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Fast Rates of Convergence in the Central Limit Theorem

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1. Introduction and Summary

This paper concerns rates of convergence in the central limit theorem which are, in a sense, the fastest possible. Our results generalise earlier ones due to Rozovskii [23, 24] and the author [6], but a crucial difference is that here we derive the leading term in an asymptotic expansion of the error from normality. The leading term may be obtained under very general conditions on the summands, and this approach makes possible a more informative account of rates of convergence.

The leading term approach to rates of convergence may be applied very generally. For example, it can be used to derive upper and lower bounds to the rate of convergence in the Lindeberg-Feller theorem, the only requisite assumption being Lindeberg's condition. However, a general account of the leading term method is well beyond the scope of this paper. A more expansive development of this technique will be published elsewhere [8].

As in the earlier papers [23 and 6], we confine our attention to sums of independent and identically distributed random variables. The literature contains a variety of approaches to rates of convergence in this context; see Chap. V of Petrov [22] for an excellent introduction to such problems. Probably the best known are the inequalities of Berry and Esséen [1, 4], which admit many refinements and sharpenings. An approach of a very different nature may be found in the characterisations given by Ibragimov [14] and Heyde [9]. These provide simple descriptions of the influence of moment conditions and order-of-magnitude conditions on rates of convergence. They have been generalised and extended by many authors [6, 7, 10–12, 15, 17–19, 23]. Closely related to the Heyde-Ibragimov characterisations are the upper and lower bounds derived by Osipov [20, 21], Rozovskii [23, 24] and Hall [6, 7]. Most of the characterisations may be derived from these bounds, and on the other hand, the characterisations are strongly suggestive of upper and lower bounds. The technique of deriving characterisations from upper and lower bounds is illustrated in the proofs of Corollaries 1 and 2 of [6].

A very different way of describing the rate of convergence is to derive the first term in an asymptotic expansion; see for example the results of Cramér

[2, 3] and Höglund [13]. Such results usually hold only under stringent additional assumptions, such as moment and smoothness conditions (for Cramér's expansion) or regular variation of the distribution tails (for Höglund's expansion). One advantage of our leading term approach to rates of convergence is that it enables the theorems of Cramér and Höglund to be viewed as special cases of more general results. It also leads to many of the results described in the previous paragraph, and so provides a unified account of rates of convergence.

In the next section we derive a lower bound to the fastest rate of convergence which can be achieved in the central limit theorem. We also examine the rate of convergence for a specific sequence of norming constants, designed to achieve the order of magnitude of the lower bound. The latter result describes the rate of approximation to the normal error by the leading term in a series expansion. Properties of the leading term are discussed in Sect. 3, where we show that under appropriate conditions, this term is asymptotically equivalent to the first term in Cramér's expansion, or to the first term in Höglund's expansion. In Sect. 4 we outline extensions of our results to rates of convergence in local limit theorems. All proofs are deferred until Sect. 5.

We close this section with some notation. Throughout this paper we let $X, X_1, X_2, \ldots$ denote a sequence of independent and identically distributed random variables from the domain of attraction of the normal law; that is, such that for suitable constants a_n and b_n,

$$P\left(\sum_{j=1}^{n} X_j \leqq a_n x + b_n\right) \to \Phi(x), \quad -\infty < x < \infty,$$

where Φ denotes the standard normal distribution function. Set $S_n = \sum_{j=1}^{n} X_j$. A necessary and sufficient condition for the distribution of X to belong to the domain of attraction of the normal law is that the function

$$V(x) = E\{X^2 I(|X| \leqq x)\}, \qquad x > 0,$$

be slowly varying at infinity; see page 83 of Ibragimov and Linnik [14]. (We use the notation $I(E)$ to denote the indicator function of an event E.) This implies that all moments below the second are finite, and so there is no loss of generality in supposing that $E(X) = 0$. We make this assumption throughout.

Since $x^{-2} V(x) \to 0$ as $x \to \infty$, the function

$$a(x) = \sup\{a : a^{-2} V(a) \geqq x^{-1}\}$$

is well defined for all large x. For such values of x we have $x\{a(x)\}^{-2} V\{a(x)\} = 1$, even for discontinuous distributions. Let $c_n = a(n)$, for large n, and $\mu_n = E\{X I(|X| \leqq c_n)\}$.

If the variance of X is finite then $c_n \sim n^{\frac{1}{2}} (\operatorname{var} X)^{\frac{1}{2}}$ as $n \to \infty$, while if the variance is infinite, $c_n = n^{\frac{1}{2}} U(n)$, where U is a slowly varying function diverging to infinity.

2. Rates of Convergence

Before tackling the problem of rates of convergence for specific sequences of norming constants, it is desirable to determine the "fastest" rate of convergence, using optimal constants. This allows us to set a benchmark by means of which rates of convergence using specific constants may be compared. To this end, define

$$D_n(c,d) = \sup_{-\infty < x < \infty} |P(S_n \leqq cx+d) - \Phi(x)|$$

for arbitrary constants $c>0$ and d. For the special sequence of constants $\{c_n\}$ defined in Sect. 1, set

$$\delta_n = nP(|X|>c_n) + nc_n^{-4}E\{X^4 I(|X| \leqq c_n)\} + nc_n^{-3}|E\{X^3 I(|X| \leqq c_n)\}|.$$

Note that $\delta_n \to 0$ as $n \to \infty$, provided X is in the domain of attraction of the normal law. This fact may be verified by routine analytic methods, but it is also a corollary of the following theorem.

Theorem 1. *If the distribution of X is in the domain of attraction of the normal law, then*

$$\liminf_{n\to\infty} [\{\inf_{c>0,d} D_n(c,d) + n^{-1}\}/\delta_n] > 0.$$

That is, up to terms of order n^{-1} the rate of convergence in the central limit theorem can be no better than the order of δ_n.

This result may also be derived from Theorem 1 of Rozovskii [24]. We present a somewhat different argument here, which we use in deriving Theorem 3 and the local limit theorems of Sect. 4.

