k}{j} \int y^j\, v_a(dy) \left(-\int \eta\, v_a(d\eta)\right)^{k-j}\end{aligned} \tag{4.23}$$

where

$$\nu_a(dy) = \frac{e^{-ay}\,\nu(dy)}{\int\limits_0^\infty e^{-a\eta}\,\nu(d\eta)} \tag{4.24}$$

Lemma 4.4. *Suppose that μ satisfies Condition* 2.2(i). *Then*

$$\sup_{|\alpha| \geqq \delta}\ \sup_{a \geqq b} |\hat{\mu}_a(a\alpha)| < 1 \tag{4.25}$$

for any $\delta > 0$ and some $b \in A$.

Proof. Again using the fact that ν varies regularly at 0 one finds that $\nu_a(dy/a)$ converges weakly to $y^{\sigma-1} e^{-y}\, dy/\Gamma(\sigma)$ as $a \to \infty$, and hence that $\hat{\nu}_a(a\alpha) \to 1/(1-i\alpha)$ uniformly in α. But $\hat{\mu}_a(\alpha) = e^{ir\alpha}\,\hat{\nu}_a(-\alpha)$, and hence (4.25) holds for all large b.

5. The Functions τ and ρ

By the definition

$$\tau(\lambda) = e^{\lambda/2} \int\limits_{\sqrt{\lambda}}^\infty e^{-x^2/2}\, \frac{dx}{\sqrt{2\pi}}. \tag{5.1}$$

The substitution $x^2 \to x + \lambda$ therefore gives

$$\tau(\lambda) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty \frac{e^{-x/2}}{\sqrt{x+\lambda}}\, dx \tag{5.2}$$

and hence

$$\frac{(-1)^n\, \tau^{(n)}(\lambda)}{n!} = \frac{(-1)^n}{2\sqrt{2\pi}} \binom{-1/2}{n} \int\limits_0^\infty \frac{e^{-x/2}\, dx}{(x+\lambda)^{n+1/2}} > 0. \tag{5.3}$$

The inequalities (2.3) are (1.8) p. 166 of [4].

It follows from (2.2) that

$$\rho(\lambda, s) = \sum_{n=0}^\infty \sum_{m=0}^\infty e^{-\lambda} \frac{\lambda^n}{n!}\, s^m \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} e^{i\xi(n+m-\lambda)}\, d\xi \tag{5.4}$$

$$= \sum\sum e^{-\lambda} \frac{\lambda^n}{n!}\, s^m \frac{\sin \pi(n+m-\lambda)}{(n+m-\lambda)}$$

and hence in particular that

$$\rho(\lambda, s) = \sum_{n+m=\lambda} e^{-\lambda} \frac{\lambda^n}{n!}\, s^m \tag{5.5}$$

when $\lambda \in \mathbb{N}$.

Lemma 5.1. $\lim_{l\to 0} \rho(v/l^2, e^{-la}) = \tau(va^2)$ *for* $v>0$ *and* $a>0$.

Proof. Make the substitution $\xi \to l\xi$ in (2.2). Then by dominated convergence

$$\lim_{l\to 0} \rho\left(\frac{v}{l^2}, e^{-la}\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-v\xi^2/2}}{a-i\xi}\, d\xi = \tau(va^2). \tag{5.6}$$

Here we also used Parseval's relation.

We will not take space to prove (2.5).

Lemma 5.2. *There are positive numbers* c *and* λ_0 *such that*

$$\rho(\lambda, s) \geqq c \min\left(1, \frac{1}{\sqrt{\lambda(1-s)^2}}\right) \tag{5.7}$$

for $\lambda \geqq \lambda_0$ *and* $0 \leqq s < 1$.

Proof. Put

$$\tilde{\rho}(\lambda, s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-\lambda\alpha^2/2}}{1-se^{i\alpha}}\, d\alpha \tag{5.8}$$

and note that

$$|\rho(\lambda, s) - \tilde{\rho}(\lambda, s)| \leqq \frac{\text{Const.}}{\sqrt{\lambda}} \min\left(1, \frac{1}{\sqrt{\lambda(1-s)^2}}\right). \tag{5.9}$$

We may replace $(1-se^{i\alpha})^{-1}$ in (5.8) by

$$\operatorname{Re}(1-se^{i\alpha})^{-1} \geqq \frac{1-s}{(1-s)^2 + 2s(1-\cos\alpha)}. \tag{5.10}$$

But $1-\cos\alpha \leqq \alpha^2/2$ and hence

$$\tilde{\rho}(\lambda, s) \geqq \frac{1}{\sqrt{s}}\, g_{\pi\sqrt{\lambda}}((1-s)\sqrt{\lambda/s}), \tag{5.11}$$

where

$$g_L(t) = \frac{1}{2\pi} \int_{-L}^{L} e^{-\alpha^2/2} \frac{t}{t^2+\alpha^2}\, d\alpha \geqq c \min\left(1, \frac{1}{t}\right) \tag{5.12}$$

for some $c>0$, provided $L>1$.

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