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Uniform Amarts: A Class of Asymptotic Martingales for which Strong Almost Sure Convergence Obtains

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Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n)_{n\in\mathbb{N}}$ an increasing sequence of sub-σ-fields of $\mathscr{F}$. The general almost sure convergence theorem for vector-valued asymptotic martingales proven in [7] may be stated as follows:

Theorem I. *Let E be a Banach space with the Radon-Nikodym property and a separable dual. Let $(X_n, \mathscr{F}_n)_{n\in\mathbb{N}}$ be an E-valued asymptotic martingale of class (B), that is such that*

$$\sup_{\tau\in T} \int \|X_\tau\| \, dP < \infty.$$

Then there is an E-valued random variable X such that the sequence $(X_n(\omega))_{n\in\mathbb{N}}$ converges weakly to $X(\omega)$ for almost every $\omega \in \Omega$.

The conclusion of this theorem cannot be improved: It is known (see [2]) that whenever E is infinite dimensional, one can always construct an E-valued asymptotic martingale of class (B) (in fact even uniformly bounded) for which strong convergence fails almost surely.

Thus while the above theorem is elegant and very general, it has the drawback that it is not a proper extension of the Doob almost sure convergence theorem for vector-valued martingales.

The purpose of this paper is to introduce a somewhat smaller class of vector-valued asymptotic martingales – namely the "uniform amarts" – for which strong almost sure convergence obtains. This class is wide enough to include: the martingales, the quasi-martingales, as well as the dominated (by an L^1-function) sequences of random variables which are strongly convergent to a limit almost surely.

I am indebted to Louis Sucheston for several useful comments concerning this paper. He also persuaded me to abandon the term "asymptotic martingale" in favor of the term "amart", as more convenient: thus esoteric considerations of an esthetic nature sometimes have to give way to considerations of a more practical nature. The principal results of this paper were announced in [4].

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1. The Uniform Amart; Notation, Definitions and Preliminaries

We begin by recalling the necessary terminology. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n)_{n\in\mathbb{N}}$ an increasing sequence of sub-σ-fields of $\mathscr{F}$; here $\mathbb{N}=\{1,2,3,\ldots\}$. A *stopping time* (with respect to the sequence $(\mathscr{F}_n)_{n\in\mathbb{N}}$) is a mapping $\tau: \Omega \to \mathbb{N}\cup\{+\infty\}$ such that $\{\tau=n\}\in\mathscr{F}_n$ for all $n\in\mathbb{N}$. Let T be the set of all *bounded stopping times.* With the definition $\tau\leqq\sigma$ if $\tau(\omega)\leqq\sigma(\omega)$ for all $\omega\in\Omega$, T is a directed set "filtering to the right". For $\tau\in T$ recall that

$$\mathscr{F}_\tau=\{A\in\mathscr{F}\mid A\cap\{\tau=n\}\in\mathscr{F}_n \text{ for all } n\in\mathbb{N}\}$$

and that $\tau\leqq\sigma$ implies $\mathscr{F}_\tau\subset\mathscr{F}_\sigma$.

Let E be a *Banach space. In this article we shall only consider random variables with values in E that are strongly measurable and strongly integrable* (in the sense of Bochner).

A sequence $(X_n)_{n\in\mathbb{N}}$ of E-valued random variables is called *adapted* (with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$) if $X_n\colon \Omega\to E$ is (Bochner) $\mathscr{F}_n$-measurable for each $n\in\mathbb{N}$.

Since in what follows we shall only deal with sequences of E-valued random variables, *we may and shall assume that*

E is a separable Banach space;

$D\subset\{x'\in E'\mid \|x'\|\leqq 1\}$ is a countable set with the property that

$$\|x\|=\sup\{|\langle x',x\rangle|\mid x'\in D\}, \text{ for each } x\in E. \tag{1}$$

Let now $\mathscr{A}$ be an algebra of subsets of Ω. If $v\colon \mathscr{A}\to E$ is a *finitely additive* set function, we denote by $\|v\|$ the *total variation of* v (see [9], p. 97), that is

$$\|v\|=\sup\sum_i\|v(A_i)\|$$

(the supremum being taken over all finite sequences (A_i) of disjoint sets in $\mathscr{A}$), whenever this supremum is finite.

For any Banach space F, we denote by $\mathscr{S}_F(\Omega,\mathscr{A})$ the set of all $g\colon\Omega\to F$ of the form

$$g=\sum_i x_i 1_{A_i}$$

where the sum is finite, the A_i's are disjoint and belong to $\mathscr{A}$ and $x_i\in F$ for each i. We also write:

$$\mathscr{S}_F^1(\Omega,\mathscr{A})=\{g\in\mathscr{S}_F(\Omega,\mathscr{A})\mid \|g(\omega)\|\leqq 1 \text{ for all } \omega\in\Omega\}.$$

We now recall the definition of amart (=asymptotic martingale; see [7, 5]):

Definition 1. An adapted sequence $(X_n)_{n\in\mathbb{N}}$ of E-valued random variables is called a strong amart, or simply an amart, if X_n is integrable for each $n\in\mathbb{N}$ and if the net

$$\left(\int X_\tau\, dP\right)_{\tau\in T}$$

converges in the strong topology of E.

