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Upper Bounds for k-th Maximal Spacings

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Summary. If K_n denotes the k-th maximal spacing generated by an i.i.d. sequence of random variables uniformly distributed on $(0, 1)$, we show that for any $p \geqq 3$,

$$P(nK_n \geqq \operatorname{Log} n + (2\operatorname{Log}_2 n + \operatorname{Log}_3 n + \ldots + (1+\varepsilon)\operatorname{Log}_p n)/k \text{ i.o.}) = 1 \quad \text{or} \quad 0$$

according as $\varepsilon \leqq 0$ or $\varepsilon > 0$. We also obtain strong limiting bounds for the record times and inter-record times of K_n.

1. Introduction

Let $U_1, U_2, \ldots$ be a sequence of independent uniformly distributed random variables on $[0, 1]$. Let $U_0^{(n)} = 0 < U_1^{(n)} < \ldots < U_n^{(n)} < U_{n+1}^{(n)} = 1$ be the order statistic corresponding to $0, 1, U_1, \ldots, U_n$. The corresponding spacings are defined by

$$S_i^{(n)} = U_i^{(n)} - U_{i-1}^{(n)}, \qquad 1 \leqq i \leqq n+1.$$

Let $K_{n,1} > K_{n,2} > \ldots > K_{n,n+1}$ be the order statistic of the spacings. For any $k \leqq n$, $K_{n,k}$ will be called the k-th largest spacing of order n, and $M_n = K_{n,1}$ the maximal spacing of order n.

The upper class for M_n, $n \uparrow \infty$ is yet known and given by

$$P(nM_n \geqq \operatorname{Log} n + 2\operatorname{Log}_2 n + \operatorname{Log}_3 n + \ldots + (1+\varepsilon)\operatorname{Log}_p n \text{ i.o.}) = 0 \quad \text{or} \quad 1, \tag{1}$$

according as $\varepsilon > 0$ (Devroye [4]) or $\varepsilon \leqq 0$ (Deheuvels [2]), for any $p \geqq 3$, where Log_j is the j-th times iterated logarithm.

For $k \geqq 2$, however, the best results available up to now are:

$$P\left(nK_{n,k} \geqq \operatorname{Log} n + \frac{1}{k}(2\operatorname{Log}_2 n + \operatorname{Log}_3 n + \ldots + (1+\varepsilon)\operatorname{Log}_p n) \text{ i.o.}\right) = 0 \tag{2}$$

for any $\varepsilon > 0$ and $p \geqq 3$, and

$$P\left(nK_{n,k} \geqq \operatorname{Log} n + \frac{2-\varepsilon}{k} \operatorname{Log}_2 n \text{ i.o.}\right) = 1, \tag{3}$$

for any $\varepsilon > 0$ (Devroye [4]).

The aim of the following is to make this result more precise, by proving:

Theorem 1. *For any fixed $k \geqq 1$, if $K_n = K_{n,k}$ is the k-th largest spacing of order n, then*

$$P\left(nK_n \geqq \operatorname{Log} n + \frac{1}{k}(2 \operatorname{Log}_2 n + \operatorname{Log}_3 n + \ldots + (1+\varepsilon) \operatorname{Log}_p n\right) = 0 \quad \text{or} \quad 1,$$

according as $\varepsilon > 0$ or $\varepsilon \leqq 0$.

In the proof, we shall make use of a sequence of random stopping times defined on $U_1, U_2, \ldots$ in the following way. For any $n \geqq k$, note by $I_n = I_{n,k}$ the union of the spacing intervals corresponding to $K_{n,1}, K_{n,2}, \ldots, K_{n,k}$. The length of I_n will be denoted by $L_n = K_{n,1} + \ldots + K_{n,k}$. Put now

$$\begin{aligned} &N_1 = k \\ &N_n = \operatorname{Inf}\{m > N_{n-1};\, L_m < L_{N_{n-1}}\}, \qquad n = 2, 3, \ldots. \end{aligned} \tag{4}$$

The definition of $N_1, N_2, \ldots$ corresponds to the fact that $K_{n,1}, \ldots, K_{n,k}$ remain unchanged when n varies between N_{k-1} and N_k. When n takes the value N_k, one of them decreases, U_n falling then in $I_{n,k}$.

