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Gaps in the Range of Nearly Increasing Processes with Stationary Independent Increments

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Summary. For a class of stationary independent increments processes we find a necessary and sufficient integral test on a function $0<h\in\uparrow$ for $P[R\cap(x, x+h(x))=\emptyset$ i.o. $x\uparrow\infty]=1$ and for $P[R_t\cap(x, x+h(x))=\emptyset$ i.o. $x\downarrow 0]=1$, for all $t>0$, where $R=$range of X, $R_t=$range of X up to epoch t.

§ 1. Introduction

Let $X=\{X_t,\ t\geqq 0\}$ be a process in R^1 with stationary independent increments (s.i.i.). When the time variable t is continuous one may always choose a version of the process which is strong Markov and has certain paths regularities such as right continuity. We assume this has been done and we also assume $P[X_0=0]=1$.

Let us suppose further that X is transient, i.e., that $|X_t|\to\infty$ as $t\to\infty$, a.s., then the range R of X, defined by

$$R=\{x\colon X_t=x \text{ for some } t\geqq 0\},$$

is a random set which may have holes or gaps in it. In this paper we study the structure of the large gaps far out in R for a certain subclass of such processes. More precisely let h be a positive increasing function (a standing assumption on h throughout) such that $h(\infty)=\infty$. Then in certain cases we determine the probability of the event

$$[R\cap(x, x+h(x))=\emptyset \text{ i.o. as } x\uparrow\infty]. \tag{1.1}$$

Note that, unless $X_t\to-\infty$, $t\to\infty$, $R\cap(x, x+h(x))$ is certain to be *non-empty* i.o. as $x\uparrow\infty$ and if h increases too fast $R\cap(x, x+h(x))$ may be non-empty for all x sufficiently large. Let us say that h is a *large gap* function if $P\{(1.1)\}=1$.

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Inasmuch as the set

$$R_t \equiv \{x: X_s = x \text{ for some } s \leqq t\} \cap (-\infty, \infty)$$

is a.s. bounded for each $t < \infty$, [1], p. 45, it follows from the Hewitt-Savage 0–1 law, see Fristedt [5], p. 253, that if h is not a large gap function, the probability of (1.1) is 0. In §2 we delimit the class of large gap functions for random walks (discrete time s.i.i. processes) which drift to $+\infty$. We prove

Theorem 1. *Let $\{X_n; n \geqq 0\}$ be a random walk which drifts to $+\infty$ and let q be the tail of the increment distribution: $q(x) = P[X_n - X_{n-1} > x]$ for all n. Put $m(x) = \int_0^x q(y)\,dy$. Then h is a large gap function if and only if*

$$\int^{\infty} q \circ h(x)[1 - xq(x)/m(x)]/m(x)\,dx = \infty. \tag{1.2}$$

($\circ$ denotes function composition.) Note that $[1 - xq(x)/m(x)]/m(x) = (d/dx)(x/m(x))$ at continuity points. See §2 for a reformulation in terms of the ascending ladder height distribution.

In the continuous time case, we also study the problem of small gaps in the range. As before let $0 < h \in \uparrow$ but $h(0+) = 0$ and let R_t be the range of X up to epoch t. For a class of subordinator like processes, we determine the probability of the event

$$[R_t \cap (x, x + h(x)) = \emptyset \text{ i.o. as } x \downarrow 0]. \tag{1.3}$$

Let us say h is a *small gap* function for the range if $P\{(1.3)\} = 1$ for all $t < \infty$. Let ν denote the Levy jump measure of X and suppose

$$\nu\{(-\infty, 0)\} < \nu\{(0, 1]\} = \infty, \tag{1.4}$$

$$\int_0^1 x\,\nu\{dx\} < \infty, \tag{1.5}$$

Suppose also that X is a pure jump process (no Gaussian or linear drift components, a type B process in Fristedt [5]). In §3 we prove

Theorem 2. *Under the above assumptions h is a small gap function if and only if*

$$\int_{0+} \bar{\nu} \circ h(x)[1 - x\bar{\nu}(x)/\mu(x)]/\mu(x)\,dx = \infty \tag{1.6}$$

where $\bar{\nu}(y) = \nu\{(y, \infty)\}$ and $\mu(x) = \int_0^x \bar{\nu}(y)\,dy$. If the integral in (1.6) *converges then $P\{(1.3)\} = 0$ for all $t > 0$.*

Theorem 2 follows quickly from a result concerning the finiteness of a class of discontinuous additive functionals. See Theorem 3 in §3.

In §4 we apply Theorems 1 and 2 to compute the class of gap functions for some special processes. These include the stable subordinators and random walks with regularly varying right hand tail probabilities.

Our motivation for writing this paper comes from a remarkable result of Pruitt and Taylor [7] which describes the large gaps in the range of the asymmetric Cauchy process. Unfortunately but not surprisingly our methods fail completely for this process. We discuss their result and its relations to our work in the last section.

§2. Proof of Theorem 1

Let $X=\{X_n, n\geqq 0\}$ be a random walk starting at 0: $X_0=0$, $X_n=\xi_1+\xi_2+\ldots+\xi_n$, $n\geqq 1$, where $\xi_1, \xi_2, \ldots$, are i.i.d. random variables. We assume that X drifts to $+\infty$;

$$P[\lim_{n\to\infty} X_n=+\infty]=1. \tag{2.1}$$

(This of course does not preclude negative values.) Let $\{S_n, n\geqq 0\}$ be the (strict) ascending ladder height renewal process given by

$$S_n=X_{T_n}, \qquad n=0, 1, 2, \ldots$$

where $T_0=0$,

$$T_n=\min\{k: X_k>X_{T_{n-1}}\}, \qquad n\geqq 1.$$

Let V be the common distribution of the independent variables $\{S_n-S_{n-1}, n\geqq 1\}$ and G the renewal measure;

$$G\{dx\}=\sum_{n=0}^{\infty} P[S_n\in dx]=\sum_{n=0}^{\infty} V^{n*}\{dx\},$$

$V^{n*}=n$-fold convolution of V.