Let us consider the case of infinite variance in more detail. Fix $\lambda > 0$, define

$$C(\lambda) = (1/2)\left\{\int_\lambda^\infty x^{-1} P(|X|>x)\,dx\right\}^{\frac{1}{2}},$$

and note that

$$\begin{aligned}
\int_\lambda^{c_n} xP(|X|>x)\,dx &= \int_\lambda^{c_n} \{x^3 P(|X|>x)\}^{\frac{1}{2}} \{x^{-1} P(|X|>x)\}^{\frac{1}{2}}\,dx \\
&\leqq 2C(\lambda)\left\{\int_0^{c_n} x^3 P(|X|>x)\,dx\right\}^{\frac{1}{2}} \\
&= C(\lambda)[c_n^4 P(|X|>c_n) + E\{X^4 I(|X| \leqq c_n)\}]^{\frac{1}{2}} \\
&\leqq C(\lambda)(c_n^4 \delta_n/n)^{\frac{1}{2}}. \qquad (2.1)
\end{aligned}$$

Furthermore, in the case of infinite variance we have

$$\begin{aligned}
2\int_\lambda^{c_n} xP(|X|>x)\,dx &= c_n^2 P(|X|>c_n) - \lambda^2 P(|X|>\lambda) + E\{X^2 I(\lambda < |X| \leqq c_n)\} \\
&\geqq E\{X^2 I(|X| \leqq c_n)\}/2 \qquad (2.2)
\end{aligned}$$

for all large n. Combining (2.1) and (2.2) we find that

$$\delta_n \geqq \{1/16\,C^2(\lambda)\}[E\{X^2 I(|X| \leqq c_n)\}]^2 (n/c_n^4) = 1/16\,n\,C^2(\lambda)$$

for large n, and consequently

$$\liminf_{n\to\infty} n\delta_n \geqq 1/16\,C^2(\lambda).$$

However, it is clear that $C(\lambda)\to 0$ as $\lambda\to\infty$, and therefore $n\delta_n\to\infty$ as $n\to\infty$. In the case of finite variance, it is obvious that $n\delta_n\to\infty$ whenever $E(X^4)=\infty$. We may now deduce from Theorem 1 that

$$\liminf_{n\to\infty}\{\inf_{c>0,d} D_n(c,d)\}/\delta_n > 0 \tag{2.3}$$

for any distribution in the domain of attraction of the normal law and satisfying $E(X^4)=\infty$.

A common example of a distribution in the domain of attraction of the normal law, but not necessarily with finite variance, is one which has regularly varying tails of exponent -2. Let U be a positive slowly varying function, and suppose

$$P(|X|>x) \equiv x^{-2}U(x) \quad \text{and} \quad P(X>x) \sim pP(|X|>x)$$

as $x\to\infty$, where $0\leqq p\leqq 1$. Then it may be proved that

$$\delta_n \sim Cnc_n^{-2}U(c_n) = CU(c_n)/E\{X^2 I(|X|\leqq c_n)\}$$

as $n\to\infty$, where the constant C depends only on p. Therefore δ_n is a slowly varying function of n.

We now examine rates of convergence in the central limit theorem with explicit norming constants $c(n)$ and $d(n)$, with the goal of achieving the order of magnitude of the lower bound given by (2.3). Define the leading term function $L_n(x)$ by

$$L_n(x) = nE\{\Phi(x - X/c_n) - \Phi(x)\} + n(\mu_n/c_n)\phi(x) - \tfrac{1}{2}\phi'(x),$$

where $\phi(x)=\Phi'(x)$ is the standard normal density.

Theorem 2. *If the distribution of X is in the domain of attraction of the normal law, then*

$$\sup_{-\infty<x<\infty} |P(S_n \leqq c_n x + n\mu_n) - \Phi(x) - L_n(x)| = O(\delta_n^2 + c_n^{-1}) \tag{2.4}$$

as $n\to\infty$. If in addition the distribution of X is nonlattice, the right hand side of (2.4) *may be replaced by $O(\delta_n^2)+o(c_n^{-1})$.*

It will follow from Theorem 3 below that the quantity

$$\{\sup_{-\infty<x<\infty} |L_n(x)|\}/\delta_n$$

is bounded away from zero and infinity as $n \to \infty$, and so we may deduce the following corollary. This is closely related to Lemmas 1 and 2 of Rozovskii [23].

Corollary 1. *If the distribution of X is in the domain of attraction of the normal law, then the ratio*

$$\frac{\sup_{-\infty < x < \infty} |P(S_n \leqq c_n x + n\mu_n) - \Phi(x)| + c_n^{-1}}{\delta_n + c_n^{-1}}$$

is bounded away from zero and infinity as $n \to \infty$.

Thus, the rate of convergence with the norming constants c_n and $n\mu_n$ is of precise order δ_n, up to terms of order c_n^{-1}. If $E(X^2) < \infty$ then $c_n \sim n^{\frac{1}{2}}(\operatorname{var} X)^{\frac{1}{2}}$ as $n \to \infty$, while if $E(X^2) = \infty$, $c_n \sim n^{\frac{1}{2}} U(n)$, where U is a slowly varying function diverging to infinity.

The leading term $L_n(x)$ is related to one of the terms in an expansion suggested by Osipov [21, Theorem 1], who considered rates of convergence under the additional assumptions of finite variance and Cramér's continuity condition. It is possible to derive longer expansions than (2.4), and indeed we shall examine the second term in such expansions in order to prove (2.4). However, the leading term contains the great majority of information about rates of convergence, and so we have decided to study it in isolation.

3. Properties of the Leading Term

Our first result in this section shows that the uniform measure of L_n is of precise order δ_n.