Our main results about $N_1, N_2, \ldots$ are the following:

Theorem 2. *For any fixed $k \geqq 1$, if $N_1, N_2, \ldots$ is defined by* (4), *then*

$$N_n = \exp\left(\sqrt{2n/k} + \frac{\alpha_n \sqrt{\operatorname{Log}_2 n}}{\sqrt{k}}\right), \tag{5}$$

where

$$\operatorname{Lim\,Sup}_{n\infty} \alpha_n = 1 \text{ a.s.}, \qquad \operatorname{Lim\,Inf}_{n\infty} \alpha_n = -1 \text{ a.s.},$$

and, for any $j \geqq 4$, almost surely

$$\operatorname{Lim\,Sup}_{n\infty} \left\{ k\left(\frac{N_{n+1} - N_n}{N_n}\right) \operatorname{Log} N_n - 2\operatorname{Log}_2 N_n - \operatorname{Log}_3 N_n - \ldots - \operatorname{Log}_{j-1} N_n \right\} \Big/ \operatorname{Log}_j N_n = 1 \text{ a.s.} \tag{6}$$

Remarks. 1°) Theorem 2 makes precise Theorem 1 of [2], where it was proved in the case $k=1$ that

$$N_n = \exp(\sqrt{2n} + O(\operatorname{Log} n)) \text{ a.s.}$$

2°) The study of the sequence $\{N_n, n \geqq 1\}$ has some interest in itself. In fact, it is for the maximal spacings the equivalent of the record times for maxima of i.i.d. sequences (see [3]). It behaves though in a quite different way. We shall, in the following paragraph, describe the main properties of this sequence which we have named as the *record time sequence of the k-th maximal spacing.*

2. The Record Time Sequence of the Maximal Spacings

We shall, in the following, use the notations of § 1 and assume that $k \geqq 1$ is fixed.

Lemma 1. *The sequence* $N_1 = k < N_2 < \ldots < N_n < \ldots$ *is an increasing sequence of stopping times on* $\mathfrak{U}_1 \subset \mathfrak{U}_2 \subset \ldots \subset \mathfrak{U}_n = \sigma(U_1, \ldots, U_n) \subset \ldots$. *Furthermore, if* $\mathfrak{U}(N_n)$ *denotes the σ-field generated by the events measurable on* $U_1, \ldots, U_{N_n}$, *then*

$$P(N_{n+1} - N_n \geqq r | \mathfrak{U}(N_n)) = (1 - L_{N_n})^{r-1}, \quad r = 1, 2, \ldots. \tag{7}$$

Proof. The first part of the Lemma is straightforward. For the second, note that if L_{N_n} is given, the probability that $N_{n+1} - N_n$ is greater than r is the probability that at least r independent U_i fall outside I_{N_n}. The result follows.

Lemma 2. *For any* $\varepsilon > 0$, *there exists almost surely an* n_ε *such that for any* $n \geqq n_\varepsilon$,

$$-k(1+\varepsilon) \operatorname{Log}_3 n \leqq n L_n - k \operatorname{Log} n \leqq (2+\varepsilon) \left(\sum_{i=1}^{k} \frac{1}{i} \right) \operatorname{Log}_2 n, \tag{8}$$

and

$$-k(1+\varepsilon) \operatorname{Log}_3 N_n \leqq N_n L_{N_n} - k \operatorname{Log} N_n \leqq (2+\varepsilon) \left(\sum_{i=1}^{k} \frac{1}{i} \right) \operatorname{Log}_2 N_n. \tag{9}$$

Proof. Note first that by (4), $N_n \geqq n$ and the sequence N_n increases to infinity. Hence, it suffices to prove (8), which in turn follows from (2) and (see Devroye [5])

$$\operatorname*{Lim\,Inf}_{n\infty} (n K_n - \operatorname{Log} n + \operatorname{Log}_3 n) = -\operatorname{Log} 2 \text{ a.s.} \tag{10}$$