Lemma 1. *Suppose* $0<h\in\uparrow$ *on* $(0,\infty)$. *Then*

$P[S_{n+1}>h(S_n)+S_n$ i.o. $n\uparrow\infty]=0$ *or* 1
according as

$$\int^{\infty} \bar V\circ h(y)\, G\{dy\}<\infty \qquad \text{or } =\infty, \tag{2.2}$$

where $\bar V(x)=1-V(x)=V\{(x,\infty)\}$.

Proof. Let $Y_n=S_n-S_{n-1}$, $B_0=\Omega$ and

$$B_n=[S_{n+1}>h(S_n)+S_n]=[Y_{n+1}>h(S_n)], \qquad n\geqq 1.$$

Then $B_n B_{n+m}\subset B_n\cap[Y_{n+m+1}>h(Y_{n+2}+\ldots+Y_{n+m})]$ so

$$P(B_n B_{n+m})\leqq P(B_n)\, P(B_{m-1}), \qquad n\geqq 1,\ m\geqq 1$$

Consequently if $Z_n=1_{B_0}+\ldots+1_{B_n}$,

$$EZ_n^2\leqq C_1 EZ_n+C_2(EZ_n)^2, \qquad n\geqq 1,$$

for some finite positive constants C_1, C_2. It follows from a generalized Borel Cantelli lemma, [6], that $P[\limsup Z_n=\infty]=P[B_n \text{ i.o.}]>0$ if and only if

$$\lim_n EZ_n=\sum_{n=0}^{\infty} P[Y_{n+1}>h(S_n)]$$
$$=\int_0^{\infty} \bar V\circ h(x)\, G\{dx\}=\infty.$$

Since $P[B_n$ i.o.$]$ is either 0 or 1 anyway, the lemma follows.

Remark 1. The probabilities $P[Y_{n+1}>h(S_n)]$ may also be computed thus:

$$P[Y_{n+1}>h(S_n)]=P[S_n<h^{-1}(Y_{n+1})]$$
$$=\int_0^\infty V^{n*}(h^{-1}(y)-)\,V\{dy\}.$$

Using the fact

$$G(z)=\sum_{n=0}^\infty V^{n*}(z)\asymp\Delta(z)=z\Big/\int_0^z \bar{V}(x)\,dx \tag{2.3}$$

for all $z>0$, see Erickson [2], p. 377, ($\asymp$ means that the ratio of both sides is bounded away from 0 and ∞), we obtain the

Corollary. *The integral* (2.2) *converges or diverges according as*

$$\int^\infty \Delta\circ h^{-1}(y)\,V\{dy\} \qquad \textit{converges or diverges.} \tag{2.4}$$

Remark 2. Note also that since $\Delta(\cdot)$ is a continuous function the conclusion of Lemma 1 and the Corollary remains valid when $>$ is replaced with $\geqq$. Moreover since $\Delta(cx)\asymp\Delta(x)$, again Erickson [2], it follows that for $c>0$

$$P[S_{n+1}>h_c(S_n)+S_n \text{ i.o.}]=P[S_{n+1}>h(S_n)+S_n \text{ i.o.}]$$

where $h_c(x)=h(cx)$.

Proof of Theorem 1. Since the T_n are stopping times for X and since X has stationary independent increments, the process $X'=\{X_{j+T_{n+1}}-S_{n+1},\ j\geqq 0\}$, n fixed, is a probabilistic replica of X independent of $\{X_0, X_1, \ldots, X_{T_{n+1}}, T_{n+1}\}$. Let us put

$$M_{n+1}=\inf\{X_{j+T_{n+1}}-S_{n+1;\,j\geqq 0}\}\leqq 0.$$

Then $\delta\equiv P[M_{n+1}=0]$ does not depend on n and by (2.1) and Feller [3], chapters 12 and 18,

$$\delta=P[X_n\geqq 0 \text{ for all } n]$$
$$=\exp\left(-\sum_{n=1}^\infty (1/n)\,P[X_n<0]\right)>0.$$

Put $B_n=[S_{n+1}>h(S_n)+S_n]$ as in Lemma 1, then $[M_{n+1}=0]$ is independent of $B_1, \ldots, B_n$ and an elementary calculation gives

$$P[B_n\cap(M_{n+1}=0) \text{ for some } n\geqq m]\geqq\sum_{n=m}^\infty P[B_m^c\ldots B_{n-1}^c B_n\cap(M_{n+1}=0)]$$
$$=\delta\sum_{n=m}^\infty P[B_m^c\ldots B_{n-1}^c B_n]=\delta P\left[\bigcup_{n=m}^\infty B_n\right].$$

Hence, letting $m\to\infty$, we get

$$P[S_{n+1}>h(S_n)+S_n \text{ i.o.}]$$
$$\geqq P[M_{n+1}=0,\ S_{n+1}>h(S_n)+S_n \text{ i.o.}]$$
$$\geqq\delta P[S_{n+1}>h(S_n)+S_n \text{ i.o.}].$$

Since both probabilities are 0 or 1 by the Hewitt-Savage 0–1 law, it follows that both are simultaneously 0 or simultaneously 1.