Theorem 3. *There exists a universal constant C such that for all n,*

$$\sup_{-\infty < x < \infty} |L_n(x)| \leqq C\delta_n. \tag{3.1}$$

Furthermore,

$$\liminf_{n \to \infty} \{\sup_{-\infty < x < \infty} |L_n(x)|\}/\delta_n > 0. \tag{3.2}$$

In order to interprete this result, let us suppose that $c_n \delta_n \to \infty$ as $n \to \infty$. This will be the case if, for example,

$$E\{X^2 I(|X| \leqq x)\}/x^3 P(|X| > x) \to 0$$

as $x \to \infty$. Then it follows from Theorems 1, 2 and 3 that

$$0 < \liminf_{n \to \infty} \{\inf_{c > 0, d} D_n(c, d)\}/\delta_n \leqq \limsup_{n \to \infty} \{\inf_{c > 0, d} D_n(c, d)\}/\delta_n \leqq C_0 < \infty,$$

where C_0 is a universal constant.

Next we examine approximations to our leading term by the leading terms derived by other authors. We consider first the leading term in a Chebyshev-Edgeworth-Cramér expansion. Suppose $E(|X|^3) < \infty$, and (without loss of generality) that $E(X)=0$ and $E(X^2)=1$. Let $\tau=E(X^3)$, and note that $c_n \sim n^{\frac{1}{2}}$ as $n\to\infty$. The first term in a Chebyshev-Edgeworth-Cramér expansion is given by

$$P_n(x)=\frac{1}{6}\frac{\tau}{n^{\frac{1}{2}}}(1-x^2)\phi(x)=-\frac{1}{6}\frac{\tau}{n^{\frac{1}{2}}}\phi''(x).$$

Theorem 4. *Under the conditions above,*

$$\sup_{-\infty<x<\infty}|L_n(x)-P_n(x)|=o(n^{-\frac{1}{2}}) \tag{3.3}$$

as $n\to\infty$.

An immediate corollary of Theorems 2 and 4 is that when $E(|X|^3)<\infty$ and the distribution is nonlattice,

$$P(S_n \leqq c_n x+n\mu_n)=\Phi(x)-\frac{1}{6}\frac{\tau}{n^{\frac{1}{2}}}\phi''(x)+o(n^{-\frac{1}{2}}) \tag{3.4}$$

uniformly in x as $n\to\infty$. It is easy to see that the same result holds if c_n is replaced by $n^{\frac{1}{2}}$ and μ_n by zero.

Next we examine the leading term derived by Höglund [13], who considered the case of distributions with regularly varying tails. For the sake of brevity we examine only regular variation of order -2. Let $\tilde{\omega}_0$ denote the operator defined in relation (4) of [13], and set $Q(x)=\tilde{\omega}_0\Phi(x)$. Note that Q does not depend on n, but that Q depends on a parameter p where $0\leqq p\leqq 1$.

Theorem 5. *Suppose the function* $P(|X|>x)$ *is regularly varying at infinity with exponent* -2, *and that*

$$P(X>x)/P(|X|>x)\to p$$

as $x\to\infty$, *where* $0\leqq p\leqq 1$. *Then*

$$\sup_{-\infty<x<\infty}|L_n(x)-2nP(|X|>c_n)Q(x)|=o\{nP(|X|>c_n)\}$$

as $n\to\infty$.

The proof of Theorem 5 is straightforward, and uses well known properties of regularly varying functions. The proof will not be given here. See Seneta [25] for an account of the theory of regular variation.

4. Local Limit Theorems

Our purpose in this section is to show that the results of the earlier sections have immediate analogues in the case of convergence in local limit theorems. We consider first the local limit theorem for densities. The next result provides analogues of Theorems 1, 2 and 3.

Theorem 6. *Suppose that for some* n, S_n *has a bounded density. Let* $p_n(x;c,d)$ *denote the density of* $(S_n-d)/c$. *Then*

$$\liminf_{n\to\infty}[\inf_{c>0,d}\{\sup_{-\infty<x<\infty}|p_n(x;c,d)-\phi(x)|\}+n^{-1}]/\delta_n>0,$$

and also

$$\sup_{-\infty<x<\infty} |p_n(x; c_n, n\mu_n) - \phi(x) - L_n'(x)| = O(\delta_n^2 + n^{-1})$$

as $n\to\infty$. Furthermore, there exists a universal constant C such that for all n,

$$\sup_{-\infty<x<\infty} |L_n'(x)| \leqq C\delta_n,$$

and

$$\liminf_{n\to\infty}\{\sup_{-\infty<x<\infty} |L_n'(x)|\}/\delta_n > 0.$$

Note that the condition that S_n have a bounded density is both necessary and sufficient for the uniform convergence of densities; see Theorem 7, p. 198 of Petrov [22].

Now we examine lattice distributions.

Theorem 7. *Suppose X takes only values of the form $r+Ns$ ($N=0, \pm 1, \pm 2, \ldots$), where $s>0$ is the maximal span of the lattice. Then*

$$\liminf_{n\to\infty}(\inf_{c>1,d}[\sup_{-\infty<N<\infty} |s^{-1}cP(S_n=nr+Ns) - \phi\{(nr+Ns-d)/c\}|] + n^{-1}) / \delta_{n1} > 0, \tag{4.1}$$

and also

$$\sup_{-\infty<N<\infty} |s^{-1}c_nP(S_n=nr+Ns) - \phi\{(nr+Ns-n\mu_n)/c_n\} - L_n'\{(nr+Ns-n\mu_n)/c_n\}| = O(\delta_n^2 + n^{-1}) \tag{4.2}$$

as $n\to\infty$. Furthermore, there exists a universal constant C such that for all n,

$$\sup_{-\infty<x<\infty} |L_n'(x)| \leqq C\delta_n, \tag{4.3}$$

and

$$\liminf_{n\to\infty}[\sup_{-\infty<N<\infty} |L_n'\{(nr+Ns-n\mu_n)/c_n\}|]/\delta_n > 0. \tag{4.4}$$

It is appropriate here to mention an improvement of Theorem 2 which can be obtained for lattice distributions. Define the "rounding error" function, R, by $R(x)=[x]-x+\frac{1}{2}$, where $[x]$ denotes the integer part of x, and set

$$R_n(x) = R[\{c_nx + n(\mu_n - r)\}/s]\,\phi(x)s/c_n.$$

The techniques used to prove Theorem 7 may be modified to show that under the same conditions,

$$\sup_{-\infty<x<\infty} |P(S_n \leqq c_nx + n\mu_n) - \Phi(x) - L_n(x) - R_n(x)| = O(\delta_n^2) + o(c_n^{-1}) \tag{4.5}$$

as $n\to\infty$ (a sharpening of (2.4)). If $E(|X|^3)<\infty$, $E(X^2)=1$ and $E(X)=0$, and the distribution is lattice, we may deduce an analogue of (3.4) from (3.3) and (4.5):

$$P(S_n \leqq c_nx + n\mu_n) = \Phi(x) - \frac{1}{6}\frac{\tau}{n^{\frac{1}{2}}}\phi''(x) + R_n(x) + o(n^{-\frac{1}{2}})$$

uniformly in x as $n\to\infty$.