We shall repeatedly make use below of the following useful stability property of E-valued amarts (this is Theorem 1 in [3], or see Lemma 2 of [7]):

Theorem II. *Let E be a Banach space. Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued amart. For each $\tau\in T$ set*

$$\mu_\tau(A)=\int_A X_\tau\, dP, \qquad \text{for } A\in\mathscr{F}_\tau.$$

Then the family $(\mu_\tau(A))_\tau$ converges to a limit $\mu(A)$ in E, for each $A\in\bigcup_{n\in\mathbb{N}}\mathscr{F}_n=\bigcup_{\tau\in T}\mathscr{F}_\tau$, and the convergence is "uniform", in the sense that for each $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\sigma\in T, \qquad \sigma\geqq n_0 \Rightarrow \sup_{A\in\mathscr{F}_\sigma}\|\mu_\sigma(A)-\mu(A)\|\leqq\varepsilon. \tag{2}$$

Throughout this paper we shall make constant use of the notation introduced in Theorem II.

The following observation is now an immediate consequence of Theorem II:

Remark. Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued amart. Then

$$\lim_{\tau\in T}\int\langle g, X_\tau\rangle\, dP=L(g)$$

exists for each $g\in\mathscr{S}_{E'}(\Omega, \bigcup_{n\in\mathbb{N}}\mathscr{F}_n)$.

This suggests the following:

Definition 2. The E-valued amart $(X_n)_{n\in\mathbb{N}}$ is called a uniform amart if the previous convergence is "uniform", in the sense that for each $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\sigma\in T, \qquad \sigma\geqq n_0 \Rightarrow \sup_{g\in\mathscr{S}^1_{E'}(\Omega,\mathscr{F}_\sigma)}\left|\int\langle g, X_\sigma\rangle dP-L(g)\right|\leqq\varepsilon. \tag{3}$$

Remark. It is easily seen that the above statement is equivalent with the following: For each $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\sigma\in T, \qquad \sigma\geqq n_0 \Rightarrow \|\mu_\sigma-(\mu\mid\mathscr{F}_\sigma)\|\leqq\varepsilon. \tag{3'}$$

We also give the following definition:

Definition 3. An adapted sequence $(X_n)_{n\in\mathbb{N}}$ of E-valued random variables is called a uniform potential if X_n is integrable for each $n\in\mathbb{N}$ and if

$$\lim_{\tau\in T}\int\|X_\tau\|\, dP=0.$$

Remarks. 1) The general convergence theorem for real-valued amarts [1] (or see [10, 3]; see also [6] for a related result) implies that if $(X_n)_{n\in\mathbb{N}}$ is an E-valued uniform potential, then $\lim_{n\in\mathbb{N}} X_n(\omega)=0$ strongly almost surely.

2) It is clear that if $(X_n)_{n\in\mathbb{N}}$ is an E-valued *uniform potential*, then $(X_n)_{n\in\mathbb{N}}$ is a *potential* in the sense of [11]. The converse is not true: the asymptotic martingale constructed in [2] provides an example in every infinite-dimensional Banach space,

of a potential which is even uniformly bounded, but which is *not* a uniform potential.

Before beginning the study of uniform amarts, we recall an elementary but very useful result from [7] (in the Lemma below we denote by $\mathscr{F}_\infty$ the σ-field spanned by $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$, i.e.,

$$\mathscr{F}_\infty = \sigma(\bigcup_{n\in\mathbb{N}} \mathscr{F}_n)).$$

Lemma 1. *Let* $(X_n)_{n\in\mathbb{N}}$ *be an E-valued amart such that* $X^* = \sup_{n\in\mathbb{N}} \|X_n\| \in L^1_R$. *Then*

$$\mu(A) = \lim_{n\in\mathbb{N}} \int_A X_n \, dP = \lim_{\tau\in T} \int_A X_\tau \, dP \tag{4}$$

exists in E for each $A \in \mathscr{F}_\infty$, *and the set function* $\mu\colon \mathscr{F}_\infty \to E$ *is countably additive, absolutely continuous with respect to P and of bounded variation.*

For completeness we include the proof:

Proof. By Theorem II, we know that (4) holds for each $A \in \bigcup_{n\in\mathbb{N}} \mathscr{F}_n$. Let now $A \in \mathscr{F}_\infty = \sigma(\bigcup_{n\in\mathbb{N}} \mathscr{F}_n)$; we show that $(\int_A X_\tau \, dP)_{\tau\in T}$ is a Cauchy net in E. Let $\varepsilon > 0$. There is then $\delta = \delta(\varepsilon) > 0$ such that:

$$B \in \mathscr{F}_\infty, \quad P(B) \leqq \delta \Rightarrow \int_B X^* \, dP \leqq \varepsilon.$$

Choose $A_0 \in \bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ such that $P(A \,\triangle\, A_0) \leqq \delta$, then choose $n_0 \in \mathbb{N}$ large enough that $A_0 \in \mathscr{F}_{n_0}$ and that

$$\left.\begin{array}{l} \sigma \in T,\ \sigma \geqq n_0 \\ \tau \in T,\ \tau \geqq n_0 \end{array}\right\} \Rightarrow \left\| \int_{A_0} X_\sigma \, dP - \int_{A_0} X_\tau \, dP \right\| \leqq \varepsilon.$$

Note also that for *each* $\tau \in T$

$$\left\| \int_A X_\tau \, dP - \int_{A_0} X_\tau \, dP \right\| \leqq \int_{A \triangle A_0} \|X_\tau\| \, dP \leqq \int_{A \triangle A_0} X^* \, dP \leqq \varepsilon.$$

We deduce

$$\left.\begin{array}{l} \sigma \in T,\ \sigma \geqq n_0 \\ \tau \in T,\ \tau \geqq n_0 \end{array}\right\} \Rightarrow \left\| \int_A X_\sigma \, dP - \int_A X_\tau \, dP \right\| \leqq 3\varepsilon.$$

This proves the existence of the limit in (4). The countable additivity of μ follows by the Vitali-Hahn-Sacks Theorem (see [9], p. 321); the absolute continuity of μ with respect to P and the fact that μ is of bounded variation follow from the observation that

$$\|\mu(A)\| \leqq \int_A X^* \, dP, \qquad \text{for each } A \in \mathscr{F}_\infty.$$

This completes the proof of the Lemma.