Now suppose the integrals in (2.2) or in (2.4) diverge. Then, by Lemma 1 and the preceding, w.p. 1 (with probability 1 = a.s.) infinitely many of the events

$$[S_{n+1} > S_n + h(S_n), M_{n+1} = 0]$$

occur. Plainly, in view of the definitions of S_n, M_{n+1}, this says that w.p. 1 the walk X never lands in $(S_n, S_n + h(S_n))$ for infinitely many n. In other words, (1.1) has probability 1.

Now let us suppose $P\{(1.1)\} = 1$. Since $R' = \{x: S_n = x \text{ for some } n \geqq 0\} \subset R$, we have

$$P[R' \cap (x, x+h(x)) = \emptyset \text{ i.o. } x \uparrow \infty] = 1,$$

and there exists random variables $X_1^0 < X_2^0 < \ldots \uparrow \infty$ such that

$$P[S_n \notin (X_k^0, X_k^0 + h(X_k^0)) \text{ for all } n, k] = 1.$$

Put $\rho_k = \max\{n: S_n < X_k^0\}$. Then a.s.

$$S_{\rho_k+1} - S_{\rho_k} > h(X_k^0) > h(S_{\rho_k}), \qquad k = 1, 2, \ldots,$$

i.e., $P[S_{n+1} > h(S_n) + S_n \text{ i.o.}] = 1$. Thus by Lemma 1, the integrals in (2.2) and (2.4) diverge.

To complete the proof of Theorem 1, we show that the integrals in (1.2) and (2.2) are equivalent, i.e., that they diverge or converge together. First we note the inequalities

$$q(x) \leqq \bar{V}(x) \leqq q(x)\, E(T_1), \qquad x > 0, \tag{2.5}$$

which we prove below. Recall that $q(x) = P[\xi_1 > x]$. Next we note that $E(T_1) < \infty$ since the walk drifts to $+\infty$. (See Feller, p. 610.) From these facts we see that the integral in (2.2) is equivalent to the integral $\int^\infty q \circ h(x)\, G\{dx\}$. An integration by parts shows this to be equivalent to $\int^\infty G(x)\, d(-q \circ h(x))$. Replacing $G(x)$ by $x \Big/ \int_0^x \bar{V}(y)\, dy$, as in (2.3), then estimating $\bar{V}$ with (2.5) and integrating by parts once again, we arrive at the integral in (1.2). Note that $x/m(x)$ is a nondecreasing function since m is concave, consequently the integrand in (1.2) is nonnegative as it should be.

Proof of (2.5). The left side of (2.5) is obvious from a consideration of the first step of the walk. Fix $x > 0$ and define random integers

$$Q_n = \#\{k: S_k > x + S_{k-1}, k \leqq n\},$$
$$R_n = \#\{j: X_j > x + X_{j-1}, j \leqq T_n\}.$$

The strong law of large numbers implies that as $n \to \infty$, $Q_n/n \to \bar{V}(x)$, $R_n/T_n \to q(x)$, and $T_n/n \to E(T_1)$. On the other hand $Q_n \leqq R_n$ for each n because for any k, $S_k = X_{T(k)} \geqq X_i$ for any $i < T_{k+1}$. These four facts yield the right side of (2.5).

§ 3. Proof of Theorem 2

Our first goal is a criterion for the finiteness of a class of discontinuous additive functionals of a process X. We confine our attention to the case that X is a subordinator without a deterministic linear part but one can easily generalize the result.

Let X be an s.i.i. process on $(\Omega, \mathscr{F}, P)$ with the usual path regularity such that for $\lambda,\ t \geqq 0$

$$E e^{-\lambda X_t} = \exp\left(-t \int_0^\infty (1-e^{-\lambda x})\, v\{dx\}\right), \tag{3.1}$$

where v, the Levy measure, satisfies

$$\int_0^\infty \min(1, x)\, v\{dx\} < \infty, \tag{3.2}$$

$$v\{(0,1)\} = \infty. \tag{3.3}$$

We may suppose, without loss of generality, that X is of function space type [1], p. 24. Thus θ_t is the usual shift. Also we write $P^x[\cdot]$, $E^x[\cdot]$ for $P[\cdot \mid X_0 = x]$, $E[\cdot \mid X_0 = x]$ and p, E for P^0, E^0. From (3.1) and (3.2) one may show that $P[X_t \geqq 0 \text{ for all } t \geqq 0] = 1$.

Let us write

$$\bar{v}(x) = v\{(x, \infty)\},$$

$$\Delta_s = \Delta(s) = X_s - X_{s-} = X_{s+} - X_{s-}.$$

Recall that (3.3) implies $\Delta_s > 0$ for (countably) infinitely many s in $[t, t+\delta]$ for every $t \geqq 0$ and $\delta > 0$. Moreover

$$X_t = \sum_{0 \leqq s \leqq t} \Delta_s \in \uparrow \tag{3.4}$$

and the random variable $\#(t, \varepsilon) =$ the number of jumps $> \varepsilon$ up to epoch t is Poisson with mean $t\bar{v}(\varepsilon)$.

Let ψ be a real Borel function on $[0, \infty) \times [0, \infty)$ subject to one or more of the following

$\psi 1)$ $\psi \geqq 0$, ψ is bounded and $\psi(0, x) = 0$ for all $x \geqq 0$.