The results of Theorems 6 and 7 may be applied in the same manner as those of Theorems 1–3. For example, they can be used to derive local analogues of the result (2.3), Corollary 1 and Theorems 4 and 5. Only minor modifications of the earlier techniques are required.

5. Proofs

The symbol C denotes a positive generic constant, not depending on n or t, and the characteristic function of X is denoted by α. It follows from the slow variation of $E\{X^2 I(|X|\leqq x)\}$ as $x\to\infty$ that

$$E\{|X|I(|X|>x)\}=o[x^{-1}E\{X^2 I(|X|\leqq x)\}] \tag{5.1}$$

as $x\to\infty$. This result will be used during the proofs.

Proof of Theorem 1. Choose $a_n>0$ and b_n such that

$$D_n(a_n, b_n)\leqq 2 \inf_{c>0,d} D_n(c,d),$$

and let β_n denote the characteristic function of $(S_n-b_n)/a_n$. The techniques used to derive the inequality (3.20) of [14] lead to the result

$$\left|\int_0^1 (1-t)\{\beta_n(tz)e^{(tz)^2/2}-1\}\,dt\right|\leqq C(z)D_n(a_n,b_n). \tag{5.2}$$

In the next few lines we shall use this inequality to derive more serviceable lower bounds.

Observe that

$$\begin{aligned}\log\{\beta_n(t)e^{t^2/2}\}&=n\{\alpha(t/a_n)-1\}+t^2/2-itb_n/a_n+r_{n1}(t)\\&=\Delta_n(t)+r_{n1}(t),\end{aligned} \tag{5.3}$$

say, where $|r_{n1}(t)|\leqq n|\alpha(t/a_n)-1|^2$ whenever $|\alpha(t/a_n)-1|\leqq\frac{1}{2}$. But

$$|\alpha(t)-1|\leqq 3[t^2E\{X^2I(|X|\leqq c_n)\}+tE\{|X|I(|X|>c_n)\}]$$

for all $t\geqq 0$, and so $|\alpha(t)-1|\leqq C(t^2+c_n^{-1}t)E\{X^2I(|X|\leqq c_n)\}$, using (5.1). Consequently,

$$|r_{n1}(t)|\leqq Ct^2(1+t^2)/n \tag{5.4}$$

when $|\alpha(t/a_n)-1|\leqq\frac{1}{2}$. Since $\beta_n(t)e^{t^2/2}\to 1$ uniformly on compact intervals,

$$|\log\{\beta_n(t)e^{t^2/2}\}-\{\beta_n(t)e^{t^2/2}-1\}|\leqq C(z)|\beta_n(t)e^{t^2/2}-1|^2$$

uniformly in $0\leqq t\leqq z$. Combining the estimates from (5.3) down we obtain

$$|\beta_n(t)e^{t^2/2}-1-\Delta_n(t)|\leqq C(z)\{|\beta_n(t)e^{t^2/2}-1|^2+n^{-1}\},$$

which when substituted into (5.2) yields

$$\left|\int_0^1 (1-t)\Delta_n(tz)\,dt\right| \leqq C_1(z)\left\{D_n(a_n, b_n) + \int_0^1 |\beta_n(tz)e^{(tz)^2/2} - 1|^2\,dt + n^{-1}\right\}$$
$$\leqq C_2(z)\{D_n(a_n, b_n) + n^{-1}\}, \tag{5.5}$$

the last inequality following via Parseval's identity.

The proof of Theorem 1 may be completed by using techniques from the proof of Theorem 1 of [6]. In particular, by taking real parts in (5.5) for $z=1$ and $z=2$, we may deduce that

$$n[P(|X|>a_n) + a_n^{-4}E\{X^4 I(|X|\leqq a_n)\}] \leqq C\{D_n(a_n, b_n) + n^{-1}\},$$

and by taking imaginary parts, that

$$a_n^{-3}|E\{X^3 I(|X|\leqq a_n)\}| \leqq C\{D_n(a_n, b_n) + n^{-1}\}.$$

Since $a_n/c_n \to 1$ as $n\to\infty$, these two inequalities yield Theorem 1.

Proof of Theorem 2. Redefine $\Delta_n(t) = n\{\alpha(t/c_n) - 1\} + t^2/2 - itn\mu_n/c_n$. Arguing as in the proof of Theorem 1, we may write

$$\log\{\alpha^n(t/c_n)\exp(t^2/2 - itn\mu_n/c_n)\} = \Delta_n(t) + r_{n1}(t),$$

where $|r_{n1}(t)| \leqq n|\alpha(t/c_n) - 1|^2$ whenever $|\alpha(t/c_n) - 1| \leqq \frac{1}{2}$. Thus,

$$\alpha^n(t/c_n)\exp(t^2/2 - itn\mu_n/c_n) = \sum_{k=0}^{2}\{\Delta_n(t) + r_{n1}(t)\}^k/k! + r_{n2}(t), \tag{5.6}$$

where $|r_{n2}(t)| \leqq |\Delta_n(t) + r_{n1}(t)|^3 \exp\{|\Delta_n(t) + r_{n1}(t)|\}$. We shall prove shortly that there exist constants ε_3, $C_3 > 0$ such that

$$|\Delta_n(t) + r_{n1}(t)| \leqq -C_3 t^{\frac{3}{2}} + t^2/2 + t + 2 \tag{5.7}$$

