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The Spectrum of Dynamical Systems Arising from Substitutions of Constant Length

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Summary. Minimal flows and dynamical systems arising from substitutions are considered. In the case of substitutions of constant length the trace relation of the flow is calculated and is used to determine the spectrum of the dynamical system. Several methods are indicated to obtain new substitutions from given ones, leading among other things to a description of the behaviour of powers of the shift homeomorphism on the system arising from any substitution.

In the domain of topological dynamics, minimal flows play an important role as building blocks for more complicated flows. A large class of examples of minimal flows are given by those arising from substitutions. The systematic investigation of substitution minimal flows began with Gottschalk [5], and has been extended more recently by Coven and Keane [2], Martin [9, 10] and Kamae [7], the goal being topological and measure-theoretic classification of these objects.

In the first part of the paper we synthesize, extend and simplify the known results on topological classification and give a complete spectral classification in the case of substitutions of constant length. An interesting side result is that if (X, T, μ) is a dynamical system arising from a substitution of constant length then T^n is minimal if and only if T^n is ergodic. (This implies for instance that if T has a rational eigenvalue, then the eigenfunction corresponding to this eigenvalue can be chosen to be continuous.)

The second part is concerned with substitutions of non-constant length. We analyse the behaviour of T^n in this case and display a class of flows (generated by substitutions of non-constant length) which are topologically isomorphic to constant length flows, thus reducing the structure problem (both measure-theoretical and topological) to one that can be handled by the results of the first part. In general, the measure-theoretic structure of substitution dynamical systems of non-constant length is unknown, and seems to be difficult to determine (see [12]).

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I. Definitions and Preliminaries

1. Symbols, Blocks and Sequences

Let I be a finite alphabet with r symbols, $r \geqq 2$. We usually suppose that $I=\{0, 1, \ldots, r-1\}$. Elements of

$$I^* = \bigcup_{k \geqq 1} I^k$$

are called *blocks* (over I). If

$$B = b_0 \ldots b_{n-1} \in I^n$$

is a block we call $N(B)=n$ the length of B and we denote by $N_i(B)$ the number of times the symbol $i \in I$ appears in B. It is sometimes convenient to write $B(k)$ for the symbol b_k. Elements of $I^{\mathbb{Z}}$ will be called *sequences.* If

$$x = \ldots x_{-1} x_0 x_1 \ldots$$

is a sequence, then $x[k, n]$ denotes the block $x_k x_{k+1} \ldots x_n$. If $(B_k)_{k \in \mathbb{Z}}$ is a sequence of blocks, then we can form in an obvious manner a sequence

$$x = \ldots B_{-1} \dot{B}_0 B_1 \ldots$$

where the dot over B_0 indicates that $x[0, N(B_0)-1] = B_0$.

2. The Space, the Topology and the Homeomorphism

Provide $I^{\mathbb{Z}}$ with the metric d defined by $d(x, x)=0$ and $d(x, y) = 1/(\min\{|k|: x_k \neq y_k\} + 1)$ if $x \neq y$. If B is a block, the set

$$[B] = \{x \in I^{\mathbb{Z}}: x[0, N(B)-1] = B\}$$

is called a *cylinder*. Let T be the shift homeomorphism on $I^{\mathbb{Z}}$; i.e., if $x \in I^{\mathbb{Z}}$ then $(Tx)_k = x_{k+1}$ for all $k \in \mathbb{Z}$. The collection of cylinders and their translates under T form an open and closed base for the topology induced by d. The *orbit* of x under T is the set

$$\mathrm{Orb}(x) = \mathrm{Orb}(x; T) = \{T^k x: k \in \mathbb{Z}\}.$$

3. Substitutions

A substitution θ is a map $\theta: I \to I^*$. Many of its properties can be obtained from the *θ-matrix*

$$L(\theta) = (l_{ij})_{i,j=0}^{r-1}$$

defined by $l_{ij} = N_j(\theta i)$. The *length* of a substitution is the vector $(l_0, \ldots, l_{r-1})$, where $l_i = \sum_{j \in I} l_{ij} = N(\theta i)$. If all l_i are equal then θ is of *constant length.*

The map θ extends to I^* and $I^{\mathbb{Z}}$ by defining

$$\theta B = \theta b_0 \theta b_1 \ldots \theta b_n \qquad \text{if } B = b_0 b_1 \ldots b_n \text{ is a block,}$$
$$\theta x = \ldots \theta(x_{-1})\, \dot{\theta}(x_0)\, \theta(x_1) \ldots \qquad \text{if } x \text{ is a sequence.}$$

In this way we can consider the substitution θ^n defined by $\theta^n i = \theta^{n-1}(\theta i)$ for each $n \geqq 1$, and we see that

$$L(\theta^n) = L(\theta)^n.$$

Throughout this paper we shall assume that $l_i^{(n)} = N(\theta^n i) \geqq 2$ for some $n \geqq 1$ and all $i \in I$. We call a substitution θ *primitive* if $L(\theta)$ is a primitive matrix (i.e., if $L(\theta)^n$ is strictly positive for some $n \geqq 1$).

4. *Flows*

A pair (X, T) with X a non-empty, compact metric space and T a homeomorphism will be called a *flow*. A non-empty, closed T-invariant subset of X is called *minimal* if it contains no proper closed T-invariant subsets. It is well known that X is minimal iff $\overline{\mathrm{Orb}}(x) = X$ for all $x \in X$ iff all $x \in X$ are almost periodic.

Recall that an element $x \in I^{\mathbb{Z}}$ is almost periodic if any block $B \in I^*$ which appears in x occurs with bounded gap.

5. *Flows Generated by Substitutions*

With any substitution θ we can find two symbols p and q and an $n \geqq 1$ such that the last symbol of $\theta^n p$ is equal to p and the first symbol of $\theta^n q$ is equal to q ([5, 3]). We call pq a *cyclic pair* for θ. Any cyclic pair generates a sequence $w = w^{pq}$ defined by

$$w[-N(\theta^{nk} p), N(\theta^{nk} q) - 1] = \theta^{nk} p\, \theta^{nk} q \qquad \text{for } k = 0, 1, 2, \ldots.$$

Among the cyclic pairs there is always at least one such that $w = w^{pq}$ is almost periodic ([5, 3]). In this case we call the minimal flow $(\overline{\mathrm{Orb}}(w), T)$ the flow generated by θ (and pq). It can be shown ([3]) that with no loss of generality (since we are only interested in minimal flows) we may assume θ to be primitive. If θ is primitive then all cyclic pairs pq (such that w^{pq} is almost periodic) generate the same flow, which we denote by $(X(\theta), T)$. We shall repeatedly use the fact that in this case $X(\theta) = X(\theta^n)$ for any $n \geqq 1$, so that we may replace θ by a power of θ without changing the flow.

The minimal flow $(X(\theta), T)$ will be called the *substitution flow* generated by θ (θ a primitive substitution), and we adjoin the words "*of constant length*" if θ has constant length.

Note that this flow may be finite since it can happen that the sequence generated by θ is periodic. In this case we call θ *periodic*. A simple characterization of periodic substitutions of constant length is given in 2.9 (iii).

6. Dynamical Systems and Spectra

A triple (X, T, μ) where (X, T) is a flow and μ a T-invariant probability measure is called a dynamical system. If μ is unique then (X, T, μ) is called *uniquely ergodic.*

Given two flows (X, T) and (Y, T'), a continuous map ϕ from X onto Y such that $\phi \circ T = T' \circ \phi$ is called a *homomorphism.* If ϕ is also one-to-one ϕ is called an *isomorphism.* If (X, T, μ) and (Y, T', μ') are uniquely ergodic dynamical systems then any homomorphism is measure preserving and therefore a measure-theoretic homomorphism.

The homeomorphism T induces a unitary operator in $L^2(X, \mu)$ by $f \to f \circ T$. The spectrum of this operator is an invariant for measure-theoretic isomorphism and is called the spectrum of (X, T, μ).

7. Dynamical Systems Generated by Substitutions

If θ is a primitive substitution, then the flow $(X(\theta), T)$ admits a unique T-invariant Borel probability measure μ ([11]). We call the triple $(X(\theta), T, \mu)$ a *substitution dynamical system,* and call the spectrum of $(X(\theta), T, \mu)$ the *spectrum* of θ.