$\psi 2)$ $\psi 1)$ holds and for each fixed x, $y \mapsto \psi(x, y)$ is non-increasing.

$\psi 3)$ $\psi 1)$ holds and for each $y > 0$ and every $t > 0$

$$P\Big[\sum_{0 \leqq r \leqq t} \psi(\Delta_r, y) < \infty\Big] = 1. \tag{3.5}$$

Remark 3. If (3.5) holds for one $t = t_0$ and $\psi 1)$ holds, then (3.5) holds for every $t \geqq 0$. To see this note that for $t_0 < t \leqq 2t_0$,

$$\begin{aligned} P\Big[\sum_{0 \leqq r \leqq t} \psi(\Delta_r, y) < \infty\Big] &= P\Big[\sum_{0 \leqq r \leqq t - t_0} \psi(\Delta_r \circ \theta_{t_0}, y) < \infty\Big] \\ &= E P^{X(t_0)}\Big[\sum_{0 \leqq s \leqq t - t_0} \psi(\Delta_s, y) < \infty\Big] = P\Big[\sum_{0 \leqq s \leqq t - t_0} \psi(\Delta_s, y) < \infty\Big] = 1 \end{aligned}$$

by, respectively, $(3.5)_{t_0}$, Markov property, independent increments, $(3.5)_{t_0}$.

Remark 4. $\psi 3)$ is satisfied if $x \mapsto \psi(x, y)$ is non-decreasing for each $y>0$ and

$$\int_{0+} \bar{v}(x)\, d_x \psi(x, y) < \infty \qquad \text{for all } \; y>0. \tag{3.6}$$

In particular $\psi 1)$–$\psi 3)$ are satisfied by the function

$$\begin{aligned} \psi(x, y) &= 1 \qquad \text{for } \; x>h(y) \\ &= 0 \qquad \text{for } \; x \leqq h(y) \end{aligned}$$

where $0<h\in\uparrow$, $h(0+)=0$. (To see that (3.6) and $x \mapsto \psi(x, y)$ non-decreasing give $\psi 3)$, note that the process $Y_t = \sum_{0\leqq s\leqq t} \psi(\Delta_s, y)$, $t\geqq 0$, if finite is a non-decreasing s.i.i. process. As such its Levy measure v_Y satisfies (3.2). Conversely, if v_Y satisfies (3.2), then Y is finite. An intergration by parts and a change of variables shows that (3.2) is equivalent t_0 (3.6). Note $\bar{v}_Y(z) = \bar{v} \circ \psi_y^{-1}(z)$, ψ_y^{-1} $=$inverse to $x \mapsto \psi(x, y)$.)

Put $\psi_\varepsilon(x, y) = \psi(x, y)\, I_{(\varepsilon, \infty)}(x)$ and define

$$\begin{aligned} A_t &= \sum_{0<s\leqq t} \psi(\Delta_s, X_{s-}) \\ A_t(\varepsilon) &= \sum_{0<s\leqq t} \psi_\varepsilon(\Delta_s, X_{s-}). \end{aligned}$$

Note that for $\varepsilon>0$, $A_t(\varepsilon)<\infty$ a.s. and that both A_t and $A_t(\varepsilon)$ are (discontinuous) additive functionals:

$$A_{t+s} = A_t + A_s \circ \theta_t \text{ a.s.,} \qquad s, t \geqq 0. \tag{3.7}$$

On the event $[A_{t_0}<\infty]$ A is right continuous on $[0, t_0)$.

Lemma 2. *Suppose $\psi 1)$ holds. Then for each t as $\varepsilon \downarrow 0$, $A_t(\varepsilon) \nearrow A_t$ (finite or not) a.s., and $EA_t(\varepsilon) \nearrow EA_t$ (finite or not). If $EA_{t_0}<\infty$, the convergence is uniform (a.s. in the first limit) on $[0, t_0]$.*

Proof. Clearly $0 \leqq \psi_\varepsilon \nearrow \psi$ so both limits are consequences of the Monotone convergence theorem. The uniform convergences follow from

$$|A_t - A_t(\varepsilon)| \leqq |A_{t_0} - A_{t_0}(\varepsilon)| \leqq A_{t_0}, \qquad t \leqq t_0, \text{ on } [A_{t_0}<\infty].$$

Let us now define the continuous additive functionals (if finite)

$$\begin{aligned} B_t &= \int_0^t \int_0^\infty \psi(x, X_s)\, v\{dx\}\, ds \\ B_t(\varepsilon) &= \int_0^t \int_\varepsilon^\infty \psi(x, X_s)\, v\{dx\}\, ds. \end{aligned}$$

Of course B_t may not be finite but $B_t(\varepsilon)$ is finite a.s. for $\varepsilon>0$ under $\psi 1)$.