Lemma 3. *On the same probability space where* $U_1, U_2, \ldots$ *is defined (eventually extended), there exists a sequence* $\zeta_1, \zeta_2, \ldots$, *independent of* $U_1, U_2, \ldots$, *of independent uniformly distributed on* $[0, 1]$ *random variables such that if for* $n = 1, 2, \ldots$,

$$\omega_n = (-\operatorname{Log}(1 - L_{N_n}))(N_{n+1} - N_n - 1) - \operatorname{Log}(1 - \zeta_n(1 - L_{N_n})), \tag{11}$$

then the sequence $\{\omega_n, n \geqq 1\}$ *is an i.i.d. sequence of exponentially* $E(1)$ *distributed random variables such that, for any* $n \geqq 1$, ω_n *is independent of* $\mathfrak{U}(N_n)$ *and satisfies*

$$N_{n+1} - N_n = \frac{\omega_n}{-\operatorname{Log}(1 - L_{N_n})} + 1 - \frac{\operatorname{Log}(1 - \zeta_n L_{N_n})}{\operatorname{Log}(1 - L_{N_n})} = \left[\frac{\omega_n}{-\operatorname{Log}(1 - L_{N_n})} \right] + 1, \tag{12}$$

where $[u]$ *denotes the integer part of* u.

Proof. It follows as a direct corollary of [3], Theorem 1. Note that Lemma 3 is essentially the same as [1], Lemma 2.

In the following, we shall use the notations

$$\lambda_n = L_{N_n}, \quad n = 1, 2, \ldots.$$

Lemma 4. *Let*, for $0 < z$, $\zeta < 1$, $\phi(z) = \dfrac{1}{-\operatorname{Log}(1-z)} - \dfrac{1}{z} + \dfrac{1}{2}$ *and*

$$\psi(\zeta,z)=-\frac{\operatorname{Log}(1-\zeta z)}{\operatorname{Log}(1-z)}+\zeta, \quad \textit{then,}$$

$$-\frac{z}{2}<\phi(z)<0 \quad \textit{and} \quad 0<\psi(\zeta,z)<\zeta z.$$

Proof. See [3], Lemma 3 and Lemma 4.

Lemma 5. *For any* $r>0$, $\operatorname{Lim}_{n\infty} N_n/n^r=+\infty$.

Proof. The result is clearly true for any $r: 0<r<1$, since by (4) for any $n\geqq 1$, $N_n\geqq n$. Let us assume that it is true for some $r>0$. If we use the fact that (see (9)) as $n\uparrow\infty$, $\lambda_n\sim k(\operatorname{Log} N_n)/N_n$ a.s., we obtain that $\lambda_n=o(\operatorname{Log} n)/n^r)$ a.s. Next, by (12) and Lemma 4,

$$N_{n+1}-N_n\geqq\frac{\omega_n}{-\operatorname{Log}(1-\lambda_n)}\geqq\omega_n\left\{\frac{1}{\lambda_n}-\frac{1}{2}-\frac{\lambda_n}{2}\right\}.$$

By summing up and by Kronecker's lemma, it follows that

$$N_n=k+\sum_{i=1}^{n-1}(N_{i+1}-N_i)=\frac{n^{r+1}}{(\operatorname{Log} n)\,o(1)} \quad \text{a.s.}$$

Hence, the result is still true for $r+\frac{1}{2}$ and also for any $r>0$.

Lemma 6. *Almost surely as* $n\uparrow\infty$,

$$N_{n+1}-N_n\sim\frac{\omega_n N_n}{k\operatorname{Log} N_n}.$$

Proof. It suffices to prove by (12) and (9) that $\operatorname{Lim}_{n\infty}\frac{\omega_n N_n}{\operatorname{Log} N_n}=+\infty$ a.s.

This follows from Lemma 5 and:

Lemma 7. *For any* $p\geqq 1$,

$$P(\omega_n\geqq\operatorname{Log} n+\ldots+(1+\varepsilon)\operatorname{Log}_p n \ \text{ i.o.})=P(\omega_n\leqq(n(\operatorname{Log} n)\ldots(\operatorname{Log}_p n)^{1+\varepsilon})^{-1} \ \text{ i.o.})$$
$$=0 \ \textit{or} \ 1$$

according as $\varepsilon>0$ *or* $\varepsilon\leqq 0$.