II. Minimality and Ergodicity of Powers of the Shift (Constant Length)

In the first part of this section, we shall give some definitions and simple results for an arbitrary minimal flow (X, T).

Definition 1. A cyclic partition of X is a partition $(X_u)_{u=0}^{m-1}$ of X into disjoint subsets such that

$$X_{u+1} = TX_u \quad \text{for } 0 \leqq u < m-1 \quad \text{and} \quad TX_{m-1} = X_0.$$

Let $n \geqq 1$. A *T^n-invariant partition* of X is a partition of X whose elements are all closed and T^n-invariant. A *T^n-minimal partition* of X is a partition of X whose elements are all T^n-minimal.

Lemma 2 ([7, L. 15]). *Let (X, T) be a minimal flow and n a positive integer. There exists a cyclic T^n-minimal partition. This partition is unique up to cyclic permutations of its members.*

In the sequel the number of elements of a cyclic T^n-minimal partition will be denoted by $\gamma(n)$ for each $n \geqq 1$. The equivalence relation whose classes are the members of the cyclic T^n-minimal partition will be denoted by Λ_n.

Lemma 3. *The function $\gamma(\cdot)$ and the relation Λ_n have the following properties*

(i) $1 \leqq \gamma(n) \leqq n$ *and* $\gamma(n)$ *divides* n.

(ii) $\Lambda_{\gamma(n)} = \Lambda_n$ *and thus* $\gamma(\gamma(n)) = \gamma(n)$.

(iii) *If* m *divides* n *then* $\Lambda_m \supset \Lambda_n$; *moreover if* $\gamma(n) = n$ *then* $\gamma(m) = m$.

(iv) *If* $(m, n) = 1$ *then* $\Lambda_{mn} = \Lambda_m \cap \Lambda_n$ *and* $\gamma(mn) = \gamma(m)\gamma(n)$.

(v) *If* $\gamma(n) > 1$ *then there is an* $m > 1$ *dividing* n *with* $\gamma(m) = m$.

(vi) *If* $\gamma(n) < n$ *then* $\lim_{k \to \infty} \dfrac{\gamma(n^k)}{n^k} = 0$.

Proof. Cf. [7, p. 296] for proofs of (i)–(iv). To prove (v) take $m=\gamma(n)$ and use (i) and (ii). We shall prove (vi). If $\gamma(n)<n$ then we can find a prime p and an $a>0$ (by (iv)) such that $n=p^a s$, $(p,s)=1$ and $\gamma(p^a)<p^a$. We claim that $\gamma(p^{ka})<p^a$ for all $k\geqq 1$. Indeed, $\gamma(p^{ka})=p^b$ for some $b\geqq 0$ (by (i)), $\gamma(p^b)=\gamma(\gamma(p^{ka})=p^b$ (by (ii)) and therefore $b<a$ (by (iii)). The multiplicativity of γ yields now

$$\frac{\gamma(n^k)}{n^k}=\frac{\gamma(p^{ka})\,\gamma(s^k)}{p^{ka}\,s^k}<\frac{1}{p^{(k-1)a}}\to 0 \quad \text{if } k\to\infty.$$

Definition 4. ([5]) The *trace relation* Λ of (X,T) is defined by $\Lambda=\bigcap_{n\geqq 1}\Lambda_n$. If we want to emphasize the dependence on T we write $\Lambda=\Lambda^T$.

Lemma 5. *Let* (X,T) *be a minimal flow. Then* $\Lambda=\bigcap_{n:\,n=\gamma(n)}\Lambda_n$.

Proof. By 3(ii), $\Lambda=\bigcap_{n\geqq 1}\Lambda_n=\bigcap_{n\geqq 1}\Lambda_{\gamma(n)}=\bigcap_{n:\,n=\gamma(n)}\Lambda_n$.

We shall now turn to substitution flows $(X(\theta),T)$ generated by a substitution θ of constant length l. Our aim is to determine Λ for any such flow.

In the course of the proof of Lemma 7 we shall need the following combinatorial lemma.

Lemma 6. *Let* $x\in I^{\mathbb{Z}}$ *and* $J(n)=\{B\in I^*: N(B)=n,\ B$ *appears in* $x\}$ *for each* $n\geqq 1$. *If for some* $n\geqq 1$

$$\operatorname{Card}(J(n))\leqq n$$

then x *is periodic.*

Proof. [1, Th. 2.06, 2.11].

Lemma 7. *Let* $(X(\theta),T)$ *be a substitution flow of constant length* l. *Then either* $\gamma(l^n)=l^n$ *for all* $n\geqq 1$ *or* θ *is periodic.*

Proof. Let θ be a substitution of constant length l on r symbols. Let w be such that $X(\theta)=\overline{\mathrm{Orb}}(w)$. We shall first establish that for all $n\geqq 0$ there appear at most $r+r^2(\gamma(l^n)-1)$ different blocks of length l^n in w.

Let $X_0=\theta^n X$. Then X_0 is a T^{l^n}-minimal set (the mapping $\theta^n:(X,T)\to(X_0,T^{l^n})$ is a homomorphism). Since $\theta w=w$ we have $w\in X_0$. Therefore

$$T^{k\gamma(l^n)}w\in X_0\subset\bigcup_{i\in I}[\theta^n i] \qquad \text{for all } k\in\mathbb{Z}.$$

Thus w is composed of overlapping blocks of the form $\theta^n i$ (of length l^n) spaced at intervals $\gamma(l^n)$. Since there are r blocks $\theta^n i$ and at most r^2 blocks $\theta^n i\theta^n j$ we obtained the desired result.

Let us now suppose that $\gamma(l^n)<l^n$ for some $n\geqq 1$. Since $X(\theta^n)=X(\theta)$ we may assume $n=1$. By Lemma 3(vi), $\lim_{n\to\infty}\gamma(l^n)/l^n=0$. We can therefore find an n such that $r+r^2(\gamma(l^n)-1)<l^n$. But then there are fewer than l^n blocks of length l^n in w, and w is periodic by Lemma 6.

We shall now search for integers n relatively prime to l such that $\gamma(n)=n$.

Definition 8. (Cf. [9, 4.06]) Let $(X(\theta), T)$ be a substitution flow of constant length l and let w be such that $\overline{\text{Orb}}(w)=X(\theta)$. The number

$$h(\theta)=\max\{n\geqq 1\colon (n, l)=1,\ n \text{ divides } \gcd\{a\colon w_a=w_0\}\}$$

will be called the *height* of θ(or $X(\theta)$).

Remark 9. Let θ be a primitive substitution of constant length on r symbols. Then

(i) $1\leqq h(\theta)\leqq r$.

Let k be an integer, let $S_k=\{a\colon w_{a+k}=w_k\}$ and $g_k=\gcd S_k$. The upper bound on $h(\theta)$ follows directly from

$$\{n\geqq 1\colon (n, l)=1,\ n \text{ divides } g_0\}=\{n\geqq 1\colon (n, l)=1,\ n \text{ divides } g_k\}.$$

We shall show that the set on the right contains the set on the left. (The other inclusion is proved similarly.)

Let $g_k=\gcd\{a_1, \dots, a_m\}$ where $a_1\in S_k, \dots, a_m\in S_k$. Choose N such that the symbol w_0 appears at some place, say p, in $\theta^N w_k$. It follows from $\theta^N w=w$ that

$$k l^N+p\in S_0 \quad \text{and} \quad (k+a_s) l^N+p\in S_0 \qquad \text{for } s=1, \dots, m.$$

Hence

$$a_s l^N\in S_0-S_0 \qquad (s=1, \dots, m)$$

and

$$\gcd(S_0-S_0) \mid \gcd\{a_1 l^N, \dots, a_m l^N\}=l^N g_k.$$

Therefore if $(n, l)=1$ and n divides g_0 then n divides g_k. (For any set A of integers, $\gcd A$ divides $\gcd(A-A)$).

(ii) We shall describe an algorithm to calculate $h(\theta)$.

Apply the following labeling procedure for those $n=r, r-1, \dots, 1$ such that $(n, l)=1$.