Lemma 3. *Suppose $\psi 1)$ hold. Then for each t, $\varepsilon>0$*

$$EA_t(\varepsilon) = EB_t(\varepsilon) < \infty, \tag{3.8}$$

$$E(A_t(\varepsilon) - B_t(\varepsilon))^2 \leqq m EA_t(\varepsilon), \qquad m = \sup \psi, \tag{3.9}$$

$$EA_t = EB_t \quad \textit{finite or not.} \tag{3.10}$$

Proof. Let $m = \sup\psi < \infty$, then $A_t(\varepsilon) \leqq m \#(t, \varepsilon)$ and $B_t(\varepsilon) \leqq m t \bar{v}(\varepsilon)$ a.s. So $EB_t(\varepsilon)$ and $EA_t(\varepsilon)$ are both bounded by $m t \bar{v}(\varepsilon) < \infty$. Our process X satisfies assumptions (A.1) and (A.2) of Watanabe [8], p. 68. (To establish (A.2ii) use

$$\lim_{t \to 0+} (1/t) P[X_t > \varepsilon] = \bar{v}(\varepsilon)$$

at continuity points of $\bar{v}$, cf. Fristedt [4], p. 32). It follows therefore from Theorem 4.2 of [8], p. 68, that (n^*, t) is the Levy system for X (the death time $\zeta = \infty$ a.s. in our case) where

$$n^*(x, E) = v\{E - x\}.$$

If we write $f(x, y) = \psi_\varepsilon(y - x, x)$, then $f \geqq 0$, $f(x, x) = 0$ for all x, by $\psi 1$, and

$$A_t(\varepsilon) = \sum_{0 < s \leqq t} f(X_{s-}, X_s).$$

Theorem 3.1 and other calculations on p. 63 of [8] imply

$$EA_t(\varepsilon) = E \int_0^t f(X_s, y)\, n^*(X_s, dy)\, ds = EB_t(\varepsilon)$$

which gives us (3.8). The inequality (3.9) follows from formula (3.11) in [8], p. 63. Finally, (3.10) follows from (3.8), Lemma 2 and Monotone convergence.

Lemma 4. *Suppose $\psi 2$) hold and for some t, $\varepsilon_0 > 0$, $EA_t(\varepsilon_0) = a_0 > 0$. Then*

$$\limsup_{\varepsilon \to 0+} (EA_t(\varepsilon))^2 / EA_t^2(\varepsilon) > 0. \tag{3.11}$$

Proof. First we estimate $EB_t^2(\varepsilon)$. Put

$$W(y) = \int_\varepsilon^\infty \psi(x, y)\, v\{dx\}.$$

Then we have

$$\begin{aligned} EB_t^2(\varepsilon) &= 2 \int_0^t \int_s^t E[W(X_s)\, W(X_r)]\, dr\, ds \\ &= 2 \int_0^t \int_s^t E[W(X_s)\, E^{X(s)}\, W(X_{r-s})]\, dr\, ds \\ &\leqq 2 \int_0^t \int_s^t [EW(X_s)][EW(X_{r-s})]\, dr\, ds \\ &\leqq 2(EB_t(\varepsilon))^2 = 2(EA_t(\varepsilon))^2. \end{aligned} \tag{3.12}$$

The first inequality in (3.12) comes from $\psi 2$) (and $P[X(s) > 0] = 1$) as follows

$$\begin{aligned} E[W(X_s)\, E^{X(s)}\, W(X_{r-s})] &= E[W(X_s)\, EW(X_{r-s} + y)|_{y = X(s)}] \\ &= EW(X_s)\, E \int_\varepsilon^\infty \psi(x, X_{r-s} + y)\, v\{dx\}|_{y = X(s)} \\ &\leqq EW(X_s)\, EW(X_{r-s}). \end{aligned}$$

From (3.12) and (3.9) we get

$$\begin{aligned} EA_t^2(\varepsilon) &\leqq mEA_t(\varepsilon) + 2EA_t(\varepsilon)\,B_t(\varepsilon) \\ &\leqq mEA_t(\varepsilon) + 2^{3/2}(EA_t^2(\varepsilon))^{1/2}(EA_t(\varepsilon)). \end{aligned}$$

Or, if we let $R_\varepsilon^2 = (EA_t(\varepsilon))^2/EA_t^2(\varepsilon)$, then

$$(R_\varepsilon^{-1} - 2^{\frac{1}{2}})^2 \leqq m(EA_t(\varepsilon))^{-1} + 2 \leqq m a_0^{-1} + 2, \qquad \varepsilon \leqq \varepsilon_0,$$

and (3.11) follows easily.

Theorem 3. *Suppose* $\psi 1)$–$\psi 3)$ *hold. Then either* (a) $P[A_t < \infty] = 1$ *for all* $t < \infty$ *or else* (b) $P[A_t = \infty] = 1$ *for all* $t > 0$. *Alternative* (a) *obtains if and only if* $EA_1 < \infty$ *and alternative* (b) *obtains if and only if* $EA_1 = \infty$.

Proof. 1° From $\psi 2)$, $\psi 3)$ and the fact that $P[X_t > 0] = 1$ for any $t > 0$ (by (3.8) and (3.4)), we get for $0 < t_1 < t_2$

$$\begin{aligned} &P\Big[\sum_{t_1 < s \leqq t_2} \psi(\Delta_s, X_{s-}) < \infty\Big] \\ &\quad \geqq P\Big[\sum_{t_1 < s \leqq t_2} \psi(\Delta_s, X(t_1)) < \infty\Big] \\ &\quad = EP\Big[\sum_{0 < s \leqq t_2 - t_1} \psi(\Delta_s, y) < \infty\Big]\Big|_{y = X t_1)} \\ &\quad = 1. \end{aligned}$$

Since $A_{t_2} = A_{t_1} + \sum_{t_1 < s \leqq t_2} \psi(\Delta_s, X_{s-})$, we find that

$$P[A_t < \infty] = P[A_s < \infty] = P[A_r < \infty \text{ for all } r < \infty],$$

for any s, $t > 0$. But the events $[A_t < \infty]$, $t > 0$, are nested and $[A_t < \infty] \in \mathscr{F}_t = \sigma(X_s, s \leqq t)$. Hence for any t

$$P[A_t < \infty] = \lim_{s \to 0+} P[A_s < \infty] = 0 \quad \text{ or } 1$$

by Blumenthal's 0–1 law. Thus one of (a) or (b) must hold.

