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Almost Sure Invariance Principles When $EX_1^2=\infty$

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Summary. Let $\{Y_i\}$ be *iid* with $EY_1=0$, $EY_1^2=1$. Let $\{X_i\}$ be *iid* normal mean zero, variance one random variables. According to Strassen's first almost sure invariance principle $\{X_i\}$ and $\{Y_i\}$ can be reconstructed on a new probability space without changing the distribution of each sequence such that $\sum_{i=1}^n Y_i-\sum_{i=1}^n X_i=o(n\log_2 n)^{1/2}$ a.s., thus improving on the trivial bound obtainable from the law of the iterated logarithm: $\sum_{i=1}^n Y_i-\sum_{i=1}^n X_i=O(n\log_2 n)^{1/2}$ a.s. In this work we establish analogous improvements for symmetric $\{Y_i\}$ in the domain of normal attraction to a symmetric stable law with index $0<\alpha<2$. (We make this assumption of symmetry in order to avoid messy details concerning centering constants.) Let $\{X_i\}$ be *iid* symmetric stable random variables with index $0<\alpha<2$. Then, for example, hypotheses are stated which imply for a given γ satisfying $2>\gamma\geqq\alpha$ that $\sum_{i=1}^n Y_i-\sum_{i=1}^n X_i=o(n^{1/\gamma})$ a.s., thus improving on the trivial bound: $\sum_{i=1}^n Y_i-\sum_{i=1}^n X_i=o(n^{(1/\alpha)+\varepsilon})$ a.s., $\varepsilon>0$.

1. Introduction

Let $\{Y_i\}$ be independent identically distributed (*iid*) random variables with $EY_1=0$ and $EY_1^2=1$. From the viewpoint of this paper, Y_1 is in the domain of normal attraction to the normal law. (See [2] p. 180 for a discussion of normal attraction.) Let $\{X_i\}$ be *iid* normal random variables with mean zero and variance one. According to Strassen's first almost sure invariance principle [8], there exists a probability space with $\{X_i\}$ and $\{Y_i\}$ redefined on it such that each sequence has the original distribution and that

$$\sum_{i=1}^n Y_i-\sum_{i=1}^n X_i=o(n\log_2 n)^{1/2}\quad\text{a.s.}\tag{1.1}$$

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This improves the trivial bound obtained from the law of the iterated logarithm when $\{X_i\}$ and $\{Y_i\}$ are say independent of each other:

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = O(n \log_2 n)^{1/2} \quad \text{a.s.} \tag{1.2}$$

It is the purpose of this paper to establish results analogous to (1.1) for random variables in the domain of normal attraction to a stable law with index $0<\alpha<2$. In Sect. 2 as a preliminary step we establish results which yield bounds analogous to (1.2) for $0<\alpha<2$. The results of Sect. 2 may be of independent interest. In Sect. 3 we establish almost sure invariance principles analogous to (1.1) for $0<\alpha<2$.

2. The Strong Law for Random Variables Outside the Domain of Partial Attraction to the Normal Law

Throughout Sect. 2 and 3, all random variables are assumed to be outside the domain of partial attraction to the normal law unless specifically stated otherwise. (See [2], p. 183 for a discussion of partial attraction.) Further, in order to avoid messy details about centering constants, all random variables are assumed to be symmetric. Similar results can be obtained when the assumption of symmetry is dropped.

Let $\{Y_i\}$ be such a sequence of *iid* random variables. Let $0<a_n\uparrow\infty$. The following theorem is a restatement of a result of Heyde [3].

Theorem 2.1. *We have that*

$$\sum_{i=1}^{n} Y_i = o(a_n) \text{ a.s.} \quad or \quad \limsup \frac{\sum_{i=1}^{n} Y_i}{a_n} = \infty \text{ a.s.} \tag{2.1}$$

according as

$$\sum_{n=1}^{\infty} P[|Y_1|>a_n]<\infty \quad or \quad \sum_{n=1}^{\infty} P[|Y_1|>a_n]=\infty \tag{2.2}$$

(see also Feller [1] *for a closely related result).*

Definition. Let $\{X_i\}$ and $\{Y_i\}$ each be sequences of *iid* random variables. Suppressing subscripts, we say that X and Y obey the same strong law of large numbers if for each $0<a_n\uparrow\infty$ either

$$\text{(i)} \quad \sum_{i=1}^{n} X_i = o(a_n), \qquad \sum_{i=1}^{n} Y_i = o(a_n) \quad \text{a.s.}$$

or

$$\text{(ii)} \quad \limsup \sum_{i=1}^{n} X_i/a_n = \infty, \qquad \limsup \sum_{i=1}^{n} Y_i/a_n = \infty \quad \text{a.s.}$$

When (i) holds for $\{X_i\}$ we say that the strong law holds for X.