Let $w_0=q$. Label $\theta q(m)$ with the number m modulo n for $m=0, \dots, l-1$. If for some i the symbol i appears at more than one place in θq and has obtained different labels then $n\neq h(\theta)$. Otherwise let L_i be this unique label for each i appearing in θq. (For example: $L_q=0$.) Label $\theta i(m)$ with the number lL_i+m modulo n. If for some j the symbol j appears at more than one place in θi or θq and has obtained different labels then $n\neq h(\theta)$. Now continue in this manner. If at some step, a symbol j has obtained different labels, then $n\neq h(\theta)$. On the other hand, if we can continue until $\theta i(m)$ is labeled consistently for all i and m, then $n=h(\theta)$.

(iii) It is easily seen that $h(\theta)=r$ implies that θ is periodic. Combining this observation with [7, L.5] we obtain that if θ is one-to-one, then θ is periodic iff $h(\theta)=r$.

If θ is not one-to-one we associate with θ a substitution η that is one-to-one by identifying i and j iff there is a positive integer k such that $\theta^k i=\theta^k j$. Then θ is periodic iff η is periodic.

Example. The substitution defined by $0\to 010$, $1\to 201$, $2\to 102$ has height 2.

Lemma 10. $h(\theta)=\max\{n\geqq 1\colon (n,l)=1$ *and* $\gamma(n)=n\}$.

Proof. Let $X=X(\theta)=\overline{\mathrm{Orb}}(w)$ and $d=\gcd\{a\colon w_a=w_0\}$.

1. If n divides d then $\gamma(n)=n$.

Let $Y=\bigcup_{m\in\mathbb{Z}} T^{md}[w_0]\cap X$. Then $T^d Y=Y$ and $T^u Y\cap T^v Y=\emptyset$ if $0\leqq u<v<d$ by the definition of d. Since $\bigcup_{u=0}^{d-1} T^u Y=X$, Y is closed. Hence $(T^u Y)_{u=0}^{d-1}$ is a cyclic T^d-invariant partition. Since it has maximal cardinality it is a cyclic T^d-minimal partition, so $\gamma(d)=d$. Therefore $\gamma(n)=n$ if n divides d (Lemma 3 (iii)).

2. If $\gamma(n)=n$ and $(n,l)=1$ then n divides d.

Let $\lambda=\exp(2\pi i/n)$. Then λ is an eigenvalue corresponding to a continuous eigenfunction – we call such a λ a *continuous eigenvalue* for short – $\left(f=\sum_{t=0}^{n-1} 1_{T^t X_0},\right.$ X_0 a T^n-minimal set$\left.\right)$. By Lemma 11 (ii) n divides any a such that $w_a=w_0$. Hence n divides $\gcd\{a\colon w_a=w_0\}=d$.

In the proof of the preceding lemma we needed the second part of the following lemma.

Lemma 11. *Let $(X(\theta),T)$ be a substitution flow of constant length l. Then*

(i) *$(X(\theta),T)$ has no continuous irrational eigenvalues.*

(ii) *If for some positive n with $(n,l)=1$ $\exp(2\pi i/n)$ is a continuous eigenvalue and if a is an integer such that $w_a=w_0$ then n divides a.*

Proof. (Cf. [9, 4.08].) (i) Take any continuous $f\neq 0$ such that $f(Tx)=\exp(2\pi i\alpha)f(x)$ for all $x\in X(\theta)$.

Fix an $a\neq 0$ such that $w_{a-1}w_a=w_{-1}w_0$. Since $\theta^k w=w$ for all $k\geqq 1$ we have $\lim_{k\to\infty} T^{a\cdot l^k}w=w$ and therefore

$$f(w)=\lim_{k\to\infty} f(T^{a\cdot l^k}w)=\lim_{k\to\infty}\exp(2\pi i a l^k\alpha)f(w).$$

Therefore $a l^k\alpha=0 \pmod 1$ for k large enough. Hence α is rational.

(ii) We proceed as in (i), but now we can only conclude that $\lim_{k'\to\infty} T^{a l^{k'}}w=w$ for some subsequence (k') of the integers and an x such that $x_m=w_m$ for all $m\geqq 0$. The latter implies that $f(x)=f(w)$, so as in (i) we have $a l^{k'}/n=0 \pmod 1$ for k' large enough. Since $(n,l^{k'})=1$ n has to divide a.

Theorem 12. *Let $(X(\theta),T)$ be a substitution flow, where θ is a non-periodic substitution of constant length l and height $h=h(\theta)$. Let Λ be the trace relation of $(X(\theta),T)$. Then*

$$\Lambda=\bigcap_{n\geqq 1}\Lambda_{l^n}\cap\Lambda_h.$$

Proof. According to Lemma 5 $\Lambda=\bigcap_{n\colon n=\gamma(n)}\Lambda_n$ and thus by Lemma 7 and 10 $\Lambda=\bigcap_{n\geqq 1}\Lambda_{l^n}\cap\Lambda_h$. Let n be any other integer such that $\gamma(n)=n$. Decompose $n=ms$ with $(m,l)=1$ and s divides l^k for some $k\geqq 1$. By Lemma 3(iv) $\Lambda_n=\Lambda_m\cap\Lambda_s$. Since s

divides l^k we have by Lemma 3 (iii) that $\Lambda_s \supset \Lambda_{l^k}$. We finish the proof by showing that m divides $h = h(\theta)$ and hence $\Lambda_m \supset \Lambda_h$.

Let m' be any factor of m such that $(m', h) = 1$. Then $\gamma(m'h) = \gamma(m')\gamma(h) = m'\gamma(h)$ (by Lemma 3 (iv) and (iii)). Hence by Lemma 10 $m' = 1$. So m and h have the same prime factors. But a similar argument shows that any prime factor of m cannot appear with a higher exponent in m than in h. Therefore m divides h.

Let $\mathbb{Z}(l)$ be the topological group of l-adic numbers and let τ be the homeomorphism of $\mathbb{Z}(l)$ corresponding to addition of the unit element. Then $(\mathbb{Z}(l), \tau)$ is a minimal flow. Similarly we define the minimal flow $(\mathbb{Z}_n, \tau_n)$ where $\mathbb{Z}_n$ is the cyclic group of order n. If $(n, l) = 1$ the product flow $(\mathbb{Z}(l) \times \mathbb{Z}_n, \tau \times \tau_n)$ is a minimal flow.

Theorem 13. *Let $(X(\theta), T)$ be a substitution flow, where θ is a non-periodic substitution of constant length l. Then*

$$(X/\Lambda, T_\Lambda) \simeq (\mathbb{Z}(l) \times \mathbb{Z}_{h(\theta)}, \tau \times \tau_{h(\theta)}).$$

Proof. Theorem 13 is an immediate consequence of Theorem 12, Lemma 7 and Lemma 10.

Remark. Let Σ be the least closed invariant equivalence relation such that $(X/\Sigma, T_\Sigma)$ is equicontinuous ([4]). This flow is called the structure system of (X, T). For any minimal flow, $\Sigma = \Lambda$ iff all continuous eigenvalues of T are rational. It follows therefore from Lemma 11 (i) and Theorem 13 that the structure system of a non-periodic substitution of constant length l is $(\mathbb{Z}(l) \times \mathbb{Z}_{h(\theta)}, \tau \times \tau_{h(\theta)})$. This result has been obtained in [9, Theorem 5.09] with the restriction that θ be one-to-one.

We shall now dwell for a moment on the opposite case $\gamma(n) = 1$.