We may now state and prove the following:

Proposition 1. *Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued amart such that $X^* = \sup_{n\in\mathbb{N}} \|X_n\| \in L^1_{\mathbb{R}}$. Then the assertions* a) *and* b) *below are equivalent.*

a) *There is an E-valued random variable X such that*

$$\lim_{n\in\mathbb{N}} \langle x', X_n(\omega)\rangle = \langle x', X(\omega)\rangle$$

almost surely, for each $x' \in D$.

b) *There is an E-valued random variable $X \in L^1_E(\Omega, \mathscr{F}_\infty, P)$ such that* (with the notation of formula (4) in Lemma 1)

$$\mu(A) = \int_A X\, dP, \qquad \textit{for all } A \in \mathscr{F}_\infty,$$

that is, $X = \dfrac{d\mu}{dP}$.

If in addition $(X_n)_{n\in\mathbb{N}}$ is a uniform amart, then statements a) *and* b) *above are also equivalent to:*

c) *There is an E-valued random variable X such that*

$$\lim_{n\in\mathbb{N}} X_n(\omega) = X(\omega)$$

strongly almost surely.

Proof. We note first that, for each $x' \in D$, $(\langle x', X_n\rangle)_{n\in\mathbb{N}}$ is an L^1-bounded real amart, dominated in fact by the L^1-function X^* and hence by [1] (or see [10, 3]) converges almost surely to a limit. As D is countable, we can find $N(D) \in \mathscr{F}$, $P(N(D)) = 0$ such that for each $\omega \notin N(D)$ and each $x' \in D$,

$$\lim_{n\in\mathbb{N}} \langle x', X_n(\omega)\rangle = \Phi(x', \omega) \quad \text{exists}$$

$$\sup_{n\in\mathbb{N}} |\langle x', X_n(\omega)\rangle| \leqq X^*(\omega) \quad \text{and} \quad \Phi(x', \omega) \leqq X^*(\omega).$$

It follows that for each $A \in \mathscr{F}_\infty$,

$$\langle x', \int_A X_n\, dP = \int_A \langle x', X_n\rangle\, dP \to \int_A \Phi(x', \omega)\, dP(\omega).$$

On the other hand, by Lemma 1, we also have, for each $A \in \mathscr{F}_\infty$

$$\int_A X_n\, dP \to \mu(A) \quad \text{strongly,}$$

whence

$$\langle x', \int_A X_n\, dP\rangle \to \langle x', \mu(A)\rangle.$$

We deduce

$$\langle x', \mu(A)\rangle = \int_A \Phi(x', \omega)\, dP(\omega) \qquad \text{for } x' \in D, \qquad A \in \mathscr{F}_\infty. \tag{5}$$

a) $\Rightarrow$ b). By assumption there is a set $N'(D) \in \mathscr{F}$, $P(N'(D)) = 0$, and we may assume $N'(D) \supset N(D)$, such that for each $\omega \notin N'(D)$ and each $x' \in D$,

$$\langle x', X(\omega)\rangle = \Phi(x', \omega).$$

From (5) we deduce

$$\langle x', \mu(A)\rangle = \int_A \langle x', X(\omega)\rangle\, dP(\omega) = \langle x', \int_A X\, dP\rangle$$

for each $A \in \mathscr{F}_\infty$ and $x' \in D$; hence

$$\mu(A) = \int_A X\, dP \qquad \text{for every } A \in \mathscr{F}_\infty$$

(if X is not $\mathscr{F}_\infty$-measurable, we replace X by its conditional expectation $E(X \mid \mathscr{F}_\infty)$).

b) $\Rightarrow$ a). By assumption, there is $X \in L^1_E(\Omega, \mathscr{F}_\infty, P)$ such that

$$\mu(A) = \int_A X\, dP \qquad \text{for all } A \in \mathscr{F}_\infty.$$

Using (5) we deduce that for each $x' \in D$,

$$\int_A \langle x', X\rangle\, dP = \int_A \Phi(x', \omega)\, dP(\omega) \qquad \text{for all } A \in \mathscr{F}_\infty;$$

hence there is a set $N(x') \in \mathscr{F}$, $P(N(x')) = 0$ such that

$$\langle x', X(\omega)\rangle = \Phi(x', \omega) \qquad \text{for } \omega \notin N(x').$$

Then $B = (\bigcup_{x' \in D} N(x')) \cup N(D) \in \mathscr{F}$, $P(B) = 0$, and for each $x' \in D$ and $\omega \notin B$ we have

$$\lim_{n \in \mathbb{N}} \langle x', X_n(\omega)\rangle = \Phi(x', \omega) = \langle x', X(\omega)\rangle,$$

which proves a). Thus a) $\Leftrightarrow$ b).