Theorem 2.2. *X and Y obey the same strong law of large numbers if and only if*

$$1 \ll \frac{P[|Y|>x]}{P[|X|>x]} \ll 1 \quad \text{as } x \to \infty. \tag{2.3}$$

(Here $\ll$ is a convenient substitute for the "big 0" notation.)

Proof. Assume (2.3). Choose $a_n \uparrow \infty$. Then $\sum_{n=1}^{\infty} P[|X|>a_n]<\infty$ if and only if $\sum_{n=1}^{\infty} P[|Y|>a_n]<\infty$. Apply Theorem 2.1 to obtain the desired conclusion.

Now assume (2.3) fails. Then $P[|Y|>x]/P[|X|>x]$ has either 0 or ∞ as a limit point. Because of the interchangeability of X and Y it suffices to consider the case where ∞ is a limit point. By Theorem 2.1 it suffices to construct $0<a_n\uparrow\infty$ such that $\sum_{n=1}^{\infty} P[|X|>a_n]<\infty$, $\sum_{n=1}^{\infty} P[|Y|>a_n]=\infty$. By assumption there exists $x_i\uparrow\infty$, $b_i \geqq i$ and $i^3<y_i\uparrow\infty$ such that $P[|X|>x_i]=1/y_i$, $P[|Y|>x_i]=b_i/y_i$. We will choose the first few a_n to be some constant, the next few a_n to be some other constant, etc. Let the first ($[\cdot]$ denotes the greatest integer function) $[y_1/1^2]$ of the a_n equal x_1, the next $[y_2/2^2]$ of the a_n equal $x_2, \ldots$, the jth $[y_j/j^2]$ of the a_n equal x_j, etc. Thus

$$\sum_{n=1}^{\infty} P[|X|>a_n]=\sum_{j=1}^{\infty}\left[\frac{y_j}{j^2}\right]\frac{1}{y_j} \leqq \sum_{j=1}^{\infty}\frac{1}{j^2}<\infty$$

and

$$\sum_{n=1}^{\infty} P[|Y|>a_n]=\sum_{j=1}^{\infty}\left[\frac{y_j}{j^2}\right]\frac{b_j}{y_j} \geqq \sum_{j=1}^{\infty}\left[\frac{y_j}{j^2}\right]\frac{j}{y_j}=\infty. \quad \square$$

If Y is in the domain of normal attraction to a stable law with index α we write $Y \in N(\alpha)$.

Corollary 2.1. *Let X and Y satisfy $X\in N(\alpha_x)$, $Y\in N(\alpha_y)$ for indices $\alpha_x<2$, $\alpha_y<2$ respectively. Then X and Y obey the same strong law if and only if $\alpha_x=\alpha_y$. In particular X stable with index $\alpha_x<2$ and $Y\in N(\alpha_x)$ implies X, Y obey the same strong law of large numbers.*

Proof. By a result of Gnedenko, ([2], p. 181) it follows that (2.3) holds if and only if $\alpha_x=\alpha_y$. $\square$

Corollary 2.1 raises the question whether a random variable Y can fail to be in the domain of normal attraction to a stable law X and yet obey the same strong law of large numbers as X. The answer is "yes":

Example 2.1. Let X be Cauchy and F_Y for large x be given by

$$1-F_Y(x)=\frac{2+\sin(\log x)}{x}.$$

Note that F_Y is a distribution function and that (2.3) holds. Thus X and Y obey the same strong law of large numbers. According to a result of Gnedenko ([2], p. 181), Y is in the domain of normal attraction to a Cauchy distribution only if

$$P[|Y|>x]=\frac{A+\beta(x)}{x}$$

for some $A>0$, $\beta(x)\to 0$ as $x\to\infty$. Thus Y is not in the domain of normal attraction to the Cauchy distribution.