Theorem 14. *Let $(X(\theta), T)$ be a substitution flow of constant length, and let $n \geqq 1$. If $X(\theta)$ is T^n-minimal then there exists a substitution flow $(X(\eta), \hat{T})$ such that*

$$(X(\theta), T^n) \simeq (X(\eta), \hat{T}).$$

Proof. Let θ be a substitution of constant length l, w such that $\theta w = w$ and $X(\theta) = \overline{\text{Orb}}(w)$. Let J be the collection of all blocks of length n appearing in w. Let ϕ: $\hat{I} \to J$ be a bijection between a finite set $\hat{I}$ and J. We extend ϕ to $\hat{I}^*$ and $\hat{I}^{\mathbb{Z}}$ by defining

$$\phi(\hat{i}_0 \dots \hat{i}_k) = \phi(\hat{i}_0) \dots \phi(\hat{i}_k) \quad \text{for } \hat{i}_0 \dots \hat{i}_k \in \hat{I}^*,$$

$$\phi(y) = \dots \phi(y_{-1})\, \dot{\phi}(y_0)\, \phi(y_1) \dots \quad \text{for } y \in \hat{I}^{\mathbb{Z}}.$$

Define a substitution η: $\hat{I} \to \hat{I}^l$ by

$$\eta(\hat{i}) = \phi^{-1}\theta(\phi(\hat{i})) \quad \text{for all } \hat{i} \in \hat{I}.$$

Let $\hat{p} = \phi^{-1}(w_{-n} \dots w_{-1})$, $\hat{q} = \phi^{-1}(w_0 \dots w_{n-1})$. Then

$$\eta(\hat{q}) = \phi^{-1}(\theta(w_0 \dots w_{n-1})) = \phi^{-1}(w_0 \dots w_{nl-1}) = \hat{q} \dots.$$

Analogously $\eta(\hat{p})$ ends with $\hat{p}$. Hence $\hat{p}\hat{q}$ is a cyclic pair. Let $\hat{w}$ be the sequence generated by $\hat{p}\hat{q}$ under η. Then $\hat{w} = \phi^{-1}(w)$ so that $\hat{w}$ is almost periodic since w is T^n-

almost periodic. (Note that w is T^n-almost periodic – by the T-almost periodicity of w – whether T^n is minimal or not.) Since $\hat{w}$ is almost periodic and contains all symbols of $\hat{I}$ (by the T^n-almost periodicity of w), η is a primitive substitution. Let $(X(\eta), \hat{T})$ be the substitition flow generated by η, where $\hat{T}$ denotes the shift on $\hat{I}^{\mathbb{Z}}$. The map ϕ: $X(\eta) \to \phi(X(\eta))$ is bijective, open and satisfies $\phi\hat{T} = T^n\phi$. Since both $\phi(X(\eta))$ and $X(\theta)$ are T^n-minimal sets with at least w in their intersection ϕ is an isomorphism ϕ: $(X(\eta), \hat{T}) \to (X(\theta), T^n)$.

Example. Let θ be defined by $0 \to 011$, $1 \to 101$. Then $l=3$ and $h(\theta)=1$, so by Lemma 10 T^2 is minimal. $J=\{01, 11, 10\}$ and if we take $\hat{I}=\{a, b, c\}$, ϕ with $\phi(a)=01$, $\phi(b)=11$, $\phi(c)=10$ then η is defined by

$$a \to aba, \quad b \to cba, \quad c \to ccb.$$

Theorem 15. *Let $(X(\theta), T, \mu)$ be a substitution system of constant length and let $n \geqq 1$. Then T^n is minimal iff T^n is ergodic (w.r.t. μ).*

Proof. Since T is minimal and the results at the beginning of this section apply ergodicity of T^n implies minimality of T^n. Let T^n be minimal. Apply Theorem 14 to obtain a (measure) isomorphism ϕ: $(X(\eta), \hat{T}, \hat{\mu}) \to (X(\theta), T^n, \phi\hat{\mu})$. Since $\hat{\mu}$ is uniquely ergodic so is $\phi\hat{\mu}$. Hence $\phi\hat{\mu}=\mu$ by the T^n-invariance of μ.

In the last part of this section we take a closer look at those substitutions with a height greater than 1.

Definition 16. Let (X, T) be an arbitrary flow and $n \geqq 1$. By the *stack* of *height n* over (X, T) we mean the flow $(X \times \mathbb{Z}_n, \sigma)$ where σ is defined by $\sigma(x, k)=(x, k+1)$ if $0 \leqq k < n-1$ and $\sigma(x, n-1)=(Tx, 0)$. The flow (X, T) is called the *base* of $(X \times \mathbb{Z}_n, \sigma)$.

Lemma 17. *Let $(X(\theta), T)$ be a substitution flow of constant length and $n \geqq 1$ such that $\gamma(n)=n$. Let X_0 be a T^n-minimal subset of $X(\theta)$. Then there exists a finite set $\hat{I}$ and a primitive substitution η of length l on $\hat{I}$ such that $(X_0, T^n) \simeq (X(\eta), \hat{T})$ and therefore $(X(\theta), T) \simeq (X(\eta) \times \mathbb{Z}_n, \sigma)$.*

Proof. Let $X(\theta)=\overline{\mathrm{Orb}}(w)$. We shall prove the lemma for the T^n-minimal set X_0 which contains w. Copy the proof of Theorem 14 with J being the collection of all blocks of length n appearing at places kn ($k \in \mathbb{Z}$) in w. This yields an isomorphism: ϕ: $(X(\eta), \hat{T}) \to (X_0, T^n)$. ($\phi$ and η are defined as in the proof of Theorem 14.) Let σ be the transformation of the stack with height n and base $(X(\eta), \hat{T})$. Define ψ: $X(\theta) \to X(\eta) \times \mathbb{Z}_n$ by $\psi(x)=(\phi^{-1}(T^{-k}x), k)$ if $x \in T^k X_0$. Then it is easily shown that ψ is an isomorphism between $(X(\theta), T)$ and $(X(\eta) \times \mathbb{Z}_n, \sigma)$.

Note that the substitution η is "essentially" unique.

Example. Let θ be a non-periodic substitution of constant length $n \geqq 1$. Then $\gamma(l^n)=l^n$ by Lemma 7. We see that we can take $\hat{I}=I$ and $\eta=\theta$. Hence

$$(X(\theta), T) \simeq (X(\theta) \times \mathbb{Z}_{l^n}, \sigma).$$

Definition 18. A substitution θ is *pure* if $h(\theta)=1$.

Lemma 19. *Let $(X(\theta), T)$ (where $X(\theta)=\overline{\mathrm{Orb}}(w)$) be a substitution flow of constant length with $h=h(\theta)>1$. Let $X_0=\overline{\mathrm{Orb}}(w;\, T^h)$ and let η be the substitution (given by*

Lemma 17) *such that* $(X_0, T^h) \simeq (X(\eta), \hat{T})$. *Then*

(i) η *is pure*,

(ii) $\Lambda^{T^h} = \Lambda^T$ *on* $X_0 \times X_0$.

Proof. (i) According to Lemma 17 there exist a substitution η and an isomorphism ϕ between $(X(\eta), \hat{T})$ and (X_0, T^h). Let $Y_0 \subset X(\eta)$ and $n \geqq 1$ be such that $(n, l) = 1$ and such that $(\hat{T}^u Y_0)_{u=0}^{n-1}$ is a cyclic T^n-minimal partition of $X(\eta)$. Then $(T^{u+v}(\phi Y_0))_{u=0}^{n-1}{}_{v=0}^{h-1}$ is a cyclic T^{nh}-minimal partition of $X(\theta)$, so by Lemma 10 $n = 1$. Hence by the same lemma η is pure.

(ii) By (i) and Theorem 12 $\Lambda^{\hat{T}} = \bigcap_{n \geqq 1} \Lambda^{\hat{T}}_{l^n}$ and therefore

$$\Lambda^{T^h} = \bigcap_{n \geqq 1} \Lambda^{T^h}_{l^n} = \bigcap_{n \geqq 1} \Lambda^T_{h\,l^n} = \bigcap_{n \geqq 1} \Lambda^T_{l^n} \cap \Lambda^T_h = \Lambda^T$$

on $X_0 \times X_0$, using Lemma 3 (iv) and that X_0 is an equivalence class of Λ^T_h.

Definition 20. Let $(X(\theta), T)$ be a substitution flow of constant length with $h(\theta) > 1$. Then we call the substitution η (substitution flow $(X(\eta), \hat{T})$) given by Lemma 17 the *pure base* of θ(of $(X(\theta), T)$). If $h(\theta) = 1$ then the pure base of θ is equal to θ.

Example. The substitution flow generated by $0 \to 010$, $1 \to 102$, $2 \to 201$ is (isomorphic to) a stack of height 2 with a pure base generated by

$$a \to aab, \qquad b \to aba.$$

III. The Spectrum of Substitutions of Constant Length

Let θ be a substitution of length l on r symbols. For each $n \geqq 1$ and k with $0 \leqq k < l^n$ we call the set $\{\theta^n 0(k), \theta^n 1(k), \ldots, \theta^n (r-1)(k)\}$ a *column* of the substitution θ. In this section we shall show that the nature of the spectrum of θ is determined by the presence (respectively absence) of a column consisting of one symbol in its pure base.