Assume now in addition that the amart $(X_n)_{n \in \mathbb{N}}$ is uniform. It is clear that c) $\Rightarrow$ a). Thus the proof is finished if we show b) $\Rightarrow$ c):

Hence suppose the existence of $X \in L^1_E(\Omega, \mathscr{F}_\infty, P)$ such that

$$X = \frac{d\mu}{dP}.$$

Then formula (3) in the definition of the uniform amart becomes:

$$\sigma \in T,\ \sigma \geqq n_0 \Rightarrow \sup_{g \in \mathscr{S}^1_{E'}(\Omega, \mathscr{F}_\sigma)} \left|\int \langle g, X_\sigma - X\rangle\, dP\right| \leqq \varepsilon.$$

Now there is a simple function $X_0 \in \mathscr{S}_E(\Omega, \mathscr{F}_m)$ for some $m \geqq n_0$, such that

$$\int \|X - X_0\|\, dP \leqq \varepsilon. \tag{6}$$

We deduce, for $\sigma \geqq m$ and *all* $g \in \mathscr{S}_{E'}^1(\Omega, \mathscr{F}_\sigma)$

$$|\int \langle g, X_\sigma - X_0 \rangle| \leqq 2\varepsilon,$$

whence

$$\int \|X_\sigma - X_0\| \leqq 2\varepsilon \quad \text{for } \sigma \geqq m. \tag{7}$$

From (6) and (7) follows that

$$\sigma \in T, \ \sigma \geqq m \Rightarrow \int \|X_\sigma - X\| \, dP \leqq 3\varepsilon.$$

This shows that the net $(X_\sigma)_{\sigma \in T}$ converges to X in L_E^1. The proof is now completed by making use of the equivalence i) $\Leftrightarrow$ ii) of Proposition 3 (Section 2).

Remark. If E has the Radon-Nikodym property, then (in view of Lemma 1) it is clear that assertion b) holds.

2. General Properties of Uniform Amarts; Stability, Structure, Convergence

We begin by observing that the notion of uniform amart is a natural extension of the real-valued amart to the vector-valued case:

Proposition 2. *Every real amart is a uniform amart.*

Proof. We recall that if a real-valued additive set function ν defined on an algebra $\mathscr{A}$ of subsets of Ω is bounded, then it is of bounded variation and (see [9], p. 97, Lemma 5):

$$\|\nu\| \leqq 2 \sup_{A \in \mathscr{A}} |\nu(A)|.$$

With the notation of Theorem II, applied to the R-valued amart $(X_n)_{n \in \mathbb{N}}$, it follows immediately for $\sigma \in T$, $\sigma \geqq n_0$,

$$\|\mu_\sigma - (\mu \mid \mathscr{F}_\sigma)\| \leqq 2 \sup_{A \in \mathscr{F}_\sigma} |\mu_\sigma(A) - \mu(A)| \leqq 2\varepsilon$$

and hence the assertion is proved.

We next prove the following:

Proposition 3. *Let $(X_n)_{n \in \mathbb{N}}$ be an adapted sequence of E-valued random variables such that $X^* = \sup_{n \in \mathbb{N}} \|X_n\| \in L_R^1$. Consider the following assertions:*

i) *The net $(X_\sigma)_{\sigma \in T}$ converges in L_E^1.*

ii) *There is $X \in L_E^1$ such that*

$$\lim_{n \in \mathbb{N}} X_n(\omega) = X(\omega)$$

strongly almost surely.

iii) *$(X_n)_{n \in \mathbb{N}}$ is a uniform amart.*

Then i) $\Leftrightarrow$ ii) $\Rightarrow$ iii).

Proof. ii) $\Rightarrow$ i). Clearly $(X_\sigma)_{\sigma\in T}$ converges to X in L^1_E; this is just an application of the Lebesgue Dominated Convergence.

i) $\Rightarrow$ ii). Choose a strictly increasing sequence of positive integers $(n_k)_{k\geqq 1}$ such that

$$\sigma\in T,\ \sigma\geqq n_k \Rightarrow \int \|X_\sigma - X_{n_k}\|\, dP \leqq \frac{1}{2^{2k}}.$$

Let $U_n^{(k)} = X_n - X_{n_k}$ for $n\geqq n_k$. The sequence $(U_n^{(k)})_{n\geqq n_k}$ is adapted with respect to $(\mathscr{F}_n)_{n\geqq n_k}$ and

$$U_\tau^{(k)} = X_\tau - X_{n_k} \qquad \text{for all } \tau\in T, \qquad \tau\geqq n_k.$$

By the Maximal Lemma given in [7] we have:

$$P\left(\left\{\sup_{n\geqq n_k} \|U_n^{(k)}\| > \frac{1}{2^k}\right\}\right) \leqq 2^k (\sup_{\substack{\sigma\in T\\ \sigma\geqq n_k}} \int \|U_\sigma^{(k)}\|\, dP) \leqq \frac{1}{2^k}.$$

Let now

$$A_k = \left\{\sup_{n\geqq n_k} \|U_n^{(k)}\| > \frac{1}{2^k}\right\},$$

and

$$B_j = \bigcup_{k\geqq j} A_k \qquad \text{for each } j\in\mathbb{N}.$$

Then $P(B_j) \leqq \frac{1}{2^{j-1}}$ and it is easily seen that for $\omega\notin B_j$ the sequence $(X_n(\omega))_{n\in\mathbb{N}}$ is Cauchy in E. Thus i) $\Leftrightarrow$ ii) is proved.

It remains to show i) $\Rightarrow$ iii). By assumption there is $X\in L^1_E$ such that $(X_\sigma)_{\sigma\in T}$ converges to X in L^1_E. Thus given $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\sigma\in T,\ \sigma\geqq n_0 \Rightarrow \int \|X_\sigma - X\|\, dP \leqq \varepsilon.$$

We deduce for $\sigma\in T$, $\sigma\geqq n_0$:

$$\sup_{g\in\mathscr{L}^1_{E'}(\Omega,\mathscr{F}_\sigma)} \left|\int \langle g, X_\sigma - X\rangle\, dP\right| \leqq \int \|X_\sigma - X\|\, dP \leqq \varepsilon.$$

Hence $(X_n)_{n\in\mathbb{N}}$ is a uniform amart and Proposition 3 is proved.