2° Suppose $EA_1 < \infty$. Then $EA_t = EB_t < \infty$ for all $t > 0$ by (3.7), (3.10), definition of B_t and $\psi 2)$. Consequently (a) must hold.

3° Suppose $EA_1 = \infty$. By Lemma 2 it follows that $EA_1(\varepsilon) > 0$ for all $\varepsilon > 0$ sufficiently small. Hence, by Lemma 4 and Kochen-Stone, [6], we have with positive probability ($= \alpha$ say)

$$\limsup_{\varepsilon \to 0+} A_1(\varepsilon)/EA_1(\varepsilon) \geqq 1.$$

But then, by Lemma 2,

$$\begin{aligned} A_1 = \lim_{\varepsilon \to 0+} A_1(\varepsilon) &\geqq (\limsup_{\varepsilon \to 0+} A_1(\varepsilon)/EA_1(\varepsilon))(\liminf_{\varepsilon \to 0+} EA_1(\varepsilon)) \\ &= \infty \end{aligned}$$

w.p. $\alpha>0$. But then $\alpha=1$ and (b) holds by the first part of the proof. (Note that this also shows that $EA_1=\infty$ if and only if $EA_t=\infty$ for all $t>0$.) This completes the proof of Theorem 3.

Proof of Theorem 2. Suppose first that X is as in Theorem 3. Put

$$\begin{aligned} \psi(x,y)&=1 \quad \text{for } x>h(y)\\ &=0 \quad \text{for } x\leqq h(y). \end{aligned} \tag{3.14}$$

Clearly

$$\begin{aligned} &P[R_t\cap(x,x+h(x))=\emptyset \text{ i.o. } x\downarrow 0]\\ &\qquad =P\Big[\sum_{0<s\leqq t}\psi(\Delta_s,X_{s-})=\infty\Big]. \end{aligned}$$

As noted in Remark 4, this ψ satisfies $\psi 1)$–$\psi 3)$. Applying Theorem 3 and (3.10) we see that h is a small gap function if and only if

$$E\int_0^1\int_0^\infty \psi(x,X_s)\,v\{dx\}\,ds=E\int_0^1 \bar{v}\circ h(X_s)\,ds=\infty. \tag{3.15}$$

Writing H_1 for the truncated "renewal" measure

$$H_1(y)=H_1\{[0,y]\}=\int_0^1 P[X_s\leqq y]\,ds,$$

and noting that for $\delta>0$

$$\int_\delta^\infty \bar{v}\circ h(y)\,H_1\{dy\}\leqq \bar{v}\circ h(\delta)\int_0^1 P[X_s>\delta]\,ds<\infty,$$

we find that (3.15) is equivalent to

$$\int_0^\delta \bar{v}\circ h(y)\,H_1\{dy\}=\infty \tag{3.16}$$

for any $0<\delta<\infty$.

Now consider the general case of Theorem 2. We may write $X=X^{(+)}-X^{(-)}$ where $X^{(+)}(-X^{(-)})$ is the sum-of-positive(negative)-jumps process. Under (1.4) there are only finitely many negative jumps in $[0,t]$ for any t. Hence if n is this (random) number of negative jumps then

$$R_t=\bigcup_{k=0}^n (R_k^+ + y_k)$$

where the R_k^+ coincide locally with independent copies of the range of $X^{(+)}$ near 0, and $y_1,\dots,y_n$ are random points (the values of X at the negative jump epochs). In particular $R_0^+=R_s(X^{(+)})$ where $s=\min\{t,\sigma_1\}$, $\sigma_1=\min\{r\colon \Delta_r<0\}$. Under (1.4)–(1.5) X does not hit points, so $P[0\in R_k+y_k$ for some $1\leqq k\leqq n]=0$. These facts and right continuity of X make plain that

$$\begin{aligned} &P[R_t\cap(x,x+h(x))=\emptyset \text{ i.o. } x\downarrow 0]\\ &\qquad =P[R_0^+\cap(x,x+h(x))=\emptyset \text{ i.o. } x\downarrow 0]\\ &\qquad =1 \end{aligned}$$

if and only if h is a small gap for the range of $X^{(+)}$ (a fact that was intuitively obvious from the start). Note that the Levy measure for $X^{(+)}$ is just $\nu|_{(0,\infty)}$.

We now return to the case that $\nu\{(-\infty,0)\}=0$. To finish up it suffices to show that (3.16) and (1.6) are equivalent. Clearly (3.16) is equivalent to

$$\int_{0+} H_1(y)\, d\bar{\nu}\circ h(y) = -\infty.$$

(Integrate by parts.) But, as we establish below,

$$H_1(y) \asymp y/\mu(y), \qquad y\to 0+. \tag{3.17}$$

This estimate and another integration by parts shows that (3.16) and (1.6) are indeed equivalent. Let $F_s(x)=P[X_s\leqq x]$, $U_r^\delta=\sum_{n\leqq r} F_\delta^{n*}=\sum_{n\leqq r} F_{n\delta}$ (* denotes convolution). If we integrate the identity $U_r^\delta*(1-F_\delta)=1-F_{(r+1)\delta}$ we get

$$M_{(r+1)\delta}(y)=\int_0^y (1/\delta)\, M_\delta(y-x)\, V_\delta\{dx\}, \tag{3.18}$$