Recall that the purpose of Sect. 2 is to obtain a bound analogous to (1.2) with $\{X_i\}$ being *iid* stable with index $\alpha<2$ and $\{Y_i\}$ being *iid* and in the domain of normal attraction to X_1. Since X and Y obey the same strong law of large numbers, it follows from the results of this section that when $\sum_{n=1}^{\infty} P[|X_1|>a_n]<\infty$ (equivalently $\sum_{n=1}^{\infty} P[|Y_1|>a_n]<\infty$) we have for $\{X_i\}$ and $\{Y_i\}$ independent say of each other

$$\sum_{i=1}^{n} Y_i-\sum_{i=1}^{n} X_i=o(a_n) \quad \text{a.s.} \tag{2.4}$$

It is the goal of Sect. 3 to construct $\{X_i\}$, $\{Y_i\}$ such that (2.4) is improved just as (1.1) improves (1.2) in the $\alpha=2$ case. It should be noted that (2.4) is the proper analogue to (1.2) in the sense that there is no analogue to the *LIL* for random variables in the domain of normal attraction to a stable law with index $\alpha<2$.

3. Almost Sure Invariance Principles

In order to improve (2.4) we make use of the concept of the quantile transform, so successfully exploited by the Hungarian school of probabilists (see [4] for instance).

Lemma 3.1. *Let $Y\in N(\alpha)$ with index $0<\alpha<2$. Then there exists $A>0$, $\beta(x)\to 0$ as $x\to\infty$ such that*

$$P[Y>x]=\frac{A+\beta(x)}{x^\alpha}, \quad x>0. \tag{3.1}$$

Let W have a distribution function F_W which is strictly increasing and continuous and

$$P[W>x]=\frac{A}{x^\alpha}, \quad x \text{ large.} \tag{3.2}$$

Then, as $Y\to\infty$

$$|Y-F_W^{-1}F_Y(Y)|\ll Y|\beta(Y)|. \tag{3.3}$$

Moreover $F_W^{-1}F_Y(Y)$ has the same distribution as W provided Y has a continuous and strictly increasing distribution function.

Proof. (3.1) is Gnedenko's result ([2], p. 181). We have for some w_0,

$$F_W^{-1}(w)=\left(\frac{A}{1-w}\right)^{1/\alpha}, \quad 1>w\geqq w_0.$$

Hence, there exists $y_0>0$ such that for $Y>y_0$,

$$Y-F_W^{-1}F_Y(Y)=Y\,\frac{\{A+\beta(Y)\}^{1/\alpha}-A^{1/\alpha}}{\{A+\beta(Y)\}^{1/\alpha}}.$$

An application of the mean value theorem now yields

$$|Y-F_W^{-1}F_Y(Y)|\ll Y|\beta(Y)|. \quad \square$$

We no can state the improvement (2.4).

Theorem 3.1. *Let $\alpha<2$. Let $Y\in N(\alpha)$, Y thus satisfying* (3.1). *Let X be stable of index α, with X scaled such that*

$$P[X>x]=\frac{A+\beta'(x)}{x^\alpha}, \quad x>0$$

where $\beta'(x)\to 0$ as $x\to\infty$. Let $g(\cdot)$ be non-decreasing and satisfy

$$g(x)\geqq x\,\delta(x), \quad x \text{ large}$$

where $\delta(\cdot)$ is computed from $\beta(\cdot)$, $\beta'(\cdot)$ and satisfies $0<\delta(x)\to 0$ as $x\to\infty$. Let $0<a_n\uparrow\infty$ be such that the strong law of large numbers holds for X (hence for Y). Suppose

$$g(a_n)/n^{1/r}\uparrow\infty \text{ for some } r<2 \text{ and that } \{g(a_n)/n\} \text{ is either non-decreasing or non-increasing.} \tag{3.4}$$

Then there exists a probability space with $\{X_i\}$ and $\{Y_i\}$ sequences of iid random variables defined on it with distributions F_X and F_Y respectively such that

$$\sum_{i=1}^n Y_i-\sum_{i=1}^n X_i=o(g(a_n)) \quad \text{a.s.} \tag{3.5}$$

Moreover, if $0<x\,\delta(x)\uparrow\infty$ and (3.4) *is satisfied for $g(x)=x\,\delta(x)$,*

$$\sum_{i=1}^n Y_i-\sum_{i=1}^n X_i=o(a_n\,\delta(a_n)) \quad \text{a.s.} \tag{3.6}$$

Remark. Note when we can take $g(y)=\sup_{0\leqq x\leqq y} x\,\delta(x)$ that (3.5) and (3.6) are stronger than (2.4) since $\delta(a_n)\to 0$ and hence $g(a_n)/a_n\to 0$.