Proof. By Borel-Cantelli.

Lemma 8. *Almost surely as* $n\uparrow\infty$,

$$\operatorname{Log} N_n\sim\sqrt{2n/k}.$$

Proof. By Lemma 6 and Lemma 7,

$$\operatorname{Log} N_{n+1}=\operatorname{Log} N_n+\frac{\omega_n}{k\operatorname{Log} N_n}(1+o(1)) \ \text{ a.s.}$$

By taking squares,

$$\operatorname{Log}^2 N_{n+1} - \operatorname{Log}^2 N_n = \frac{2\omega_n}{k}(1+o(1)) \text{ a.s.}$$

It follows by summing up and using the law of large numbers that

$$\operatorname{Log}^2 N_n \sim 2n/k \text{ a.s.} \quad \text{as } n \uparrow \infty.$$

Lemma 9. *Let for* $n \geqq 1$

$$N_{n+1} = N_n \left\{ 1 + \frac{\omega_n}{k \operatorname{Log} N_n}(1+\delta_n) \right\}.$$

Then

$$\operatorname*{Lim\,Sup}_{n\infty} \delta_n \frac{\operatorname{Log} N_n}{\operatorname{Log}_2 N_n} \leqq \frac{4}{k} \sum_{i=1}^{k} \frac{1}{i} \text{ a.s.}$$

Proof. By (9), if $C = (2+\varepsilon) \sum_{i=1}^{k} \frac{1}{i}$, as $n \uparrow \infty$, we get

$$\left| \frac{1}{N_n \lambda_n} - \frac{1}{k \operatorname{Log} N_n} \right| \leqq \frac{C \operatorname{Log}_2 N_n}{k N_n \lambda_n \operatorname{Log} N_n} \sim \frac{C \operatorname{Log}_2 N_n}{k^2 \operatorname{Log}^2 N_n}.$$

Next, from (12), using the notations of Lemma 4,

$$N_{n+1} - N_n = \frac{\omega_n}{\lambda_n} - \frac{1}{2}\omega_n + 1 - \zeta_n + \omega_n \phi(\lambda_n) + \psi(\zeta_n, \lambda_n).$$

Clearly, by Lemma 4 and Lemma 7, since by (9), $\lambda_n \sim \frac{k \operatorname{Log} N_n}{N_n}$ a.s.,

$$N_{n+1} - N_n = N_n \left\{ \frac{\omega_n}{k \operatorname{Log} N_n} + \omega_n \left(\frac{1}{N_n \lambda_n} - \frac{1}{k \operatorname{Log} N_n} \right) + O\left(\frac{\operatorname{Log}_2 N_n}{N_n} \right) \right\} \text{ a.s.}$$

Finally, we use the fact that, as $n \uparrow \infty$, $\omega_n \leqq (1+\varepsilon) \operatorname{Log} n \sim 2 \operatorname{Log}_2 N_n$ a.s.

Lemma 10. *Let for* $n \geqq 1$

$$\operatorname{Log} N_{n+1} = \operatorname{Log} N_n + \frac{\omega_n (1+\delta_n)}{k \operatorname{Log} N_n} - \frac{\varepsilon_n \omega_n^2}{2k^2 \operatorname{Log}^2 N_n}, \tag{13}$$

then

$$\operatorname*{Lim}_{n\infty} \varepsilon_n = 1 \text{ a.s.}$$

Proof. It follows from Lemma 9, by taking logarithms.