where $r=r(\delta)=$ greatest integer in $1/\delta$, $V_\delta=\delta U_{r(\delta)}^\delta$ and $M_s(x)=\int_0^x (1-F_s(z))\,dz$. We now let $\delta\to 0$ in (3.18). First, since V_δ approximates $\int_0^1 F_s\,ds$, we have

$$V_\delta(x)\to H_1(x)$$

at points of continuity. Next, for all $y>0$,

$$M_{(r+1)\delta}(y)\to M_1(y)$$

by continuity of F_s in s. Finally Fristedt [4], Theorem 4, p. 32, (set $h(x)=x$ there), shows that

$$(1/\delta)\, M_\delta(z)\to \int_0^z \bar{\nu}(x)\,dx=\mu(z),$$

and, since each M_δ is nondecreasing and μ is continuous, this limit is uniform on each bounded z-interval. From (3.18) and these limits we get

$$M_1(y)=\int_0^y \mu(y-x)\, H_1\{dx\}, \qquad y>0. \tag{3.19}$$

But μ is a concave nondecreasing function, so, as in Erickson [2], Lemma 1, we get from (3.19)

$$M_1(y)/\mu(y)\leqq H_1(y)\leqq 2M_1(y)/\mu(y)$$

for all $y>0$. Since $M_1(y)\sim yP[X_1>0]$ as $y\to 0$, (3.17) follows.

Remark 5. Clearly one may add a drift term bt to the right hand side of (3.4) and still obtain the conclusion of Theorem 3 provided $b>0$. By the same token Theorem 2 also remains valid when a nonnegative linear drift is added to X.

§ 4. Examples

(a) To Theorem 1. *Though h be a gap function $2h$ need not be.* In Remark 2 in § 2 we showed that if $h(x)$ is a gap function then so is $h_c(x) = h(cx)$ for any $c > 0$. In view of this one might suppose that $\mathrm{ch}(x)$ is also a gap function. This is obviously true for $c \leqq 1$ but for $c > 1$ it is not as the following example shows.

Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. with $P[Y_i > x] = q(x) = e^{-x}$, $x > 0$, and X the random walk with $X_n = Y_1 + Y_2 + \ldots + Y_n$. Then $xq(x)/m(x) = xe^{-x}/(1 - e^{-x}) \to 0$, $x \to \infty$, and from Theorem 1 we see that h is a large gap function if and only if

$$\int^{\infty} \exp(-h(x))\, dx = \infty.$$

For example $h(x) = \log x$ is a gap function but $2\log x$ is not.

In spite of this example we can prove the

Proposition. *Assume the set up of Sect. 2. If h is a convex gap function then ch is a gap function for any $c > 0$.*

Proof. It suffices to prove this in case $c = 2$. But h convex implies h^{-1} is concave. We may suppose $h^{-1}(0) \geqq 0$, then $h^{-1}(x) \leqq 2h^{-1}(x/2)$, and it follows that $\int^{\infty} \Delta \circ h^{-1}(x/2)\, V\{dx\} = \infty$ by (2.4). But the inverse of $x \mapsto 2h(x)$ is $x \mapsto h^{-1}(x/2)$.

(b) To Theorem 1. *Finite mean. Suppose the ξ_i (notation as in § 2) have a finite mean. Then, since $X_n \to +\infty$, $E\xi_i > 0$, and $(1 - xq(x)/m(x))/m(x) \to 1/E\xi_1^+ < \infty$. Hence h is a gap function if and only if*

$$\int^{\infty} q \circ h(x)\, dx = \infty.$$

In particular $h(x) = cx$ cannot be a gap function for any $c > 0$.

(c) To Theorem 1. *Regularly varying Tails.* Let F stand for the distribution of the $\xi_n = X_n - X_{n-1}$ in Theorem 1. We will suppose that

$$P[\xi_n > x] = 1 - F(x) = q(x) = x^{-\alpha} L(x), \qquad x > 0, \tag{4.1}$$

$$\int_{-\infty} |x|/m(|x|)\, F\{dx\} < \infty, \qquad \int_{-\infty}^{\infty} |x|\, F\{dx\} = \infty, \tag{4.2}$$

where L is slowly varying and $0 \leqq \alpha \leqq 1$. According to Erickson [2], (4.2) implies that X drifts to $+\infty$. From properties of regularly varying functions, see [3], we have $xq(x)/m(x) \to 1 - \alpha$, as $x \to \infty$. Hence, for $\alpha \neq 0$, h is a gap function for the range of X if and only if

$$\int^{\infty} h(x)^{-\alpha} L \circ h(x)/m(x)\, dx = \infty. \tag{4.3}$$

Hence $h(x) = x^{\beta}$ is a large gap function when $\beta < 1$ and not when $\beta > 1$. Now suppose that $L(x)$ is asymptotically a finite positive constant as is the case for stable random walks. Suppose further that $\alpha \neq 0$ and $\alpha \neq 1$. Then we have $m(x) \asymp x^{1-\alpha}$ and (4.3) reduces to

$$\int^{\infty} (x/h(x))^{\alpha} x^{-1} dx = \infty. \tag{4.4}$$

For example $h(x) = x(\log x)^{1/\alpha}$ is a gap function (but not when $\alpha = 1$).