Definition 1. Let θ be a primitive substitution of length l on r symbols. If θ is pure we define the column number $c(\theta)$ of θ by

$$c(\theta) = \min_{n \geqq 1} \min_{0 \leqq k < l^n} \operatorname{card}\{\theta^n 0(k), \theta^n 1(k), \ldots, \theta^n (r-1)(k)\}.$$

If θ is not pure its column number is defined as the column number of its pure base.

Remark 2. That $c(\theta)$ is computable follows from the fact that

$$c(\theta) = \min \operatorname{card}\{\theta^{2^r - r - 1} 0(k), \ldots, \theta^{2^r - r - 1}(r-1)(k) \colon 0 \leqq k < l^{2^r - r - 1}\}.$$

This is implied by the following observations:

(i) If $\{i_1, \ldots, i_c\}$ is a column of cardinality c then

$$\operatorname{card}\{\theta i_1(k), \ldots, \theta i_c(k)\} \leqq c \qquad \text{for } k = 0, \ldots, l-1.$$

(ii) A substitution has at most $2^r - r - 1$ columns with a cardinality larger than 1.

Examples. (i) The substitution θ defined by

$$0\to 04,\quad 1\to 01,\quad 2\to 34,\quad 3\to 31,\quad 4\to 42$$

is primitive and pure and has only 2 different columns: $\{0, 3, 4\}$ and $\{1, 2, 4\}$. Hence $c(\theta)=3$.

(ii) Let θ be the "circulant" substitution defined by

$$i\to i(i+1)\dots(i-2)(i-1) \quad \text{for } i=0,1,\dots,r-1.$$

(The symbols in this definition are to be considered modulo r.) Then θ is primitive ($L(\theta)>0$) and pure. The only column appearing is I, so $c(\theta)=r$.

Now let $s\geqq 1$ and η a substitution on $r+s-1$ symbols defined by

$$\eta(i)=i^s\theta(i) \qquad \text{if } 0\leqq i\leqq r-2$$
$$\eta(i)=(r+s-1)\dots(r+1)\,r\,\theta(r-1) \qquad \text{if } r-1\leqq i\leqq r+s-1.$$

Then η is primitive ($L(\eta^2)>0$) and pure. Any column of η contains all symbols i with $0\leqq i\leqq r-2$ plus one of the symbols $r-1, r, \dots, r+s-1$ hence $c(\eta)=r$. (This example provides a correct proof of Theorem 6 in [7]).

Theorem 3. *Let $(X(\theta), T)$ be a substitution flow of constant length, Λ its trace relation and $x\Lambda$ the equivalence class containing $x\in X(\theta)$. Then*

i) $\min_{x\in X(\theta)} \operatorname{card}(x\Lambda)=c(\theta)$.

ii) *If* $y, z\in x\Lambda$, $y\neq z$ *and* $\operatorname{card}(x\Lambda)=c(\theta)$ *then* $d(y,z)=1$.

Proof [7, Theorem 5]. Since however our definition of $c(\theta)$ is slightly different in case θ is not pure we have to show that

$$(*)\quad \min_{x\in X(\theta)} \operatorname{card}(x\Lambda^T)=\min_{y\in X(\eta)} \operatorname{card}(y\Lambda^{\hat{T}})$$

if $(X(\eta), \hat{T})$ is the pure base of $(X(\theta), T)$.

Let X_0 be a T^h-minimal subset of $X(\theta)$. Then (1.19) (X_0, T^h) is isomorphic to $(X(\theta), \hat{T})$, therefore

$$\min_{y\in X(\eta)} \operatorname{card}(y\Lambda^{\hat{T}})=\min_{x\in X_0} \operatorname{card}(x\Lambda^{T^h}).$$

Now $(*)$ is implied by 1.19 (ii) and the T-invariance of Λ.

Theorem 4. *Let $(X(\theta), T, \mu)$ be a substitution dynamical system of constant length. Then*

$$\mu\{x\in X\colon \operatorname{card}(x\Lambda)=c(\theta)\}=1.$$

Proof. This theorem is a generalisation of a result obtained in the proof of Theorem 7 in [7] and can be proved analogously.

Let $(X(\theta), T, \mu)$ be a substitution dynamical system of constant length. We write $X=X(\theta)$. Let $L^2(X)=L^2(X, T, \mu)$ be the Hilbert space of complex-valued square integrable functions on X with inner product $\langle f, g\rangle=\int f\bar{g}\,d\mu$. Let Λ be the trace relation of (X, T), π: $X \to X/\Lambda$ the projection homomorphism and $L^2(X/\Lambda) = L^2(X/\Lambda, T_\Lambda, \pi\mu)$. Let $D = \{f \in L^2(X): f = g \circ \pi \text{ a.e. } g \in L^2(X/\Lambda)\}$. It is not difficult to see that

$$D = \{f \in L^2(X): f \text{ constant on } x\Lambda \text{ for a.e. } x \in X\}$$

For any $f \in L^2(X)$ we define a function Ef on X by

$$Ef(x) = \frac{1}{c(\theta)} \sum_{y \in x\Lambda} f(y), \qquad x \in X.$$

It will appear that Ef is a version of the conditional expectation of f w.r.t. Λ (i.e. with respect to the σ-algebra generated by $x\Lambda$, $x \in X$).

Theorem 5. (i) *D is a closed T-invariant linear subspace of $L^2(X)$.*
(ii) *E is the projection on D.*
(iii) *If $f \in D^\perp, f^2 \in D^\perp, \ldots, f^{c(\theta)} \in D^\perp$ then $f = 0$ a.e.*

Proof. By the definition of D (*i*) is obviously true. Let $c = c(\theta)$, $Z = \{z \in X: \text{card}(z\Lambda) \neq c\}$. Then $\mu(Z) = 0$ by Theorem 4.

1. If f is continuous then Ef is continuous on $X \setminus Z$.

Let $x^{(n)}$, $x \in X \setminus Z$ be such that $x^{(n)} \to x$ if $n \to \infty$. It suffices to show that $Ef(x^{(n')}) \to Ef(x)$ for a subsequence (n'), $n' \to \infty$.

Let $y^{(n,1)} = x^{(n)}$, $y^{(1)} = x$. By Theorem 3 (ii) we can choose $y^{(n,2)} \in x^{(n)}\Lambda$ such that $d(y^{(n,2)}, y^{(n,1)}) = 1$. Let $(y^{(n',2)})$ be a subsequence of $(y^{(n,2)})$ such that $y^{(n',2)} \to y^{(2)}$, $y^{(2)} \in X$. Then $y^{(2)} \in x\Lambda$ and $d(y^{(2)}, y^{(1)}) = 1$. Continuing in this way we find exactly c sequences $(y^{(n',m)})_{n'=1}^\infty$ (where we denote any subsequence of (n') again by (n')) and c points $y^{(m)} \in x\Lambda$ such that $y^{(n',m)} \in x^{(n')}\Lambda$, $y^{(n',m)} \to y^{(m)}$ for $m = 1, 2, \ldots, c$. Therefore

$$\begin{aligned} |Ef(x^{(n')}) - Ef(x)| &= \frac{1}{c} \left| \sum_{y \in x^{(n')}\Lambda} f(y) - \sum_{y \in x\Lambda} f(y') \right| \\ &= \frac{1}{c} \left| \sum_{m=1}^{c} f(y^{(n',m)}) - \sum_{m=1}^{c} f(y^{(m)}) \right| \\ &\leqq \frac{1}{c} \sum_{m=1}^{c} |f(y^{(n',m)}) - f(y^{(m)})|. \end{aligned}$$

The proof of 1. is finished since $f(y^{(n',m)}) \to f(y^{(m)})$ if $n' \to \infty$ by the continuity of f.

Since the continuous functions are dense in $L^2(X)$ and since $f_n \to f$ pointwise clearly implies $Ef_n \to Ef$ pointwise, we deduce from 1. that Ef is measurable and integrable.

2. $ET = TE$.

The relation $ET = TE$ is implied by the T-invariance of Λ.