We now take squares in (13). It gives

$$\operatorname{Log}^2 N_{n+1} - \operatorname{Log}^2 N_n$$
$$= \frac{2\omega_n(1+\delta_n)}{k} - \frac{\varepsilon_n \omega_n^2}{k^2 \operatorname{Log} N_n} - \frac{\varepsilon_n \omega_n^3 (1+\delta_n)}{k^3 \operatorname{Log}^3 N_n} + \frac{\omega_n^2 (1+\delta_n)^2}{k^2 \operatorname{Log}^2 N_n} + \frac{\varepsilon_n^2 \omega_n^4}{4k^4 \operatorname{Log}^4 N_n}.$$

Next, since $\operatorname{Log} N_n \sim \sqrt{2n/k}$ by Lemma 8, we get easily the following evaluations:

$$\sum_{i=1}^{n} \frac{\varepsilon_i \omega_i^2}{k^2 \operatorname{Log} N_i} \sim \sum_{i=1}^{n} \frac{\omega_i^2}{k^2 \sqrt{2i/k}} \sim (2/k)^{3/2} \sqrt{n} \text{ a.s.,}$$

$$\sum_{i=1}^{n} \frac{\omega_i^2}{k^2 \operatorname{Log}^2 N_i} \sim \sum_{i=1}^{n} \frac{\omega_i^2}{2ki} \sim k^{-1} \operatorname{Log} n \text{ a.s.,}$$

$$\sum_{i=1}^{\infty} \frac{\varepsilon_n \omega_n^3}{k^3 \operatorname{Log}^3 N_n} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{\varepsilon_n \omega_n^4}{4k^4 \operatorname{Log}^4 N_n} < \infty \text{ a.s.}$$

From there the following result can be proved.

Theorem 3. *If* $\{N_n, n \geqq 1\}$ *is defined by* (4), *then, on the same probability space (eventually extended), it is possible to define an i.i.d. sequence* $\{\omega_n, n \geqq 1\}$ *of exponentially* $E(1)$ *distributed random variables such that*

$$\operatorname*{Lim}_{n\infty} n^{-1/2} \left\{ \operatorname{Log}^2 N_n - \frac{2}{k} \sum_{i=1}^{n} \omega_i \right\} = R \in]-\infty, +\infty[\text{ a.s.} \tag{14}$$

Corollary 1. *If* $N(0,1)$ *denotes the standard normal distribution, then, as* $n \to \infty$,

$$\frac{k}{2\sqrt{n}} \left\{ \operatorname{Log}^2 N_n - \frac{2n}{k} \right\} - \frac{Rk}{2} \xrightarrow{w} N(0,1)$$

and

$$\sqrt{2k} \{ \operatorname{Log} N_n - \sqrt{2n/k} \} - \frac{Rk}{2} \xrightarrow{w} N(0,1).$$

Proof. It follows directly from Theorem 3 and the central limit theorem.

Proof of Theorem 2. Note first that (5) is a direct consequence of Theorem 3 and of the law of the iterated logarithm applied to $\sum_{i=1}^{n} \omega_i$. Next, we prove the following Lemma.

Lemma 11. *Almost surely as* $n \uparrow \infty$,

$$\operatorname{Log}_2 N_n = \tfrac{1}{2} \operatorname{Log} n + \tfrac{1}{2} \operatorname{Log}(2/k) + O(n^{-1/2} \operatorname{Log} n),$$
$$\operatorname{Log}_3 N_n = \operatorname{Log}_2 n - \operatorname{Log} 2 + O(1/\operatorname{Log} n),$$

and, for any $p \geqq 4$,

$$\operatorname{Log}_p N_n = \operatorname{Log}_{p-1} n + O\left(1 \Big/ \prod_{j=2}^{p-2} \operatorname{Log}_j n\right).$$

Proof. It suffices to take the p-th iterated logarithm of both sides of (5).

It is possible by Lemma 11 and Lemma 7 to prove that, for any $p \geqq 3$, $P(\omega_n \geqq 2 \operatorname{Log}_2 N_n + \operatorname{Log}_3 N_n + \ldots + (1+\varepsilon) \operatorname{Log}_p N_n \text{ i.o.}) = 0$ when $\varepsilon > 0$ and 1 when $\varepsilon < 0$. Finally, by Lemma 9, almost surely as $n \uparrow \infty$,

$$k \left\{ \frac{N_{n+1} - N_n}{N_n} \right\} \operatorname{Log} N_n = \omega_n \left(1 + 0 \left(\frac{\operatorname{Log} n}{\sqrt{n}} \right) \right). \tag{15}$$

This suffices to complete the proof of (6).