Corollary 1. *Suppose that the Banach space E has the Radon-Nikodym property. Then for an adapted sequence $(X_n)_{n\in\mathbb{N}}$ of E-valued random variables, with $X^* = \sup_{n\in\mathbb{N}} \|X_n\| \in L^1_R$, the assertions* i), ii) *and* iii) *of Proposition 3 are equivalent.*

The next theorem generalizes to the vector-valued case a result that is well known for the real amarts (this goes back to Lemma 2 of [1]; see also [10, 3]):

Theorem 1. *Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued uniform amart which is L^1-bounded, that is*

$$\sup_{n\in\mathbb{N}}\int\|X_n\|\,dP=A<\infty.$$

Then:

1) *$(X_n)_{n\in\mathbb{N}}$ is an amart of class (B), that is*

$$\sup_{\tau\in T}\int\|X_\tau\|\,dP<\infty$$

2) *$(\|X_n\|)_{n\in\mathbb{N}}$ is a real-valued L^1-bounded amart.*

Proof. 1) Choose $n_1\in\mathbb{N}$ such that

$$\sigma\in T,\ \sigma\geqq n_1\Rightarrow\|\mu_\sigma-(\mu\,|\,\mathscr{F}_\sigma)\|\leqq 1 \tag{8}$$

(we use the notation of Theorem II). Now for $n\geqq n_1$ we have

$$|\|\mu_n\|-\|\mu\,|\,\mathscr{F}_n\||\leqq\|\mu_n-(\mu\,|\,\mathscr{F}_n)\|\leqq 1;$$

in particular, for all $n\geqq n_1$

$$\|\mu\,|\,\mathscr{F}_n\|\leqq\|\mu_n\|+1=\int\|X_n\|\,dP+1\leqq A+1.$$

Since the total variation of a measure increases with the σ-field, we deduce

$$\|\mu\,|\,\mathscr{F}_\sigma\|\leqq A+1\qquad\text{for all }\ \sigma\in T. \tag{9}$$

From (8) and (9) it follows that

$$\sigma\in T,\ \sigma\geqq n_1=\|\mu_\sigma\|\leqq\|\mu\,|\,\mathscr{F}_\sigma\|+1\leqq A+2,$$

or equivalently,

$$\sigma\in T,\ \sigma\geqq n_1\Rightarrow\int\|X_\sigma\|\,dP\leqq A+2. \tag{10}$$

A standard argument now completes the proof of 1): For arbitrary $\sigma\in T$ note that

$$X_\sigma+X_{n_1}=X_{\sigma\vee n_1}+X_{\sigma\wedge n_1},$$

whence by (10):

$$\int\|X_\sigma\|\,dP\leqq\int\|X_{\sigma\vee n_1}\|\,dP+\int\|X_{\sigma\wedge n_1}\|\,dP+\int\|X_{n_1}\|\,dP$$

$$\leqq(A+2)+2\{\int(\sup_{1\leqq j\leqq n_1}\|X_j\|)\,dP\}=C<\infty.$$

Thus 1) is proved.

2) Since the net $(\|\mu\,|\,\mathscr{F}_\sigma\|)_{\sigma\in T}$ is bounded above (see (9)) and is increasing, it converges in R. Now given $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\sigma\in T,\ \sigma\geqq n_0\Rightarrow\|\mu_\sigma-(\mu\,|\,\mathscr{F}_\sigma)\|\leqq\varepsilon;$$

whence for $\sigma\in T$, $\sigma\geqq n_0$

$$|\int\|X_\sigma\|\,dP-\|\mu\,|\,\mathscr{F}_\sigma\||=|\|\mu_\sigma\|-\|\mu\,|\,\mathscr{F}_\sigma\||\leqq\varepsilon.$$

This shows that the net $(\int\|X_\sigma\|\,dP)_{\sigma\in T}$ converges in R and hence completes the proof of Theorem 1.

We next prove an "optional sampling theorem" for uniform amarts; this was suggested by the optional sampling theorem for real-valued amarts given in [10]:

Theorem 2. *Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued uniform amart for $(\mathscr{F}_n)_{n\in\mathbb{N}}$. Let $(\tau_k)_{k\in\mathbb{N}}$ be a non-decreasing sequence of bounded stopping times for $(\mathscr{F}_n)_{n\in\mathbb{N}}$ and define*

$$\mathscr{G}_k=\mathscr{F}_{\tau_k}=\{A\in\mathscr{F}\mid A\cap\{\tau_k=n\}\in\mathscr{F}_n \text{ for all } n\in\mathbb{N}\}$$

and

$$Y_k=X_{\tau_k},\qquad \text{for } k\in\mathbb{N}.$$

Then $(Y_k)_{k\in\mathbb{N}}$ is a uniform amart with respect to the sequence of σ-fields $(\mathscr{G}_k)_{k\in\mathbb{N}}$. Further if $(X_n)_{n\in\mathbb{N}}$ is L^1-bounded, then $(Y_k)_{k\in\mathbb{N}}$ is L^1-bounded.