3. $\int Ef\,d\mu = \int f\,d\mu$ for all $f \in L^2(X)$.

Let μ_0 be defined by $\mu_0(f) = \int Ef\,d\mu$. Then $\mu_0(1) = 1$ and μ_0 is T-invariant by 2. The unique ergodicity of μ implies $\mu_0 = \mu$.

4. If $f \in L^2(X)$, $g \in D$ then $E(fg) = gE(f)$.

Indeed $E(fg)(x) = \frac{1}{c} \sum_{y \in x\Lambda} f(y)\,g(y) = g(x)\,Ef(x)$ for a.e. $x \in X$.

5. $E^2 = E$ and E is hermitian.

Taking 1 and Ef in 4. we obtain $E^2 = E$. Taking $g = \overline{Ef}$ in 4. and applying 3. we obtain $\langle f, Ef \rangle = \|Ef\|^2$.

6. E is the projection on D.

By 5. E is a projection. Since range$(E) \subset D \subset \{f\colon Ef = f\}$ E is the projection on D.

We have yet to prove (iii).

Let $f \in L^2(X)$ such that $f \in D^\perp$, $f^2 \in D^\perp, \ldots, f^c \in D^\perp$. Then by 6.

$$\sum_{y \in xA} f^k(y) = 0 \quad \text{for } k = 1, 2, \ldots, c \quad \text{and} \quad x \in X \setminus Z.$$

By Lemma 6 this implies that $f(y) = 0$ if $y \in xA$. Hence $f = 0$ on $X \setminus Z$.

Lemma 6. *Let $z_1, \ldots, z_c$ be complex numbers such that*

$$\sum_{m=1}^{c} z_m^k = 0 \quad \textit{for } k = 1, 2, \ldots, c.$$

Then $z_m = 0$ for $m = 1, 2, \ldots, c$.

Proof. Consider the z_m as indeterminates. Like the elementary symmetrical functions the functions $\sum_{m=1}^{c} z_m^k$ $(k = 1, \ldots, c)$ generate all symmetrical functions ([14, p. 81]). This immediately implies the lemma.

Theorem 7. *Let $(X(\theta), T, \mu)$ be a substitution dynamical system of constant length. Then $(X(\theta), T, \mu)$ has discrete spectrum if $c(\theta) = 1$ and partly continuous spectrum if $c(\theta) > 1$.*

Proof. By Theorem 2.13 $(X/A, T_A, \mu)$ is isomorphic to a rotation on a compact topological group. Hence $L^2(X/A)$ is spanned by the continuous eigenfunctions of T_A. This implies that the subspace D is spanned by the continuous eigenfunctions of T.

If $c(\theta) = 1$, then $D = L^2(X)$ by Theorem 4 and $(X(\theta), T, \mu)$ has discrete spectrum.

Let $c = c(\theta) > 1$. We shall first suppose θ pure, i.e. $A = \bigcap_{n \geq 1} A_{l^n}$. In this case Theorem 2.13 yields that D is spanned by eigenfunctions with (rational) eigenvalue group $\{e^{2\pi i a/l^n}\colon n \geq 0, 0 \leq a < l^n\}$, where l is the length of θ. Let $Tf = \lambda f$ with $|\lambda| = 1$ and $f \in L^2(X)$, $f \neq 0$.

We shall show that $f \in D$, i.e. T has continuous spectrum on $D^\perp$. It follows from Theorem 5 (iii) (and the orthogonality of eigenfunctions) that at least one of $f, f^2, \ldots, f^c$ belongs to D. Therefore λ is necessarily rational, say $\lambda = \exp\left(2\pi i \frac{p}{q}\right)$, with $(p, q) = 1$. Decompose $q = q_1 q_2$ where $(q_1, l) = 1$ and q_2 divides l^n for an $n \geq 1$. If $q_1 = 1$ then $f \in D$ by the unicity of eigenfunctions. Therefore, suppose $q_1 > 1$. Now

$$T f^{q_2} = \lambda^{q_2} f^{q_2} = \exp\left(2\pi i \frac{p}{q_1}\right) f^{q_2},$$

so that our knowledge of the eigenvalue group on D enables us to conclude that $f^{q_2} \in D^{\perp}$. Hence f^{q_2} is not constant. But

$$T^{q_1} f^{q_2} = f^{q_2},$$

so T^{q_1} is not ergodic, and therefore not minimal by Theorem 2.15. Hence $\gamma(q_1) > 1$ (see Lemma 2.2). By Lemma 2.3 (v) there is an $m > 1$ dividing q_1 with $\gamma(m) = m$. Since $(m, l) = (q_1, l) = 1$ this is a contradiction to the purity of θ (by Lemma 2.10.).

We shall now consider the case $h = h(\theta) > 1$. Let X_0 be a T^h-minimal set, $(X(\eta), \hat{T})$ the pure base of $(X(\theta), T)$ i.e. there is an isomorphism

$$\phi: (X_0, T^h, \mu_0) \to (X(\eta), \hat{T}, \hat{\mu}), \qquad \text{where } \mu_0 = h \cdot \mu|_{X_0}.$$

Let $Tf = \lambda f$ with $|\lambda| = 1$ and $f \in L^2(X)$, $f \neq 0$. Let $f_0 = f|_{X_0}$. Then $T^h f_0 = \lambda^h f_0$, so ϕf_0 is an eigenfunction of $(X(\eta), \hat{T}, \hat{\mu})$. Since η is pure, ϕf_0 is constant on $y\Lambda^{\hat{T}}$ for $\hat{\mu}$-almost all $y \in X(\eta)$. Hence f_0 is constant on $x\Lambda^{T^h} = x\Lambda^T$ (by Lemma 2.19(ii)) for μ_0-almost all $x \in X_0$. This implies that f is constant on $x\Lambda^T$ for μ-almost all $x \in X$, i.e. $f \in D$.

Remark. By [5, 1.10] Card $x\Lambda = 1$ iff x is a regularly almost periodic point. In this context Theorem 7 can be rephrased as: $(X(\theta), T, \mu)$ has discrete spectrum iff $X(\theta)$ contains a regularly almost periodic point.

IV. Minimality and Ergodicity (Non-Constant Length)

The aim of this section is to prove the following theorem which has been proved in the case of a substitution of constant length (2.15).

Theorem 1. *Let $(X(\theta), T, \mu)$ be a substitution dynamical system and $n \geqq 1$. Then T^n is minimal iff T^n is ergodic (w.r.t. μ).*

The following example shows that we cannot copy the proof of Theorem 2.14.

Example 2. Let θ be defined by $0 \to 0011$, $1 \to 001$. Then T^2 is minimal ([10]). If we take $J = \{00, 01, 10, 11\}$ then θ does not induce a substitution on J as in the proof of 2.14.: $\theta(01) = 00\,11\,00\,1$ has odd length. (In particular cases one can get rid of this phenomenon by considering higher powers of θ.)

The following notions are introduced to deal with the problem illustrated by Example 2. (We shall only consider the case $I = \{0, 1\}$ but definitions and lemmas are easily generalised to more symbols).

Definition 3. A block A is called *n-balanced* if

$$N_0(A) = N_1(A) = 0 \qquad (\text{modulo } n).$$

An n-balanced block A is called *irreducible* if

$$A = BC$$

with B n-balanced and C arbitrary implies $B = A$.

Quickly verified is

Lemma 4. (i) *An n-balanced block has an unique decomposition in irreducible n-balanced blocks.*

(ii) *If θ is a substitution and A an n-balanced block then θA is an n-balanced block.*

Lemma 5. *Let $B = A_1 A_2 \dots A_{n^2}$, where the A_k are arbitrary blocks. Then there exists m and k such that $1 \leqq m \leqq k \leqq n^2$ and such that $A_m A_{m+1} \dots A_k$ is n-balanced.*

Proof. Let $u_k = N_0(A_1 \dots A_k) \pmod n$ and $v_k = N_1(A_1 \dots A_k) \pmod n$. Consider the pairs (u_k, v_k) for $k = 1, 2, \dots, n^2$. If all are different then there is a k such that $(u_k, v_k) = (0, 0)$. Hence $A_1 A_2 \dots A_k$ is n-balanced. If not all are different then $(u_{m-1}, v_{m-1}) = (u_k, v_k)$ for an m and k with $2 \leqq m \leqq k \leqq n^2$ and $A_m A_{m+1} \dots A_k$ is n-balanced.