Let us now consider (15), and evaluate:

$$\operatorname{Log} k+\operatorname{Log}(N_{n+1}-N_n)=\operatorname{Log} N_n-\operatorname{Log}_2 N_n+\operatorname{Log}\omega_n+O\left(\frac{\operatorname{Log} n}{\sqrt{n}}\right) \text{ a.s.}$$

An easy deduction from Theorem 2 and Lemma 7 gives the following result.

Theorem 4. *If* $\{N_n, n\geqq 1\}$ *is defined by* (4), *then,*

$$\operatorname*{Lim\,Sup}_{n\infty}(\operatorname{Log}_2 n)^{-1}\{\operatorname{Log}(N_{n+1}-N_n)-\sqrt{2n/k}+\tfrac{1}{2}\operatorname{Log} n\}=1 \text{ a.s.,}$$

and

$$\operatorname*{Lim\,Inf}_{n\infty}(\operatorname{Log}_2 n)^{-1}\{\operatorname{Log}(N_{n+1}-N_n)-\sqrt{2n/k}+\tfrac{3}{2}\operatorname{Log} n\}=-1 \text{ a.s.}$$

Further expansions of the upper and of the lower class of $N_{n+1}-N_n$ may be deduced easily from Lemma 9, Lemma 7, and Theorem 2. It may be remarked here that theses classes differ from the classes of N_n.

3. Upper Bounds for the k-th Maximal Spacing

We shall now give the proof of Theorem 1. Even though this proof follows closely the case of $k=1$, treated in [2], the extension from this case to an arbitrary k is not trivial and needs to be detailed.

We first define the sequence $\{n_l, l\geqq 1\}$ by $n_l=[\exp(\sqrt{2l/k})]$, $l=1,2,\ldots$, and put $T_l=\omega_{n_l}$. Next, we consider the random sequence defined by

$$\begin{aligned} l(1)&=\operatorname{Min}\{l\geqq 1;\, L_{n_l}>L_{n_{l+1}}\},\\ l(r)&=\operatorname{Min}\{l>l(r-1);\, L_{n_l}>L_{n_{l+1}}\},\quad r=2,3,\ldots. \end{aligned} \tag{16}$$

Lemma 12. *If* $\{l(r), r\geqq 1\}$ *is defined by* (16), *then* $\{n_{l(r)+1}, r\geqq 1\}$ *is an increasing sequence of stopping times on* $\{\mathfrak{U}_n, n\geqq 1\}$ *and if for* $r=1,2,\ldots$, $\theta_r=\omega_{l(r)+1}$, *then* $\{\theta_r, r\geqq 1\}$ *is an i.i.d. sequence of exponentially* $E(1)$ *distributed random variables.*

Proof. See [2], Lemma 4.

Lemma 13. *With the hypothesis of Lemma* 12,

$$\operatorname*{Lim}_{r\infty} l(r)/r=\frac{e^{1/k}}{e^{1/k}-1} \text{ a.s.}$$

Proof. See [2], Lemma 5.

Following exactly the proof in [2], it can be seen that Theorem 1 will be proved for an arbitrary $k\geqq 1$ if the following Lemma is true.

Lemma 14. *For any* $j\geqq 4$ *and* $c>0$,

$$\operatorname*{Lim}_{N\infty}\frac{1}{N}\sum_{l=1}^{N} I(n_l K_{n_l}-\operatorname{Log} n_l\geqq -c\operatorname{Log}_j n_l)=1 \text{ a.s.,}$$

where $I(A)$ *denotes the indicator function of the event* A.

To see that it is indeed the case, it can be verified that then, as in [2] Lemma 8 and Lemma 10, there exists a.s. an infinite set of indices n such that:

$$nK_n \geqq \operatorname{Log} n - c \operatorname{Log}_j n, \tag{17}$$

and hence, since $L_n \geqq kK_n$, that

$$nL_n \geqq k \operatorname{Log} n - ck \operatorname{Log}_j n,$$

and such that there exists an $m > n$ with $L_m = L_n$ and $K_m = K_n$, satisfying:

$$\left(\frac{m-n}{n}\right) \operatorname{Log} n \geqq \frac{1}{k}(2 \operatorname{Log}_2 n + \operatorname{Log}_3 n + \ldots + \operatorname{Log}_j n), \tag{18}$$

by an inequality analogous to (8).