Proof. Let $\delta(x)=|\beta(x)|+|\beta'(x)|$. We shall only prove

$$\sum_{i=1}^n Y_i-\sum_{i=1}^n W_i=o(g(a_n)) \quad \text{a.s.} \tag{3.7}$$

where W_i are *iid* with distribution (3.2). For, the same argument can then be used to yield, on a possibly different probability space, that

$$\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} W_i = o(g(a_n)) \quad \text{a.s.} \tag{3.8}$$

We can now construct $\{X_i, Y_i, W_i\}$ on the *same* probability space in such a way as to preserve the joint distributions of $\{Y_i, W_i\}$ and of $\{X_i, W_i\}$. This is done by first constructing $\{W_i\}$ and then constructing $\{Y_i\}$ and $\{X_i\}$ conditionally independent given $\{W_i\}$ and moreover with $\{Y_i\}$ given $\{W_i\}$ and $\{X_i\}$ given $\{W_i\}$ each having the required conditional distributions. Then (3.7) and (3.8) can be combined to yield the desired (3.5).

We now prove (3.7). By (3.4), it follows that

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} W_i = o(g(a_n)) \quad \text{a.s.}$$

if and only if

$$\sum_{i=1}^{n} (Y_i + Z_i) - \sum_{i=1}^{n} W_i = o(g(a_n)) \quad \text{a.s.}$$

where $\{Z_i\}$ is a sequence of *iid* random variables independent of $\{Y_i\}$ with $EZ_1 = 0$, $EZ_1^2 < \infty$. For, $\sum_{i=1}^{n} Z_i = O(n \log_2 n)^{1/2}$ a.s. by the law of the iterated logarithm. But $Y_1' = Y_1 + Z_1 \in N(\alpha)$ and thus satisfies (3.1) with the same A, but a different $\beta(\cdot)$ determined from the $\beta(\cdot)$ of Y_1 and the choice of distribution of Z_1. Moreover, the distribution function of Y_1' is continuous and strictly increasing; whereas, the distribution function of Y_1 may not be continuous and strictly increasing. Thus, in order to prove (3.7), we first construct Y_i *iid* and, if the distribution function of Y_1 is not continuous and strictly increasing, we then construct $Y_i' = Y_i + Z_i$ and finally construct $W_i = F_W^{-1} F_{Y'}(Y_i')$. Hence, by an in essential modification of the $\beta(\cdot)$ (that is, the modified $\beta(x) \to 0$ as well), we assume without loss of generality that Y_1 has a continuous and strictly increasing distribution function. Let

$$W_i = F_W^{-1} F_Y(Y_i) \tag{3.9}$$

define W_i, where F_W is given by (3.2).

We have $\sum_{n=1}^{\infty} P[|Y| > a_n] < \infty$. Thus noting that g is non-decreasing, we have that

$$\sum_{n=1}^{\infty} P[g(|Y|) > g(a_n)] < \infty. \tag{3.10}$$

By Lemma 3.1, as $Y_i \to \infty$,

$$|Y_i - F_W^{-1} F_Y(Y_i)| \ll Y_i |\beta(Y_i)| \leqq g(Y_i).$$

Thus, for some $C>0$ by (3.9), (3.10) and symmetry

$$\sum_{n=1}^{\infty} P[|Y_1-W_1|>C g(a_n)]<\infty. \tag{3.11}$$

Unfortunately we cannot now apply Theorem 2.1, because Y_1-W_1 may not be outside the domain of partial attraction to a normal. However, according to a result of Feller (see [6], p. 132), the conclusion of Theorem 2.1 holds for all *iid* random variables (that is, even those not outside the domain of partial attraction to a normal) provided the normalizing constants $\{b_n\}$ satisfy $b_n/n^{1/r}\uparrow$ for some $r<2$ and $\{b_n/n\}$ is either non-decreasing or non-increasing. Thus applying this result of Feller's to $b_n=Cg(a_n)$ in (3.11), (3.7) follows. The proof of (3.8) is now obvious, noting that F_X is continuous and strictly increasing. □