Lemma 6. *Let $x \in I^{\mathbb{Z}}$ be an almost periodic sequence and $n \geqq 1$. Then*

(i) *x has an unique decomposition*

$$x = \dots B_{-1} \dot{B}_0 B_1 \dots \qquad (B_0(0) = x_0)$$

where the B_k are elements of a finite set J of irreducible n-balanced blocks.

(ii) *If ϕ is a bijection between a finite set of symbols $\hat{I}$ and J then*

$$\dots \phi^{-1}(B_{-1}) \, \phi^{-1}(B_0) \, \phi^{-1}(B_1) \dots$$

is an almost periodic sequence.

Proof. Any block B appearing in x appears with bounded gap. Let $s(B)$ be the least upper bound of this gap, and let $s(k) = \max\{s(B) : N(B) = k,\ B \text{ appears in } x\}$.

Let $t_1 = s(1)$, $t_2 = s(1 + t_1)$, $t_k = s(t_{k-2} + t_{k-1})$ for $k = 3, 4, \dots, n^2$.

We shall prove (i) by showing that x is decomposable in irreducible n-balanced blocks (such that one block begins with x_0) whose length does not exceed t_{n^2}.

Let $A_1 = x_0$. Then A_1 reappears within t_1 steps i.e. $x_0 x_1 \dots = A_2 A_1 \dots$ with $N(A_2) \leqq t_1$. Now $A_2 A_1$ reappears within t_2 steps i.e. $x_0 x_1 \dots = A_3 A_2 A_1 \dots$ with $N(A_3) \leqq t_2$. Continuing in this manner for $k = 4, 5, \dots, n^2$ we obtain

$$x_0 x_1 \dots = A_k A_{k-1} \dots A_1 \dots \quad \text{with} \quad N(A_k) \leqq t_{k-1} \quad \text{for } k = 2, \dots, n^2.$$

By Lemma 5 there exist $1 \leqq m \leqq k \leqq n^2$ such that $A_k A_{k-1} \dots A_m$ is n-balanced. Let A be the first irreducible n-balanced block in $A_k A_{k-1} \dots A_m$. Then $N(A) \leqq t_{n^2}$ and $x[0, N(A) - 1] = A$. Applying the same arguments with $A_1 = x_{N(A)}$ we shall find the next irreducible n-balanced block. In this way, we obtain the unique decomposition of the positive part of x. Essentially the same procedure applies to the negative part of x, yielding (i).

We shall call any place in x where an irreducible n-balanced block of the decomposition of x begins a J-place. (Note that 0 is a J-place by definition.) Let A be any n-balanced block beginning at a J-place t. To prove (ii) we have to show that A reappears with bounded gap at J-places.

Let $A_0 = A$. Then A_0 reappears (with a gap independent of t) i.e. a block of the form $A_0 D_0 A_0$ appears at place t in x. Analogously we define for $k = 1, 2, \dots, n^2 - 1$ the block $A_{k+1} = A_k D_k A_k$, where A_k begins at place t in x and D_k is defined by the first reappearance of A_k in x.

Let $B_1 = D_0 A_0$, $B_{k+1} = B_k B_{k-1} \dots B_1 D_k A_0$ for $k = 1, \dots, n^2 - 1$. Then it is easily proved by induction that

$$A_k = A_0 B_k B_{k-1} \dots B_1 \quad \text{for } k = 1, \dots, n^2.$$

By Lemma 5 there exist $1 \leqq m \leqq k \leqq n^2$ such that $B_k B_{k-1} \dots B_m$ is n-balanced. Therefore the block

$$B_k B_{k-1} \dots B_m = B_k B_{k-1} \dots B_{m+1} B_{m-1} B_{m-2} \dots B_1 D_{m-1} A_0$$

appearing at J-place $t + N(A_0)$ in x is n-balanced. But since $A_0 = A$ is n-balanced, the block $B_k B_{k-1} \dots B_{m+1} B_{m-1} \dots B_1 D_{m-1}$ is n-balanced and therefore A reappears at a J-place within $N(B_k B_{k-1} \dots B_{m+1} B_{m-1} \dots B_1 D_{m-1}) < N(A_{n^2})$ steps. As in the proof of (i) it follows from the almost periodicity of x that this number does not depend on t.

Proof of Theorem 1. As remarked before (cf. the proof of 2.15), ergodicity of T^n implies minimality of T^n.

Let T^n be minimal and let $X(\theta) = \overline{\mathrm{Orb}}(w)$, where $\theta w = w$. Then w is almost periodic. We apply Lemma 6 (i) to w and obtain a set J of irreducible n-balanced blocks, such that

$$w = \dots B_{-1} \dot{B}_0 B_1 \dots \quad (B_k \in J).$$

Let $\phi: \hat{I} \to J$ be a bijection between J and a finite set $\hat{I}$. We extend ϕ in the usual way to $\hat{I}^*$ and $\hat{I}^{\mathbb{Z}}$.

Let $\hat{T}$ be the shift on $\hat{I}^{\mathbb{Z}}$. The behaviour of ϕ with respect to the homeomorphisms is given by

$$\phi(\hat{T} y) = T^{N(\phi \hat{\imath})}(\phi y) \quad y \in \hat{I}^{\mathbb{Z}},\ y_0 = \hat{\imath},\ \hat{\imath} \in \hat{I}.$$

Define a substitution η: $\hat{I} \to \hat{I}^*$ by

$$\eta \hat{\imath} = \phi^{-1}(\theta(\phi \hat{\imath}) \quad \text{for all } \hat{\imath} \in \hat{J}.$$

By Lemma 4 η is well defined. Let $\hat{p} = \phi^{-1}(B_{-1})$ and $\hat{q} = \phi^{-1}(B_0)$. Then $\hat{p}\hat{q}$ is a cyclic pair for η. Let $\hat{w} = w^{\hat{p}\hat{q}}$. Then $\hat{w} = \phi^{-1}(w)$, so $\hat{w}$ is almost periodic by Lemma 6 (ii). Since all symbols from $\hat{I}$ appear in $\hat{w}$ this implies that η is primitive. Hence, if $X(\eta) = \overline{\mathrm{Orb}}(\hat{w}, \hat{T})$ then $(X(\eta), \hat{T}, \hat{\mu})$ is an uniquely ergodic (substitution) dynamical system.

We now form a tower (Y, S, ν) on $(X(\eta), \hat{T}, \hat{\mu})$ by assigning $N(\phi \hat{\imath})/n - 1$ isomorphic copies to each cylinder $[\hat{\imath}] \cap X(\eta)$; S and ν are the corresponding transformation and probability measure (cf. [6] and [13]). Since $(X(\eta), \hat{T}, \hat{\mu})$ is uniquely ergodic, so is (Y, S, ν). We shall finish the proof by showing that $(X(\theta), T^n)$ is a factor of (Y, S).

The base Y_0 of Y and $X(\eta)$ will not be distinguished in the sequel. Define ψ: $Y \to X(\theta)$ by

$$\psi(y) = \begin{cases} \phi(y) & \text{if } y \in Y_0 \\ T^{nk} \phi(S^{-k} y) & \text{if } y \in S^k(Y_0[\hat{\imath}]),\ 1 \leqq k < \dfrac{N(\phi \hat{\imath})}{n},\ \hat{\imath} \in \hat{I}. \end{cases}$$

Let us verify that $\psi S = T^n \psi$.

If $y \in S^k(Y_0 \cap [\hat{i}])$, $m = N(\phi\hat{i})/n$ and if $0 \leqq k < m-1$ then

$$\psi(Sy) = T^{n(k+1)}\phi(S^{-k-1}Sy) = T^n T^{nk}\phi(S^{-k}y) = T^n\psi(y),$$

if $k = m-1$ then

$$\begin{aligned}\psi(Sy) &= \phi(\hat{T}S^{-m+1}y) = T^{N(\phi\hat{i})}\phi(S^{-m+1}y)\\ &= T^{mn}\phi(S^{-m+1}y) = T^n\psi(y).\end{aligned}$$

The continuity of ψ follows from that of ϕ, T and indicator functions of cylinders. Since $(X(\theta), T^n)$ is minimal and since $\phi\hat{w} = w$, ψ is surjective. Hence $(X(\theta), T^n, \psi v)$ is a factor of (Y, S, v) and as such uniquely ergodic. (See e.g. [8].) Therefore T^n is ergodic with respect to μ.