It follows from (17) and (18) that

$$mK_m > \operatorname{Log} m + \frac{1}{k}(2 \operatorname{Log}_2 m + \operatorname{Log}_3 m + \ldots + \operatorname{Log}_{j-1} m) + \left(\frac{1}{k} - c\right) \operatorname{Log}_j m + o(1),$$

proving Theorem 1.

An important point of this proof is to note that any index n where K_n decreases is also an index where L_n decreases. Hence, if $L_n = L_m$, we must have $K_n = K_m$.

We shall not repeat here the steps detailed in [2], which are, after the preceding remarks, identical in the case $k=1$ and $k \geqq 1$. It remains only to prove Lemma 14.

To do so, we shall use the following evaluation given by Devroye [4], Lemma 3.2:

Lemma 15. *If $a_n \to 0$ and $a_n \operatorname{Log} n \to \infty$ as $n \to \infty$, then,*

$$P(nK_n/\operatorname{Log} n - 1 < -a_n) \sim n^{(k-1)a_n} \exp(-n^{a_n})/(k-1)!. \tag{19}$$

If we put $a_n = (c \operatorname{Log}_j n)/\operatorname{Log} n$, we obtain that

$$\begin{aligned} P(nK_n < \operatorname{Log} n - c \operatorname{Log}_j n) &\sim (1/(k-1)!) \exp(-\exp(a_n \operatorname{Log} n) + (k-1)\, a_n \operatorname{Log} n)) \\ &= (1/(k-1)!) \exp(-(\operatorname{Log}_{j-1} n)^c + c(k-1) \operatorname{Log}_j n). \end{aligned}$$

Let $\eta_l = I(n_l K_{n_l} - \operatorname{Log} n_l < -c \operatorname{Log}_j n_l)$. By the preceding evaluation,

$$E(\eta_l) \sim (1/(k-1)!) \exp(-(\operatorname{Log}_{j-2} \sqrt{2l/k})^c + c(k-1) \operatorname{Log}_{j-1} \sqrt{2l/k}) \to 0 \text{ as } l \to \infty.$$

The Lemma 14 will be proved if we prove that $\operatorname{Lim}_{n\infty} \frac{1}{N} \sum_{l=1}^{n} (\eta_l - E(\eta_l)) = 0$, or equivalently (see [12]), if $\xi_n = \frac{1}{N} \sum_{l=1}^{N} \eta_l$, if there exists an $a > 1$ such that

$$\sum_{n=1}^{\infty} D^2(\xi_{[a^n]}) < \infty, \text{ where } D^2(\xi_N) = N^{-2} \sum_{i=1}^{N} \sum_{l=1}^{N} (E(\eta_i \eta_l) - E(\eta_i) E(\eta_l)).$$

Let us now put $c>1$, $j\geqq 4$, $b_i=[i/(\operatorname{Log} i)^{1+\varepsilon}]$, where $\varepsilon>0$,

$$A_N=N^{-2}\sum_{i=4}^{N}\sum_{l=i-b_i}^{i+b_i}\operatorname{Cov}(\eta_i,\eta_l),\text{ and } B_N=N^{-2}\sum_{i=4}^{N}\sum_{l=i+b_i+1}^{N}\operatorname{Cov}(\eta_i,\eta_l).$$

Clearly $E(\eta_l)=o((\operatorname{Log}_{j-3} l)^{-2})$, $l\to\infty$, and, as in [2], (26), it follows that $A_N=O((\operatorname{Log} N)^{-1-\varepsilon}(\operatorname{Log}_{j-3} N)^{-2})$ as $N\to\infty$. Consequently if $a>0$, then $\sum_{n\geqq 1} A_{[a^n]}<\infty$.