Remarks. It should be emphasized that when Y_1 does not have a distribution function which is continuous and strictly increasing that the $\beta(\cdot)$ in $\delta(x)=|\beta(x)|+|\beta'(x)|$ is the $\beta(\cdot)$ associated with the distribution of Y_1+Z_1. Of course, in applications, the distribution of Z_1 can be conveniently chosen, say normal or uniform with a small variance, such that the $\beta(\cdot)$ associated with the distribution of Y_1+Z_1 is "close" to the $\beta(\cdot)$ associated with the distribution of Y_1.

With the method of this paper, $\{Y_i-X_i\}$ is an *iid* sequence. Hence the best possible result, *using this method*, would be $\sum_1^n Y_i-\sum_1^n X_i=O(n\log_2 n)^{1/2}$ a.s. Hence an assumption of the character of (3.4) is clearly needed. For, otherwise, the conclusion of (3.5) would be stronger than possible with the method used.

Since finding an acceptable $g(\cdot)$ requires knowledge of the behavior of $\delta(\cdot)$ which in turn requires knowledge of the behavior of $\beta'(\cdot)$, it is very useful to know that there are asymptotic expansions for $P[X>x]$, x large. Indeed (see [6]) for $\alpha\neq 1$, as $x\to\infty$

$$P[X>x]=\frac{A+Bx^{-\alpha}+O(x^{-2\alpha})}{x^{\alpha}}$$

where $B\neq 0$. Thus

$$\beta'(x)=Bx^{-\alpha}+O(x^{-2\alpha}).$$

When $\alpha=1$,

$$\beta'(x)=Bx^{-2}+O(x^{-3}) \tag{3.12}$$

is easily seen.

Example 3.1. Let $P[Y>x]=1/x$ for x large and X be Cauchy and scaled such that $P[X>x]=(1+\beta'(x))/x$. Then, by (3.12), $|\beta'(x)|\leqq Cx^{-2}$ for some $C>0$ and all $x>0$. Hence $\delta(x)\leqq Cx^{-2}$ for x large. Let $a_n=n^2$. Define, for some $\gamma<2$, $x>0$,

$$g(x)=x^{1/(2\gamma)}.$$

Clearly $g(\cdot)$ satisfies the hypotheses of Theorem 3.1. Hence, one can construct

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = o(n^{1/\gamma}) \quad \text{a.s.}$$

for each $\gamma < 2$.

In the $\alpha = 2$ case for a given γ, under the assumption that higher moments are finite, a sharpening of (1.1) to $\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = o(n^{1/\gamma})$ a.s. is possible using a more sophisticated approach (see [5] for example). By analogy, in the $\alpha < 2$ case it is interesting to ask when one can conclude for a given γ that

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = o(n^{(1/\gamma)}) \quad \text{a.s.}$$

Theorem 3.2. *Let $\alpha < 2$. Assume $Y \in N(\alpha)$, Y thus satisfying* (3.1), *assuming without loss of generality (as explained in the proof of Theorem* 3.1) *that Y has a continuous and strictly increasing distribution function. Let X be stable of index α with X scaled such that*

$$P[X > x] = \frac{A + \beta'(x)}{x^{\alpha}}$$

where $\beta'(x) \to 0$ as $x \to \infty$. Let (referring to (3.1)) *$n^{1-\alpha/\gamma} \beta^*(n^{1/\gamma})$ be non-increasing for fixed $2 > \gamma \geqq \alpha$ and large n, $\beta^*(\cdot)$ be continuous, non-increasing, and $\beta^*(x) \to 0$ and $\beta^*(\cdot)$ satisfy*

$$\beta^*(x) \geqq |\beta(x)| + |\beta'(x)|, \qquad x \text{ large.}$$

Suppose that $x\beta^*(x)$ is strictly increasing as $x \uparrow \infty$ and that

$$\sum_{n=1}^{\infty} \beta^*(n^{1/\gamma})^{\gamma} / n^{\alpha/\gamma} < \infty. \tag{3.13}$$

Then the conclusion of Theorem 3.1 *holds with* (3.5) *replaced by*

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = o(n^{1/\gamma}) \quad \text{a.s.} \tag{3.14}$$

Remark. If $\{X_i\}$ and $\{Y_i\}$ are independent of each other, then of course

$$\limsup \frac{\left| \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i \right|}{n^{1/\alpha}} = \infty \quad \text{a.s.}$$

Thus (3.14) is an improvement over what can be trivially obtained.