for all large n and all $0 \leqq t \leqq \varepsilon_3 c_n$, and that

$$|\Delta_n(t)|^3 \leqq 144t(1+t^{11})\delta_n^2\delta_{n1}. \tag{5.8}$$

When these estimates and (5.4) are substituted into (5.6), we may deduce that

$$\alpha^n(t/c_n)\exp(-itn\mu_n/c_n) = \{1 + \Delta_n(t) + \Delta_n^2(t)/2\}e^{-t^2/2} + r_{n3}(t), \tag{5.9}$$

where

$$|r_{n3}(t)| \leqq C_4 t(1+t^{11})(\delta_n^2\delta_{n1} + n^{-1})\exp(-C_3 t^{\frac{3}{2}} + t) \tag{5.10}$$

for all large n and $0 \leqq t \leqq \varepsilon_3 c_n$. The proof of Theorem 2 will be completed by applying the smoothing inequality to the estimate (5.9).

To prove (5.7), observe that by [16, pp. 90–91], $1 - \alpha(t) = \frac{1}{2}t^2 V(t^{-1})\{1 + o(1)\}$ as $t \downarrow 0$. Therefore there exists $\varepsilon_1 \in (0,1)$ such that

$$|E(\sin tX)| \leqq (1/2)E(1 - \cos tX), \qquad E(1 - \cos tX) \geqq (1/4)t^2 V(t^{-1}) \tag{5.11}$$

for $0 < t \leqq \varepsilon_1$. Furthermore, $|\alpha(t) - 1| \leqq Ct^2 V(t^{-1})$ for all $t \in (0, \varepsilon_1]$. Choose $\varepsilon_2 \in (0, \varepsilon_1]$ so small that for this C,

$$C|\alpha(t)-1|\leqq 1/16 \tag{5.12}$$

when $t\in(0,\varepsilon_2]$. An argument based on truncations, using the fact that $\cos x-1+x^2/2$ is nonnegative for all x, produces the bound

$$|\Delta_n(t)|\leqq nE\{\cos(tX/c_n)-1\}+t^2/2+4nP(|X|>c_n)$$
$$+n|E\{\sin(tX/c_n)\}|+|tn\mu_n/c_n|.$$

From (5.11) and the fact that $4nP(|X|>c_n)+|tn\mu_n/c_n|\leqq 1+t$ for all $t\geqq 0$ and all large n, we may now deduce that

$$|\Delta_n(t)|\leqq -(n/2)E\{1-\cos(tX/c_n)\}+t^2/2+1+t$$
$$\leqq -(t^2/8)V(c_n/t)/V(c_n)+t^2/2+1+t$$

for all large n and $0\leqq t\leqq \varepsilon_2 c_n$. But with C as in (5.12),

$$|r_{n1}(t)|\leqq nC(t/c_n)^2 V(c_n/t)|\alpha(t/c_n)-1|\leqq (t^2/16)V(c_n/t)/V(c_n)$$

for $0\leqq t\leqq \varepsilon_2 c_n$, and so

$$|\Delta_n(t)+r_{n1}(t)|\leqq -(t^2/16)V(c_n/t)/V(c_n)+t^2/2+1+t. \tag{5.13}$$

Let $W(x)=x^{-\frac{1}{2}}V(x)$. Using an analogue of Proposition 4°, pp. 19–20 of [25], with $\gamma>0$ replaced by $\gamma<0$, it follows that $\{\sup_{x\geqq y} W(x)\}/W(y)\to 1$ as $y\to\infty$, whence $\{\sup_{x\geqq y} W(x)\}/W(y)\leqq C_1(>1)$ for $y\geqq C_2$, say. If $1\leqq t\leqq c_n/C_2$ then

$$W(c_n/t)/W(c_n)\geqq W(c_n/t)/\{\sup_{x\geqq c_n/t} W(x)\}\geqq 1/C_1, \tag{5.14}$$

and setting $\varepsilon_3=\min(\varepsilon_2, 1/C_2)$ and $C_3=1/16C_1$ we may deduce from (5.13) and (5.14) that $|\Delta_n(t)+r_{n1}(t)|\leqq -C_3 t^{\frac{3}{2}}+t^2/2+1+t$ for large n and $1\leqq t\leqq \varepsilon_3 c_n$. This inequality is trivial for $0\leqq t\leqq 1$, provided the term $-C_3 t^{\frac{3}{2}}$ is dropped. Since $C_3<1/16$, (5.7) is immediate.

In order to prove (5.8), we note that several quite simple manipulations and truncations produce the bounds $|\Delta_n(t)|\leqq 3(1+t^4)\delta_n$ and $|\Delta_n(t)|\leqq 2t(1+t^3)\delta_{n1}$, for all $t\geqq 0$. Therefore

$$|\Delta_n(t)|^3\leqq \{3(1+t^4)\delta_n\}^2\{2t(1+t^3)\delta_{n1}\}\leqq 144t(1+t^{11})\delta_n^2\delta_{n1}.$$

This proves (5.9) and (5.10). Next, define $\Psi(x)=\Phi(2^{\frac{1}{2}}x)$ and

$$M_n(x)=nE\{\Psi(x-X/c_n)-\Psi(x)\}+(n\mu_n/c_n)\Psi'(x)-\tfrac{1}{2}\Psi''(x).$$

The Fourier-Stieltjes transforms of L_n and M_n^{*2} (the two-fold convolution of M_n) are given by

$$\int e^{itx}dL_n(x)=\Delta_n(t)e^{-t^2/2} \quad \text{and} \quad \int e^{itx}dM_n^{*2}(x)=\Delta_n^2(t)e^{-t^2/2}.$$

Applying the smoothing inequality (Theorem 2, p. 109 of [22]) with $T=\varepsilon_3 c_n$, "$F(x)$"$=P(S_n\leqq c_n x+n\mu_n)$ and "$G(x)$"$=\Phi(x)+L_n(x)+\frac{1}{2}M_n^{*2}(x)$, we may deduce from (5.9) and (5.10) that

$$\sup_{-\infty<x<\infty} |P(S_n \leqq c_n x) - \Phi(x) - L_n(x) - \tfrac{1}{2} M_n^{*2}(x)|$$
$$= O(\delta_n^2 \delta_{n1} + n^{-1} + c_n^{-1}) + O(\delta_n^2 + c_n^{-1}),$$

provided

$$\sup_{x,n} |(d/dx)\{L_n(x) + \tfrac{1}{2} M_n^{*2}(x)\}| < \infty. \tag{5.15}$$

Thus, Theorem 2 will follow if (5.15) and

$$\sup_{-\infty<x<\infty} |M_n^{*2}(x)| = O(\delta_n^2) \tag{5.16}$$

hold. These estimates may be derived by routine analysis; for the methodology, see [8, pp. 105–108].