Following [2], it suffices to prove that $\sum_{n\geqq 1} B_{[a^n]}<\infty$. To do so, we have to evaluate $E(\eta_i\eta_l)$ when $l>i+b_i$. By a similar proof as in [2], we obtain the following upper bound for B_N:

$$B_N\leqq N^{-2}\sum_{i=4}^{N}\sum_{l=i+b_i+1}^{N}E(\eta_i)\{P(n_l K_{n_l-n_i}<\operatorname{Log} n_l-c\operatorname{Log}_j n_l)-E(\eta_l)\}. \tag{20}$$

Next, it follows likewise from [2] that

$$\begin{aligned} n_i(\operatorname{Log} n_l)/n_l &\leqq n_i(\operatorname{Log} n_{i+b_i})/n_{i+b_i}\\ &=c_i'=\exp(-b_i(1+o(1))/\sqrt{2ki})\to 0 \text{ as } i\to\infty. \end{aligned}$$

Hence, we get

$$P(n_l K_{n_l-n_i}<\operatorname{Log} n_l-c\operatorname{Log}_j n_l)=P((n_l-n_i)K_{n_l-n_i}<\operatorname{Log}(n_l-n_i)-c\operatorname{Log}_j n_l+O(c_i')).$$

Finally we get from (20) $B_N\leqq N^{-2}\sum_{i=4}^{N}\sum_{l=i+b_i+1}^{N}E(\eta_i)\gamma_i$, where

$$\begin{aligned}\gamma_i=\{P((n_l-n_i)K_{n_l-n_i}<\operatorname{Log}(n_l-n_i)-c\operatorname{Log}_j n_l+O(c_i'))-P(n_l K_{n_l}<\operatorname{Log} n_l\\ -c\operatorname{Log}_j n_l)\}.\end{aligned}$$

Following the bounds given in Devroye [4] (3.3) and Lemma 3.1, if L_n' is the k-th largest of n independent identically distributed random variables with exponential density and whose sum is T_n, then

$$\begin{aligned} P(L_n'<(1-a-b)\operatorname{Log} n)-P(T_n<n(1-b))&\leqq P(nK_n/\operatorname{Log} n<1-a)\\ &\leqq P(L_n'<(1-a+b)\operatorname{Log} n)+P(T_n\geqq n(1+b)), \end{aligned} \tag{21}$$

and for n large enough,

$$P(|T_n-n|\geqq bn)\leqq 2\exp(-nb^2/4).$$

Let us take now in (21) $a=a_n$ and $b=n^{-1/4}$. It follows that

$$P(|T_n-n|\geqq bn)\leqq 2\exp(-\sqrt{n}/4).\text{ Next, } P(L_n'<x)=\sum_{j=0}^{k-1}\binom{n}{j}(1-e^{-x})^{n-j}e^{-jx}.$$

Put for $j=0,\ldots,k$, $C_n^j(x)=\binom{n}{j}(1-e^{-x})^{n-j}e^{-jx}$. We get, for any $\varepsilon>0$ as $n\to\infty$,

$$C_n^j((1-a\pm b)\operatorname{Log} n)=\frac{1}{j!}\left(1+O\left(\frac{1}{n}\right)\right)\exp(-n^{a\mp b}+O(n^{\varepsilon-1})+j(a\mp b)\operatorname{Log} n).$$

If now $a=a_n \sim c(\mathrm{Log}_j n)/\mathrm{Log}\, n$, it follows that for any $\theta>0$,

$$C_n^j((1-a\pm b)\,\mathrm{Log}\, n)=n^{ja}\exp(-n^a)(1+o(n^{\theta-1/4})).$$

Hence,

$$P(nK_n/\mathrm{Log}\, n<1-a)=\sum_{j=0}^{k-1} n^{ja}\exp(-n^a)(1+O(n^{\theta-1/4})).$$

From there it follows as in [2] (27) that

$$B_N \leqq N^{-2}\sum_{i=4}^{N}\sum_{l=i+b_i+1}^{N} E(\eta_i)\,E(\eta_l)\,o(n_l^{\theta-1/4})=O(N^{-1}).$$

Hence $\sum_{n=1}^{\infty} B_{[a^n]}<\infty$ and the proof of Theorem 1 is complete.

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