Proof. We begin by observing that if σ is a stopping time for $(\mathscr{G}_k)$ then τ_σ is a stopping time for $(\mathscr{F}_n)$ and

$$\mathscr{G}_\sigma\subset\mathscr{F}_{\tau_\sigma}.$$

By the definition of the uniform amart, given $\varepsilon>0$ choose $m\in\mathbb{N}$ such that

$$\left.\begin{array}{l}\tau,\tau'\in T\\ m\leqq\tau\leqq\tau'\\ g\in\mathscr{S}^1_{E'}(\Omega,\mathscr{F}_\tau)\end{array}\right\}\Rightarrow|\int\langle g,X_\tau\rangle\,dP-\int\langle g,X_{\tau'}\rangle\,dP|\leqq\varepsilon. \tag{11}$$

Let $\tau_\infty=\lim_k\uparrow\tau_k$; then τ_∞ is a (possibly infinite) stopping time for $(\mathscr{F}_n)$. Now for *each* $\omega\in\Omega$

$$\|X_{\tau_k\wedge m}(\omega)\|\leqq\sup_{1\leqq j\leqq m}\|X_j(\omega)\|,\qquad \text{for each } k\in\mathbb{N}$$

and

$$\lim_k X_{\tau_k\wedge m}(\omega)=X_{\tau_\infty\wedge m}(\omega)\quad \text{strongly},$$

also $X_{\tau_k\wedge m}$ is $\mathscr{G}_k$-measurable for each $k\in\mathbb{N}$. By Proposition 3, $(X_{\tau_k\wedge m})_{k\in\mathbb{N}}$ is then a uniform amart for $(\mathscr{G}_k)$. Choose now $K\in\mathbb{N}$ so that if σ,σ' are bounded stopping times for $(\mathscr{G}_k)$,

$$\left.\begin{array}{l}K\leqq\sigma\leqq\sigma'\\ h\in\mathscr{S}^1_{E'}(\Omega,\mathscr{G}_\sigma)\end{array}\right\}\Rightarrow|\int\langle h,X_{\tau_\sigma\wedge m}\rangle\,dP-\int\langle h,X_{\tau_{\sigma'}\wedge m}\rangle\,dP|\leqq\varepsilon. \tag{12}$$

If σ,σ' and h are as in (12), then $\tau_\sigma\vee m$, $\tau_{\sigma'}\vee m$ are bounded stopping times for $(\mathscr{F}_n)$, that is $\tau_\sigma\vee m$, $\tau_{\sigma'}\vee m\in T$ and we have

$$m\leqq\tau_\sigma\vee m\leqq\tau_{\sigma'}\vee m$$

$$h\in\mathscr{S}^1_{E'}(\Omega,\mathscr{F}_{\tau_\sigma}),$$

whence by (11),

$$|\int\langle h, X_{\tau_\sigma \vee m}\rangle\, dP - \int\langle h, X_{\tau_{\sigma'} \vee m}\rangle\, dP| \leqq \varepsilon. \tag{13}$$

Now

$$Y_\sigma - Y_{\sigma'} = X_{\tau_\sigma} - X_{\tau_{\sigma'}};$$

but

$$\begin{cases} X_{\tau_\sigma} = X_{\tau_\sigma \vee m} + X_{\tau_\sigma \wedge m} - X_m \\ X_{\tau_{\sigma'}} = X_{\tau_{\sigma'} \vee m} + X_{\tau_{\sigma'} \wedge m} - X_m \end{cases}. \tag{14}$$

From (12), (13) and (14) it follows that if σ, σ' are bounded stopping times for $(\mathscr{G}_k)$

$$\left.\begin{matrix} K \leqq \sigma \leqq \sigma' \\ h \in \mathscr{S}^1_{E'}(\Omega, \mathscr{G}_\sigma) \end{matrix}\right\} \Rightarrow |\int\langle h, Y_\sigma\rangle\, dP - \int\langle h, Y_{\sigma'}\rangle\, dP| \leqq 2\,\varepsilon$$

and the theorem is proved.

The following theorem (and in fact to a large degree the notion of uniform amart itself) was motivated by the "Riesz decomposition" for vector-valued amarts given in [1]:

Theorem 3. *For a sequence $(X_n)_{n\in\mathbb{N}}$ of E-valued random variables the following two assertions are equivalent:*

i) *$(X_n)_{n\in\mathbb{N}}$ is a uniform amart.*

ii) *$(X_n)_{n\in\mathbb{N}}$ admits a unique decomposition, $X_n = Y_n + Z_n$, for $n\in\mathbb{N}$, where $(Y_n)_{n\in\mathbb{N}}$ is an E-valued martingale and $(Z_n)_{n\in\mathbb{N}}$ is an E-valued uniform potential.*

Proof. i) $\Rightarrow$ ii). We note first (with the notation of Theorem II) that for *each* $p\in\mathbb{N}$ we have, if $n \geqq p$

$$\|(\mu_n|\mathscr{F}_p) - (\mu|\mathscr{F}_p)\| \leqq \|\mu_n - (\mu|\mathscr{F}_n)\|$$

and that the right-hand side tends to 0 when $n\to\infty$; since for $m \geqq p$, $n \geqq p$,

$$\|(\mu_m|\mathscr{F}_p) - (\mu_n|\mathscr{F}_p)\| = \int \|E(X_m|\mathscr{F}_p) - E(X_n|\mathscr{F}_p)\|\, dP$$

it follows that the sequence $(E(X_n|\mathscr{F}_p))_{n\geqq p}$ is Cauchy in L^1_E. If we denote its limit in L^1_E by Y_p, then we have

$$Y_p = \frac{d(\mu|\mathscr{F}_p)}{d(P|\mathscr{F}_p)} \quad \text{for each } p\in\mathbb{N}.$$

We let now $Z_p = X_p - Y_p$ for each $p\in\mathbb{N}$.