Proof of Theorem 3. The upper bound (3.1) follows easily from the expansion

$$\begin{aligned} L_n(x) = nE[\{-\tfrac{1}{6}(X/c_n)^3 \phi''(x) + \tfrac{1}{24}(X/c_n)^4 \phi'''(x_n^*)\} I(|X| \leqq c_n)] \\ + nE[\{\Phi(x - X/c_n) - \Phi(x)\} I(|X| > c_n)], \end{aligned} \tag{5.17}$$

in which x_n^* denotes a random variable taking values between $x-1$ and $x+1$. To derive the lower bound (3.2), observe that if we replace $\{\beta_n(t) e^{t^2/2} - 1\} e^{-t^2/2}$ by $\Delta_n(t) e^{-t^2/2}$ (equal to the Fourier-Stieltjes transform of L_n), we may use a somewhat simpler argument than that in the proof of Theorem 1 to deduce instead of (5.5) that

$$\left| \int_0^1 (1-t) \Delta_n(tz)\, dt \right| \leqq C(z) \sup_{-\infty<x<\infty} |L_n(x)|,$$

for $z>0$. The left hand side of this inequality is equal to the left hand side of (5.5), provided we make the substitution $a_n = c_n$ and $b_n = n\mu_n$. The argument following (5.5) may now be used to prove (3.2).

Proof of Theorem 4. From the expansion (5.17) we may deduce that if $E(|X|^3) < \infty$,

$$\begin{aligned} \sup_{-\infty<x<\infty} |L_n(x) + \tfrac{1}{6} n c_n^{-3} E(X^3) \phi''(x)| \leqq Cn[c_n^{-3} E\{|X|^3 I(|X| > c_n)\} + P(|X| > c_n) \\ + c_n^{-4} E\{X^4 I(|X| \leqq c_n)\}]. \end{aligned} \tag{5.18}$$

Note that $c_n \sim n^{\frac{1}{2}} \operatorname{var}(X)$, $P(|X| > c_n) \leqq c_n^{-3} E\{|X|^3 I(|X| > c_n)\} = o(n^{-\frac{3}{2}})$, and for each $\varepsilon \in (0,1)$,

$$\begin{aligned} x^{-1} E\{X^4 I(|X| \leqq x)\} &\leqq \varepsilon E(|X|^3) + x^{-1} E\{X^4 I(\varepsilon x < |X| \leqq x)\} \\ &\leqq \varepsilon E(|X|^3) + E\{|X|^3 I(|X| > \varepsilon x)\} \to \varepsilon E(|X|^3) \end{aligned}$$

as $x \to \infty$. Therefore $x^{-1} E\{X^4 I(|X| \leqq x)\} \to 0$ as $x \to \infty$, and so the right hand side of (5.18) equals $o(n^{-\frac{1}{2}})$ as $n \to \infty$. Theorem 4 follows immediately.

The proof of Theorem 6 closely resembles the proofs of Theorems 1, 2 and 3, and will not be given here. We prove instead Theorem 7.

Proof of Theorem 7. We shall derive the results in the order (4.2), (4.3), (4.1) and (4.4). Set

$$x_{nN}=(nr+Ns-n\mu_n)/c_n, \qquad p_{nN}=P\{(S_n-n\mu_n)/c_n=x_{nN}\}$$

and $K_n(x)=M_n^{*2}(x)$, where M_n is as defined during the proof of Theorem 2. The argument leading to (5.16) may be used to prove that

$$\sup_{-\infty<x<\infty}|K_n'(x)|=O(\delta_n^2),$$

and so (4.2) will follow if we prove that

$$\sup_{-\infty<N<\infty}|s^{-1}c_nP(S_n=nr+Ns)-\phi(x_{nN})-L_n'(x_{nN})-K_n'(x_{nN})|=O(\delta_n^2\delta_{n1}+n^{-1}) \quad (5.19)$$

as $n\to\infty$.

The characteristic function of $(S_n-n\mu_n)/c_n$ is given by

$$\xi_n(t)=\alpha^n(t/c_n)\exp(-itn\mu_n/c_n)=\sum_{M=-\infty}^{\infty}p_{nM}\exp(itx_{nM}), \quad (5.20)$$

and the function $\psi_n(x)=\phi(x)+L_n'(x)+K_n'(x)$ has Fourier transform

$$\eta_n(t)=\int_{-\infty}^{\infty}e^{itx}\psi_n(x)\,dx=\{1+\Delta_n(t)+\tfrac{1}{2}\Delta_n^2(t)\}e^{-t^2/2}. \quad (5.21)$$

We shall derive an analogue of the smoothing inequality, based on these transforms. Multiply both sides of (5.20) by $\exp(-itx_{nN})$ for an integer N, and integrate from $-\pi c_n/s$ to $\pi c_n/s$; and subtract from this result an inversion of the transform (5.21) at $x=x_{nN}$; to obtain the formula

$$2\pi\{s^{-1}c_np_{nN}-\psi_n(x_{nN})\}=\int_{-\pi c_n/s}^{\pi c_n/s}\{\xi(t)-\eta_n(t)\}\exp(-itx_{nN})\,dt$$
$$-\int_{|t|>\pi c_n/s}\eta_n(t)\exp(-itx_{nN})\,dt. \quad (5.22)$$