Proof. Let Y^* be a random variable satisfying

$$1 - F_{Y^*}(x) = \frac{A + \beta^*(x)}{x^{\alpha}}, \qquad x \text{ large.}$$

Note for x large that Y^* has "fatter tails" than Y and $\beta^*(x)\downarrow 0$. Now, for some $C>0$

$$\begin{aligned} E|Y^*|^\gamma \beta^*(|Y^*|)^\gamma &= \sum_{n=0}^{\infty} E|Y^*|^\gamma \beta^*(|Y^*|)^\gamma I(n \leqq |Y^*|^\gamma < n+1) \\ &\leqq \sum_{n=0}^{\infty} (n+1)\,\beta^*(n^{1/\gamma})^\gamma P[n \leqq |Y^*|^\gamma < n+1] \\ &\leqq 2AC \sum_{n=0}^{\infty} (n+1)\,\beta^*(n^{1/\gamma})^\gamma \left(\frac{1}{n^{\alpha/\gamma}} - \frac{1}{(n+1)^{\alpha/\gamma}}\right) \\ &\quad + 2C \sum_{n=0}^{\infty} (\beta^*(n^{1/\gamma})^\gamma \{n^{1-\alpha/\gamma}\beta^*(n^{1/\gamma}) - (n+1)^{1-\alpha/\gamma}\beta^*(n+1)^{1/\gamma})\} \\ &\quad + 2C \sum_{n=0}^{\infty} \frac{\beta^*(n^{1/\gamma})^\gamma}{n^{\alpha/\gamma}} \beta^*(n^{1/\gamma}). \end{aligned}$$

But the second sum is bounded by a collapsing sum. Hence, by (3.13), $E|Y^*|^\gamma \beta^*(|Y^*|)^\gamma < \infty$. Hence, for some $\{b_n\}$, $C>0$, and $C'>0$

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} P[|Y^*|\,\beta^*(|Y^*|) > n^{1/\gamma}] \\ &= \sum_{n=1}^{\infty} P[|Y^*| > b_n] \\ &\geqq C \sum_{n=1}^{\infty} P[|Y| > b_n] \\ &= C \sum_{n=1}^{\infty} P[|Y|\,\beta^*(|Y|) > n^{1/\gamma}] \\ &\geqq C' \sum_{n=1}^{\infty} P[|Y|\,\beta(|Y|) > n^{1/\gamma}]. \end{aligned}$$

Thus, for some $C>0$ using (3.3) and symmetry

$$\sum_{n=1}^{\infty} P[|Y_1 - W_1| > C n^{1/\gamma}] < \infty.$$

Hence, $E|Y_1 - W_1|^\gamma < \infty$ and by the Marcinkiewicz strong law of large numbers, noting that $2>\gamma$,

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} W_i = o(n^{1/\gamma}) \quad \text{a.s.}$$

As in the proof of Theorem 3.1, this suffices to establish the desired

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = o(n^{1/\gamma}) \quad \text{a.s.} \quad \square$$

Example 3.2. Let us see how Theorem 3.2 applies to the situation of Example 3.1. That is, let $P[Y>x]=1/x$ for x large and X be Cauchy and scaled such that $P[X>x]=(1+\beta'(x))/x$. As before, $|\beta'(x)| \leqq C x^{-2}$ for some $C>0$ and all $x>0$. We take $\beta^*(x)=C(\log x)\, x^{-1}$ for x large. Clearly $\beta^*(\cdot)$ satisfies all hypotheses for each $\gamma<2$. Hence, one can construct

$$\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = o(n^{1/\gamma}) \quad \text{a.s.}$$

for each $\gamma<2$, the same conclusion obtained from Theorem 3.1 in Example 3.1.

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