It is clear that $(Y_n)_{n\in\mathbb{N}}$ is a martingale and thus

$$Y_\tau = \frac{d(\mu \mid \mathscr{F}_\tau)}{d(P \mid \mathscr{F}_\tau)} \quad \text{for each } \tau \in T.$$

On the other hand

$$\int \|Z_\tau\| \, dP = \int \|X_\tau - Y_\tau\| \, dP = \sup_{g \in \mathscr{S}^1_{E'}(\Omega, \mathscr{F}_\tau)} \left| \int \langle g, X_\tau - Y_\tau \rangle \, dP \right|$$
$$\leqq \|\mu_\tau - (\mu \mid \mathscr{F}_\tau)\|$$

and

$$\lim_{\tau \in T} \|\mu_\tau - (\mu \mid \mathscr{F}_\tau)\| = 0;$$

thus $(Z_n)_{n\in\mathbb{N}}$ is a uniform potential.

The uniqueness of the decomposition follows by a standard argument; if

$$X_n = Y_n + Z_n = Y'_n + Z'_n,$$

where $(Y'_n)_{n\in\mathbb{N}}$ is a martingale and $(Z'_n)_{n\in\mathbb{N}}$ a uniform potential, then $(\|Y_n - Y'_n\|)_{n\in\mathbb{N}}$ is a submartingale and hence

$$\int \|Y_n - Y'_n\| \, dP \leqq \int \|Y_{n+1} - Y'_{n+1}\| \, dP.$$

On the other hand

$$\int \|Y_n - Y'_n\| \, dP = \int \|Z'_n - Z_n\| \, dP \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\int \|Y_n - Y'_n\| \, dP = 0 \quad \text{for all } n \in \mathbb{N},$$

whence $Y_n = Y'_n$ almost surely for each $n \in \mathbb{N}$.

ii) ⇒ i) is trivial, since every martingale is a uniform amart and every uniform potential is a uniform amart. This completes the proof of Theorem 3.

In view of the characterization of Banach spaces with the Radon-Nikodym property in terms of convergence of martingales (see [8], or see the elegant treatment of vector-valued martingales given in [15]; for the earlier historical development of the subject see [13, 14]), the following is an immediate consequence of Theorem 3. We assume below that the probability space is not purely atomic:

Corollary 2. *For a Banach space E the following assertions are equivalent when holding for all E-valued uniform amarts $(X_n, \mathscr{F}_n)_{n\in\mathbb{N}}$:*

(1) *If $(X_n)_{n\in\mathbb{N}}$ is L^1-bounded, that is if*

$$\sup_{n\in\mathbb{N}} \int \|X_n\| \, dP < \infty,$$

then there is an E-valued random variable X such that

$$\lim_{n\in\mathbb{N}} X_n(\omega) = X(\omega)$$

strongly almost surely.

(2) *The space E has the Radon-Nikodym property.*

3. Examples of Uniform Amarts

1) *The Martingale.* We recall that an adapted sequence $(X_n)_{n\in\mathbb{N}}$ of E-valued random variables is a martingale (with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$) if $X_n \in L_E^1$ for each $n\in\mathbb{N}$ and if

$$E(X_m \mid \mathscr{F}_n) = X_n \qquad \text{for all } n \leqq m.$$

It follows easily that

$$E(X_\tau \mid \mathscr{F}_\sigma) = X_\sigma \qquad \text{for all } \sigma \leqq \tau,\ \sigma,\ \tau \in T.$$

But then – with the notation of Theorem II – we have for every $A\in\mathscr{F}_\sigma$ and $\tau\in T$, $\tau \geqq \sigma$:

$$\mu_\tau(A) = \int_A X_\tau\, dP = \int_A E(X_\tau \mid \mathscr{F}_\sigma)\, dP = \int_A X_\sigma\, dP = \mu_\sigma(A);$$

whence

$$\mu(A) = \mu_\sigma(A) \qquad \text{for every } A\in\mathscr{F}_\sigma.$$

Thus $\mu \mid \mathscr{F}_\sigma = \mu_\sigma$, for each $\sigma\in T$ and hence $(X_n)_{n\in\mathbb{N}}$ is a uniform amart.

2) *Uniform Potentials.* Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued uniform potential. Again with the notation of Theorem II we have

$$\mu(A) = 0 \qquad \text{for every } A\in\bigcup_{n\in\mathbb{N}} \mathscr{F}_n,$$

whence $\mu \mid \mathscr{F}_\sigma = 0$ for each $\sigma\in T$. On the other hand

$$\|\mu_\sigma - (\mu \mid \mathscr{F}_\sigma)\| = \|\mu_\sigma\| = \int \|X_\sigma\|\, dP \xrightarrow[\sigma\in T]{} 0$$

and hence $(X_n)_{n\in\mathbb{N}}$ is a uniform amart.

3) *Dominated, Almost Surely Convergent Sequences.* Let $(X_n)_{n\in\mathbb{N}}$ be an adapted sequence of E-valued random variables such that: a) $X^* = \sup_{n\in\mathbb{N}} \|X_n\| \in L_R^1$; b) There is an E-valued random variable X such that

$$\lim_{n\in\mathbb{N}} X_n(\omega) = X(\omega)$$

strongly almost surely. It follows from Proposition 3 that $(X_n)_{n\in\mathbb{N}}$ is a uniform amart.

4) *The Quasi-Martingale.* We recall that an adapted sequence $(X_n)_{n\in\mathbb{N}}$ of E-valued random variables is a quasi-martingale (with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$; see [12, 16, 17]) if

$X_n \in L^1_E$ for each $n \in \mathbb{N}$ and if

$$\sum_{n=1}^{\infty} \int \|X_n - E(X_{n+1} \mid \mathscr{F}_n)\| \, dP < \infty.$$

It was shown in [10] for the case $E = R$ that a quasi-martingale is an amart. The proof carries over verbatim to the Banach space case. We shall now show that an E-valued quasi-martingale is a *uniform* amart.