Since $c_n=n^{\frac{1}{2}}U(n)$ where U is slowly varying at infinity, then for any $\varepsilon>0$,

$$\int_{|t|>\varepsilon c_n}|\eta_n(t)|\,dt\leqq Cn^2\int_{|t|>\varepsilon c_n}^{\infty}t^4e^{-t^2/2}\,dt=O(n^{-p})$$

for all $p>0$. Therefore it follows from (5.22) that if $0<\varepsilon\leqq\pi/s$,

$$\sup_{-\infty<N<\infty}|s^{-1}c_np_{nN}-\psi_n(x_{nN})|\leqq\pi^{-1}\int_0^{\varepsilon c_n}|\xi_n(t)-\eta_n(t)|\,dt$$
$$+\pi^{-1}\int_{\varepsilon c_n}^{\pi c_n/s}|\xi_n(t)|\,dt+O(n^{-p}) \quad (5.23)$$

as $n \to \infty$, for all $p > 0$. In view of the estimates (5.9) and (5.10), we may choose $\varepsilon \in (0, \pi/s]$ such that for positive constants C_1 and C_2, all large n and all t in the range $0 \leqq t \leqq \varepsilon c_n$,

$$|\xi_n(t) - \eta_n(t)| \leqq C_1 t(1 + t^{11})(\delta_n^2 \delta_{n1} + n^{-1}) \exp(-C_2 t^{\frac{3}{2}} + t).$$

For such a choice of ε,

$$\int_0^{\varepsilon c_n} |\xi_n(t) - \eta_n(t)|\, dt = O(\delta_n^2 \delta_{n1} + n^{-1}) \tag{5.24}$$

as $n \to \infty$. Since the underlying distribution is lattice with minimal span s then α is periodic of period $2\pi/s$, and $|\alpha(t)| < 1$ for $0 < |t| < 2\pi/s$. Consequently there exists $c > 0$ such that $|\alpha(t)| < e^{-c}$ for $\varepsilon \leqq |t| \leqq \pi/s$, whence

$$\int_{\varepsilon c_n}^{\pi c_n/s} |\xi_n(t)|\, dt = c_n \int_{\varepsilon}^{\pi/s} |\alpha^n(t)|\, dt \leqq c_n e^{-nc} \pi/s = O(n^{-p}) \tag{5.25}$$

for all $p > 0$. On substituting (5.24) and (5.25) into (5.23) we may deduce (5.19), thereby completing the proof of (4.2) in Theorem 7.

The result (4.3) may be derived in the same manner as (3.1). We may now deduce that

$$\sup_{-\infty < N < \infty} P(S_n = nr + Ns) = O(c_n^{-1}) \tag{5.26}$$

as $n \to \infty$. This result puts us in a position to prove (4.1). Choose $a_n > 1$ and b_n such that

$$\sup_{-\infty < N < \infty} |s^{-1} a_n P(S_n = nr + Ns) - \phi\{(nr + Ns - b_n)/a_n\}|$$
$$\leqq 2 \inf_{c > 1, d} [\sup_{-\infty < N < \infty} |s^{-1} c P(S_n = nr + Ns) - \phi\{(nr + Ns - d)/c\}]. \tag{5.27}$$

Since $\sup_{-\infty < N < \infty} \phi\{(nr + Ns - b_n) a_n\} \geqq \phi(-s/a_n) \geqq \phi(-s) > 0$, and since (5.27) implies

$$s^{-1} a_n \sup_{-\infty < N < \infty} P(S_n = nr + Ns) - \sup_{-\infty < N < \infty} \phi\{(nr + Ns - b_n)/a_n\} \to 0, \tag{5.28}$$

we may deduce that

$$\liminf_{n \to \infty} \{a_n \sup_{-\infty < N < \infty} P(S_n = nr + Ns)\} > 0. \tag{5.29}$$

Conditions (5.26) and (5.29) together imply that c_n/a_n is bounded as $n \to \infty$, and thence that $a_n \to \infty$. Hence $\sup_{-\infty < N < \infty} \phi\{(nr + Ns - b_n)/a_n\} \sim \phi(0)$ as $n \to \infty$, and so in view of (5.28),

$$a_n \sim s\phi(0)/\{\sup_{-\infty < N < \infty} P(S_n = nr + Ns)\}. \tag{5.30}$$

It follows from (4.2) and (4.3) that the same asymptotic relation holds with a_n replaced by c_n, and therefore $a_n/c_n \to 1$ as $n \to \infty$.

Let us write $X=r+sY$ and $X_j=r+sY_j$, $j\geqq 1$, where the random variables Y and Y_j take only integer values. Set $b'_n=(b_n-nr)/s$ and $T_n=\sum_{j=1}^{n} Y_j$. In the work which follows we shall drop the subscripts n from a_n, b_n and b'_n. Observe first that

$$s^{-1}aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\}=s^{-1}aP(T_n=N)-\phi\{(N-b')/s^{-1}a\} \quad (5.31)$$

and

$$\phi\{(N-b')/s^{-1}a\}=(a/2\pi s)\int_{-\pi}^{\pi}\exp\{itb'-t^2(a/s)^2/2-itN\}\,dt+r_{nN},$$

where

$$|r_{nN}|\leqq\pi^{-1}\int_{\pi a/s}^{\infty}e^{-t^2/2}\,dt, \qquad -\infty<N<\infty. \quad (5.32)$$

Therefore in view of (5.31), with β denoting the characteristic function of Y,

$$\begin{aligned} &s^{-1}aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\} \\ &\quad=(a/2\pi s)\int_{-\pi}^{\pi}[\beta^n(t)-\exp\{itb'-t^2(a/s)^2/2\}]e^{-itN}\,dt-r_{nN}. \end{aligned} \quad (5.33)$$

Suppose $0<w\leqq\pi$, and define the function f_w on $(-\pi,\pi]$ by

$$f_w(t)=\begin{cases} t(w-t)\exp\{-itb'+t^2(a/s)^2/2\} & \text{for } 0\leqq t\leqq w,\\ 0 & \text{otherwise.}\end{cases}$$