(d) To Theorem 2. *Stable subordinators.* Suppose X in Theorem 2 satisfies

$$E e^{-\lambda X_t} = e^{-tC_1 \lambda^{\alpha}}, \qquad t \geqq 0, \ \lambda > 0.$$

For X not to be degenerate we must have $0 < \alpha < 1$. Then ν, the Levy measure, is given by

$$\nu\{dx\} = C_2 x^{-\alpha-1} dx, \qquad x > 0,$$

and $\nu\{(-\infty, 0)\} = 0$. The constant C_1 is irrelevant so we may as well choose it so that $C_2 = \alpha$. Integrating gives

$$\bar{\nu}(x) = x^{-\alpha}, \qquad \mu(x) = x^{1-\alpha}/(1-\alpha)$$

and

$$[1 - x\bar{\nu}(x)/\mu(x)]/\mu(x) = \alpha(1-\alpha)\, x^{1-\alpha}.$$

So h is a small gap function for the range if and only if

$$\int_{0+} (x/h(x))^{\alpha} x^{-1} dx = \infty.$$

Note the similarity with (4.4). In particular $h(x) = x^{\beta}$ is a small gap if and only if $\beta \geqq 1$; $h(x) = x|\log x|^{\beta}$ is small gap if and only if $\beta \leqq 1/\alpha$.

§ 5. The Pruitt-Taylor Theorem

There are transient s.i.i. processes on $(-\infty, \infty)$ which jump across the origin infinitely often on their way to infinity:

$$P[\lim_{t\to\infty} |X_t| = \infty, \liminf_{t\to\infty} X_t = -\infty, \limsup_{t\to\infty} X_t = +\infty] = 1. \tag{5.1}$$

These processes are quite wild. The range of such a process may have large holes far out to the left vastly different from those far out to the right. Let us say that $h \in \uparrow$ is a right (large) gap function if $P\{(1.1)\} = 1$ and a left gap function if $P\{(1.1)\} = 1$ when X is replaced by $-X$. In our proof of Theorem 1 we used in an essential way the fact that there is positive probability that X will never hit $(-\infty, 0)$ on $[X_0 \geqq 0]$. Our proof fails completely when X satisfies (5.1) and one needs an entirely different approach.

A special but very important class of processes satisfying (5.1) are the (not completely) asymmetric Cauchy process: the s.i.i. process X is asymmetric Cauchy if $\log E e^{iX_t\theta} = -t|\theta|(1 + i\beta \operatorname{sign}(\theta) \log|\theta|)$ where $\beta \in (-2/\pi, 2/\pi)$, $\beta \neq 0$. Let $I_n = (n, n + h(n))$ (or $(-n - h(n), -n)$ if we are considering left gaps) and $A_n = [X$ never hits $I_n]$. By making very intricate precise asymptotic estimates of

$P(A_n)$ and $P(A_n A_m)$, using general potential theory, W.E. Pruitt and S.J. Taylor [7] proved a remarkable result concerning the gaps in the range of X. Translated into our terminology it reads

Theorem (Pruitt-Taylor, 1977). *Let X be an asymmetric Cauchy process with $0<\beta<2/\pi$.*

(a) *Let $h>0$, and $x\to h(x)/x\in\uparrow$. Then h is a right large gap function iff*

$$\int^{\infty} (h(x)\log x)^{-1}\,dx=\infty. \tag{5.2}$$

(b) *Let $h>0$ and $x\mapsto x+h(x)\in\uparrow$. Then h is a left large gap function iff*

$$\int^{\infty} (\log x/\log(x+h(x)))^{\rho/(1-\rho)}(x\log x)^{-1}\,dx=\infty. \tag{5.3}$$

In (5.3) $\rho=(2-\pi\beta)/(2+\pi\beta)$.

It is interesting that one may show that (5.2) is necessary with very little effort as follows. Consider the a.f.

$$A_t=\sum_{0<s\leqq t}\psi(\Delta_s, X_{s-})$$

where, as in the proof of Theorem 2, $\psi(x,y)=1$ if $x>h(y)$, $\psi(x,y)=0$ otherwise. Let us suppose $h(0+)>0$. Then, as in Lemma 3,

$$\begin{aligned} EA_t &= E\int_0^t \bar{v}\circ h(X_s)\,ds \\ &= \int_0^{\infty} \bar{v}\circ h(y)\,\Gamma_t\{dy\} \end{aligned}$$

which is finite if $h(0+)>0$. Here $\Gamma_t\{dy\}=\int_0^t P[X_s\in dy]\,ds$. But for h to be a right large gap function one must clearly have $A_t\to A_\infty=\infty$ a.s. as $t\to\infty$. Hence

$$\int^{\infty} \bar{v}\circ h(y)\,\Gamma_\infty\{dy\}=\infty. \tag{5.4}$$

But if X is the asymmetric Cauchy process $\bar{v}\circ h(y)=C_1/h(y)$, $\Gamma_\infty\{dy\}\asymp(C_2/\log y)\,dy$ as $y\to\infty$. (See [7], p. 114.) Substituting these formulas into (5.4) yields (5.2). Unfortunately the analagous argument for the left side does not yield (5.3).

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Note. In this paper $\asymp$ means the ratio of two quantities is bounded away from 0 and ∞, and $\sim$ means that the ratio goes to 1.

Note Added in Proof

Let X be a symmetric stable process on the line of index $\alpha<1$. Philip Griffin, University of Washington, has obtained the following: An increasing function h is a right (or left) large gap function for X if and only if $\int^{\infty} (x/h(x))^{\alpha/2} x^{-1}\,dx = \infty$. It follows that, contrary to what one might expect, the gap functions in this case are larger than the gap functions in the case of an increasing stable process of the same index. See (4.4). A complete discussion will appear elsewhere.

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