Example 7. Let θ be as in Example 2. Let $w = w^{10}$. Then w decomposes in 2-balanced irreducible blocks from the set $J = \{00, 11, 1001\}$. If $\hat{I} = \{a, b, c\}$ then η is given by

$$a \to abab, \quad b \to ac, \quad c \to accc.$$

The homomorphism ψ is an isomorphism in this case: $(X(\theta), T^2, \mu)$ is isomorphic to the tower obtained from $(X(\eta), \hat{T}, \hat{\mu})$ by doubling the cylinder $[c]$.

We would like to give an example that is independent of the results of this section but is constructed in a similar way.

Example 8. Let θ be defined by $0 \to 01$, $1 \to 10$. Then $w = w^{00}$ is the Morse-Thue sequence. Let $J_0 = \{0, 01, 011\}$. It is not difficult to see that any sequence in $X(\theta) = \overline{\text{Orb}}(w)$ decomposes in a unique way into blocks belonging to J_0. Let $\hat{I} = \{a, b, c\}$ and ϕ a bijection between $\hat{I}$ and J_0 defined by $\phi(a) = 0$, $\phi(b) = 01$ and $\phi(c) = 011$.

Then as before θ induces a substitution η on $\hat{I}$. We find that η is defined by

$$a \to b, \quad b \to ca, \quad c \to cba.$$

Let $A = [0] \cap X(\theta) = ([00] \cup [010] \cup [0110]) \cap X(\theta)$. Let (A, T_A) be the flow induced on A (T_A is the first return time to $[0]$). Then it is easy to see that the usual extension of ϕ is an isomorphism between $(X(\eta), \hat{T})$ and (A, T_A). It follows from Theorem 1 of the next section that $(X(\eta), \hat{T})$ is isomorphic to a substitution flow of constant length. Calculations (and 2.13 and 3.7) show that the structure system of $(X(\eta), \hat{T})$ is $\mathbb{Z}(2)$ and that $(X(\eta), \hat{T}, \hat{\mu})$ has partly continuous spectrum.

We remark that the sequence $\hat{w} = w^{bc}$ generated by the pair bc under η is non-repetitive i.e. if B is any block over $\hat{I}$ then BB does not appear in $\hat{w}$ (cf. [5, Ex. 4.11]).

V. Substitutions of Non-Constant Length Isomorphic to Substitutions of Constant Length

Theorem 1. *Let θ be a substitution of non-constant length $(l_0, l_1, \ldots, l_{r-1})$. If $(l_0, l_1, \ldots, l_{r-1})$ is a right eigenvector of the θ-matrix, then $(X(\theta), T)$ is isomorphic to a substitution flow generated by a substitution of constant length.*

Proof. Let $X(\theta) = \overline{\text{Orb}}(w^{pq})$, where $\theta w^{pq} = w^{pq}$. We shall define an isomorphism from $(X(\theta), T)$ to a flow $(X(\eta), \hat{T})$, where η is a substitution on a set $\hat{I}$ consisting of $\sum_{i \in I} l_i$

symbols a_{ij}, where $0 \leqq i < r$ and $0 \leqq j < l_i$. To each block θi we assign the block $a_{i0} a_{i1} \dots a_{i\, l_i - 1}$. Since any sequence in $X(\theta)$ has a unique decomposition in blocks from the set $\{\theta 0, \theta 1, \dots, \theta(r-1)\}$ ([10]), this assignment extends to a continuous map ϕ from $X(\theta)$ to $\hat{I}^{\mathbb{Z}}$. Since θi and $\phi(\theta i)$ have the same length ϕ is a homomorphism. Define a substitution on $\hat{I}$ by

$$\eta(a_{ij}) = \phi(\theta i^*) \qquad \text{if } \theta i(j) = i^*.$$

(For example: $\eta(a_{q0}) = a_{q0} a_{q1} \dots a_{q\, l_q - 1}$).

The substitution η is primitive (we may assume that the θ-matrix $L(\theta)$ is strictly positive and this implies $L(\eta^2)$ strictly positive). If we take $\hat{p} = a_{p\, l_p - 1}$ and $\hat{q} = a_{q0}$ then $\hat{p}\hat{q}$ is a cyclic pair for η. Let $\hat{w} = w^{\hat{p}\hat{q}}$ and $X(\eta) = \overline{\text{Orb}}(\hat{w};\, \hat{T})$. Since ϕ is obviously injective, and surjective by the $\hat{T}$-minimality of $X(\eta)$ and the fact that $\phi w = \hat{w}$, ϕ is an isomomorphism between $(X(\theta), T)$ and $(X(\eta), \hat{T})$.

So far we apparently gained nothing since η is still a substitution of non-constant length. We shall exhibit however a substitution η' of constant length λ (where λ is the maximal eigenvalue of $L(\theta)$) on $\hat{I}$ which generates the same sequence $\hat{w}$ and hence the same flow.

Let $B_i = \eta(a_{i0} a_{i1} \dots a_{i\, l_i - 1}) = \phi(\theta^2 i)$ for all $i \in I$. We claim that $N(B_i) = \lambda l_i$ $(i = 0, \dots, r-1)$. To verify this note that $N(B_i) = N(\theta^2 i) = l_i^{(2)}$ and that $l_i^{(2)} = \lambda l_i$ since

$$(l_\cdot^{(2)}) = L(l_\cdot) = \lambda(l_\cdot).$$

(An irreducible positive matrix has only one independent positive eigenvector. Therefore the eigenvalue corresponding to $(l_0, l_1, \dots, l_{r-1})$ has to be λ.)

Decompose each B_i in $B_i = B_{i0} B_{i1} \dots B_{i\, l_i - 1}$, where $N(B_{ij}) = \lambda$ for $j = 0, 1, \dots, l_i - 1$. Define a substitution η' on $\hat{I}$ by

$$\eta'(a_{ij}) = B_{ij}.$$

Then η' has constant length λ and the same cyclic pair $\hat{p}\hat{q}$ generates the same $\hat{w}$ as η since

$$\eta(a_{i0} \dots a_{i\, l_i - 1}) = B_i = B_{i0} \dots B_{i\, l_i - 1} = \eta'(a_{i0} \dots a_{i\, l_i - 1})$$

and similarly $\eta^k(a_{i0} \dots a_{i\, l_i - 1}) = \eta'^k(a_{i0} \dots a_{i\, l_i - 1})$ for all $k \geqq 1$, $i \in I$.

Example 2. See Example 4.8.

Example 3 ([12]). Let $r = 2$ and θ defined by $0 \to 01$, $1 \to 1100$. Then

$$L(\theta) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 4 \end{pmatrix},$$

so that the condition of Theorem 1 is fulfilled. Here

$$\hat{I} = \{a_{00}, a_{01}, a_{10}, a_{11}, a_{12}, a_{13}\} = \{a, b, c, d, e, f\},$$

η and η' are defined by

$$\eta: \begin{array}{l} a \to ab \\ b \to cdef \\ c \to cdef \\ d \to cdef \\ e \to ab \\ f \to ab \end{array} \qquad \eta': \begin{array}{l} a \to abc \\ b \to def \\ c \to cde \\ d \to fcd \\ e \to efa \\ f \to bab \end{array}$$

and $(X(\theta), T)$ is isomorphic to $(X(\eta'), \hat{T})$.

Remark 4. We consider the case $r=2$ i.e. $I=\{0,1\}$. Let $\lambda_1 > \lambda_2$ be the eigenvalues of the θ-matrix $L(\theta)$. It follows from the Cayley-Hamilton theorem that $\begin{pmatrix} l_0 - \lambda_2 \\ l_1 - \lambda_2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue λ_1. This implies that

$$L \begin{pmatrix} l_0 \\ l_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} l_0 \\ l_1 \end{pmatrix} \quad \text{iff } \lambda_2 = 0.$$

According to Theorem 1 $(X(\theta), T)$ is topologically isomorphic to a substitution flow of constant length if $\lambda_2 = 0$. We conjecture that this condition is also necessary.

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