Extend f_w from $(-\pi,\pi]$ to $(-\infty,\infty)$ by periodicity, and let

$$a_N(w)=(1/2\pi)\int_{-\pi}^{\pi}f_w(t)e^{itN}\,dt.$$

Then from (5.32), (5.33) and Parseval's identity for finite Fourier transforms,

$$\begin{aligned} &\sum_{N=-\infty}^{\infty}[s^{-1}aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\}]\,a_N(w) \\ &\quad=(w^3a/2\pi s)\int_0^1[\beta^n(tw)\exp\{-itwb'+t^2w^2(a/s)^2/2\}-1]\,t(1-t)\,dt+r_n, \end{aligned} \quad (5.34)$$

where $|r_n|\leqq\left\{\sum_{N=-\infty}^{\infty}|a_N(w)|\right\}\pi^{-1}\int_{\pi a/s}^{\infty}e^{-t^2/2}\,dt$. Let $z=wa/s$. We shall take z to be a fixed positive number, which entails $w\to 0$ as $n\to\infty$. Two integrations by parts in the formula for $a_N(w)$ enable us to prove that

$$|a_N(zs/a)|\leqq C(z)(s/a)^3\min\{1,(a/s)^2(N-b')^{-2}\},$$

from which we may deduce that $\sum_{N=-\infty}^{\infty}|a_N(zs/a)|=0(a^{-2})$ as $n\to\infty$. Using the estimates from (5.34) down, and making the substitution $w=zs/a$ in (5.34), we find that

$$\sup_{-\infty<N<\infty} |s^{-1} aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\}|$$

$$\geqq C_1(z)\, a^2 \left|\sum_{N=-\infty}^{\infty} [s^{-1} aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\}]\, a_N(w)\right|$$

$$C_2(z)\left|\int_0^1 [\alpha^n(tz/a)\exp\{-itzb/a+(tz)^2/2\}-1]\, t(1-t)\, dt\right|+O(n^{-p}), \tag{5.35}$$

for all $p>0$. A slight modification of the argument leading to (5.5) may now be used to derive the following estimate:

$$\left|\int_0^1 t(1-t)\,[n\{\alpha(tz/a)-1\}+(tz)^2/2-itzb/a]\, dt\right|$$

$$\leqq C(z)\Bigg[\sup_{-\infty<N<\infty} |s^{-1} aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\}|$$

$$+\int_0^1 |\alpha^n(tz/a)\exp\{-itzb/a+(tz)^2/2\}-1|^2\, dt+n^{-1}\Bigg]. \tag{5.36}$$

In view of (5.33) and Parseval's identity, the integral on the right equals

$$(a/zs)\int_0^{zs/a} |\beta^n(t)\exp\{-itb'+t^2(a/s)^2/2\}-1|^2\, dt$$

$$\leqq C(z)\, a^{-1}\sum_{N=-\infty}^{\infty} |s^{-1} aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\}+r_{nN}|^2$$

$$\leqq 2\, C(z)\sup_{-\infty<N<\infty} |s^{-1} aP(S_n=nr+Ns)-\phi\{(nr+Ns-b)/a\}|$$

$$\times\left[s^{-1}+a^{-1}\sum_{N=-\infty}^{\infty}\phi\{(nr+Ns-b)/a\}\right]+2\, C_2(z)\, a^{-1}\sum_{N=-\infty}^{\infty}|r_{nN}|^2. \tag{5.37}$$

An integral approximation shows that $a^{-1}\sum_{N=-\infty}^{\infty}\phi\{(nr+Ns-b)/a\}$ is bounded as $n\to\infty$. To estimate the second series on the right in (5.37), observe that

$$r_{nN}=(a/\pi s)\int_{\pi}^{\infty}\cos\{t(b'-N)\}\exp\{-t^2(a/s)^2/2\}\, dt.$$

Following an integration by parts and an estimation like (5.32), we may deduce that $|r_{nN}|\leqq C(p)\, n^{-p}\min(1,|b'-N|^{-1})$ for all $p>0$. Therefore

$$\sum_{N=-\infty}^{\infty}|r_{nN}|^2=O(n^{-p})$$

as $n\to\infty$, for all $p>0$. Combining the results from (5.36) down, we obtain the following analogue of (5.5):

$$\left|\int_0^1 t(1-t)\,[n\{\alpha(tz/a_n)-1\}+(tz)^2/2-itzb_n/a_n]\,dt\right|$$
$$\leqq C(z)\,[\sup_{-\infty<N<\infty}|s^{-1}a_n P(S_n=nr+Ns)-\phi\{(nr+Ns-b_n)/a_n\}$$
$$+n^{-1}].$$

The proof of (4.1) may be completed using the argument following (5.5).

Finally we prove (4.4). By inverting the Fourier transform

$$\Delta_n(t)\,e^{-t^2/2}=\int_{-\infty}^{\infty} e^{itx}\,L'_n(x)\,dx,$$

we may deduce that

$$L'_n(x_{nN})=(c_n/2\pi s)\int_{-\pi}^{\pi}\Delta_n(tc_n/s)\exp\{itb'_n-itN-t^2(c_n/s)^2/2\}\,dt+r'_{nN}, \qquad (5.38)$$

where on this occasion we have set $b'_n=n(\mu_n-r)/s$, and where

$$|r'_{nN}|\leqq Cn\int_{\pi c_n/s}^{\infty} t^2 e^{-t^2/2}\,dt.$$

Let the function $f_w(t)$ and the constants $a_N(w)$, $-\infty<N<\infty$, have the meanings ascribed to them during the proof of (4.1), with the substitution $a=c_n$. From (5.38) we may deduce an analogue of (5.34), with the left hand side replaced by

$$\sum_{N=-\infty}^{\infty} L'_n(x_{nN})\,a_N(w).$$

This leads ultimately to an analogue of (5.35), and thence to an analogue of (5.36). Note that on this occasion, the term n^{-1} on the right in (5.36) may be replaced by n^{-p}, for any $p>0$, and that if $p>1$, $n^{-p}=o(\delta_n)$. The proof of (4.4) is now easily completed.

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