Proof. Let $(X_n)_{n \in \mathbb{N}}$ be an E-valued quasi-martingale. Let $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \int \|X_n - E(X_{n+1} \mid \mathscr{F}_n)\| \, dP \leqq \varepsilon.$$

Let $\tau \in T$, $\tau \geqq n_0$ and let p be *any* integer such that $p \geqq \tau$. For any $g \in \mathscr{S}^1_{E'}(\Omega, \mathscr{F}_\tau)$ note that $g \cdot 1_{\{\tau = k\}}$ is $\mathscr{F}_k$-measurable, for each k with $n_0 \leqq k \leqq p$; hence we have

$$\begin{aligned}
\int \langle g, X_\tau \rangle \, dP - \int \langle g, X_p \rangle \, dP &= \sum_{k=n_0}^{p} \int_{\{\tau = k\}} \langle g, X_k - X_p \rangle \, dP \\
&= \sum_{k=n_0}^{p} \sum_{j=k}^{p-1} \int_{\{\tau = k\}} \langle g, X_j - X_{j+1} \rangle \, dP \\
&= \sum_{k=n_0}^{p} \sum_{j=k}^{p-1} \int_{\{\tau = k\}} \langle g, X_j - E(X_{j+1} \mid \mathscr{F}_j) \rangle \, dP \\
&= \sum_{j=n_0}^{p-1} \sum_{k=n_0}^{j} \int_{\{\tau = k\}} \langle g, X_j - E(X_{j+1} \mid \mathscr{F}_j) \rangle \, dP.
\end{aligned}$$

We deduce

$$\begin{aligned}
\left| \int \langle g, X_\tau \rangle \, dP - \int \langle g, X_p \rangle \, dP \right| &\leqq \sum_{j=n_0}^{p-1} \sum_{k=n_0}^{j} \int_{\{\tau = k\}} |\langle g, X_j - E(X_{j+1} \mid \mathscr{F}_j) \rangle| \, dP \\
&\leqq \sum_{j=n_0}^{p-1} \sum_{k=n_0}^{j} \int_{\{\tau = k\}} \|X_j - E(X_{j+1} \mid \mathscr{F}_j)\| \, dP \\
&\leqq \sum_{j=n_0}^{p-1} \int \|X_j - E(X_{j+1} \mid \mathscr{F}_j)\| \, dP \leqq \varepsilon.
\end{aligned}$$

Letting $p \to \infty$ in the above inequality we deduce:

$$\tau \in T, \ \tau \geqq n_0 \Rightarrow \sup_{g \in \mathscr{S}^1_{E'}(\Omega, \mathscr{F}_\tau)} \left| \int \langle g, X_\tau \rangle \, dP - L(g) \right| \leqq \varepsilon.$$

Hence (see Definition 2), $(X_n)_{n \in \mathbb{N}}$ is a uniform amart.

Bibliography

1. Austin, D.G., Edgar, G.A., Ionescu Tulcea, A.: Pointwise convergence in terms of expectations. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **30**, 17–26 (1974)
2. Bellow, A.: On vector-valued asymptotic martingales. Proc. nat. Acad. Sci. USA **73**, No. 6, 1798–1799 (1976).
3. Bellow, A.: Several stability properties of the class of asymptotic martingales. Z. Wahrscheinlichkeitstheorie verw. Gebiete **37**, 275–290 (1977)
4. Bellow, A.: Les amarts uniformes. C.R. Acad. Sci. Paris **284**, Sér. A, 1295–1298 (1977)
5. Brunel, A., Sucheston, L.: Sur les amarts à valeurs vectorielles. C.R. Acad. Sci. Paris **283**, Sér. A, 1037–1040 (1976)
6. Chacon, R.V.: A "stopped" proof of convergence. Advances in Math. **14**, 365–368 (1974)
7. Chacon, R.V., Sucheston, L.: On convergence of vector-valued asymptotic martingales. Z. Wahrscheinlichkeitstheorie verw. Gebiete **33**, 55–59 (1975)
8. Chatterji, S.D.: Martingale convergence and the Radon-Nikodym Theorem. Math. Scand. **22**, 21–41 (1968)
9. Dunford, N., Schwartz, J.T.: Linear Operators. Part I. New York: Wiley 1953.
10. Edgar, G.A., Sucheston, L.: Amarts: A class of asymptotic martingales (Discrete parameter). J. multivariate Analysis **6**, 193–221 (1976)
11. Edgar, G.A., Sucheston, L.: The Riesz decomposition for vectorvalued amarts. Z. Wahrscheinlichkeitstheorie verw. Gebiete **36**, 85–92 (1976)
12. Fisk, D.L.: Quasi-martingales. Trans. Amer. math. Soc. **120**, 369–389 (1965
13. Ionescu Tulcea, A., Ionescu Tulcea, C.: Abstract ergodic theorems. Trans. Amer. math. Soc. **107**, 107–124 (1963)
14. Métivier, M.: Limites projectives de mesures; martingales; applications. Ann. Math. pura appl., IV Ser., **63**, 225–252 (1963)
15. Neveu, J.: Martingales à Temps Discret. Paris: Masson 1972.
16. Orey, S.: F-Processes. Proc. 5th Berkeley Sympos. Math. Statist. Probab. II, pp. 301–313. Univ. of Calif, 1965/66
17. Rao, K.M.: Quasi-martingales. Math. Scand. **24**, 79–92 (1969)

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