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Random Polytopes in a Convex Body

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Let K be a convex body with interior points in d-dimensional Euclidean space $\mathbb{R}^d$, and let $x_1, \dots, x_N$ be random points in K, independently and uniformly distributed. For any real function φ which is defined on the set of convex polytopes in $\mathbb{R}^d$, we denote by $E_N(\varphi)$ the expectation of $\varphi(\operatorname{conv}\{x_1, \dots, x_N\})$, provided it exists, where conv denotes the convex hull. For $d=2$ and under additional assumptions on K, Rényi and Sulanke [4], [5] have studied the asymptotic behaviour, as $N \to \infty$, of $E_N(\varphi)$ for various functions φ. In particular, for sufficiently smooth K they proved that

$$E_N(L) = L(K) - \Gamma\left(\frac{2}{3}\right) 12^{-1/3} \int_{\partial K} \kappa^{4/3}\, ds \left(\frac{N}{F(K)}\right)^{-2/3} + O(N^{-1}), \tag{1}$$

$$E_N(F) = F(K) - \Gamma\left(\frac{8}{3}\right) 12^{2/3}\, 10^{-1} \int_{\partial K} \kappa^{1/3}\, ds \left(\frac{N}{F(K)}\right)^{-2/3} + O(N^{-1}), \tag{2}$$

where L is the perimeter and F the area, and $\kappa > 0$ denotes the curvature of ∂K.

For further results on convex hulls of random points, the reader is referred to the surveys and references given in Santaló [6], pp. 22–33, and Baddeley [1], Chap. 2.

Extensions of (1) or (2) to higher dimensions, with L and F replaced by surface area and volume, respectively, are unknown, except for the case where K is a ball (Wieacker [8]). The object of this note is to extend (1) to $d > 2$, with the perimeter replaced by the mean width $\bar{b}$.

The mean width of a convex body K in $\mathbb{R}^d$ is, up to a constant factor, one of the Minkowski quermassintegrals $W_0, \dots, W_d$ of K. These can be defined as the coefficients in the Steiner polynomial

$$V(K_\varepsilon) = \sum_{i=0}^{d} \binom{d}{i} \varepsilon^i W_i,$$

where $V(K_\varepsilon)$ is the volume of the parallel body K_ε of K at distance ε (see Bonnesen-Fenchel [2], p. 49). In particular, W_0 is the volume and dW_1 the

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surface area of K, and $2\beta_d^{-1} W_{d-1}=\bar{b}$ is the mean width of K (β_d denotes the volume of the d-dimensional unit ball). If $b(u)$ denotes the width of K in direction u (a unit vector), that is, the distance between the two supporting hyperplanes of K orthogonal to u, then $\bar{b}$ is the mean value of $b(u)$ over all directions u (see Bonnesen-Fenchel [2], p. 50 and p. 63). For $d=3$ and sufficiently smooth K, the mean width is a constant multiple of the so-called integral mean curvature. For $d=2$, $\pi\bar{b}$ is equal to the perimeter L; hence the following theorem is a natural generalization of (1).

Theorem. *If K is a convex body in $\mathbb{R}^d$ with a boundary of class C^3 and of positive Gauss-Kronecker curvature κ, then*

$$E_N(\bar{b})$$
$$=\bar{b}(K)-\frac{2}{d(d+1)\beta_d}\left(\frac{d+1}{\beta_{d-1}}\right)^{\frac{2}{d+1}}\Gamma\left(\frac{2}{d+1}\right)\int_{\partial K}\kappa^{\frac{d+2}{d+1}}\,dS\left(\frac{N}{V}\right)^{-\frac{2}{d+1}}+O(N^{-\frac{3}{d+1}}), \quad (3)$$

where V is the volume of K, dS denotes the surface area element, and β_i is the volume of the i-dimensional unit ball.

We remark that for $d=2$ the proof which follows is simpler than that given by Rényi-Sulanke. This simplification, which opens the way to a higher dimensional generalization, is achieved by utilizing an observation of Efron [3] (formula (4.2)).

Proof of the Theorem. Let U denote the unit sphere of $\mathbb{R}^d$. In the following, hyperplanes H of $\mathbb{R}^d$ are to be parametrized by $u\in U$, $t\in\mathbb{R}$ via

$$H=\{x\in\mathbb{R}^d\colon \langle x,u\rangle=t\},$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product of $\mathbb{R}^d$. If for a convex body A in $\mathbb{R}^d$ we define

$$f(A,u,t)=\begin{cases}1, & \text{if } A\cap\{x\in\mathbb{R}^d\colon\langle x,u\rangle=t\}\neq\emptyset\\ 0 & \text{otherwise,}\end{cases}$$

then the mean width of A can be written as

$$\bar{b}(A)=\int_U\int_{-\infty}^{\infty} f(A,u,t)\,dt\,d\omega(u),$$

where ω denotes the rotation invariant probability measure on U.

The distribution ν of a uniform random point in K is defined by

$$\nu(B)=\lambda(K\cap B)/V$$

for Borel sets $B\subset\mathbb{R}^d$, where λ denotes Lebesgue measure in $\mathbb{R}^d$. Therefore we have

$$\begin{aligned} E_N(\bar{b}) &= \int \dots \int \bar{b}(\operatorname{conv}\{x_1, \dots, x_N\})\, dv(x_1) \dots dv(x_N) \\ &= \int \dots \int \int_U \int_{-\infty}^{\infty} f(\operatorname{conv}\{x_1, \dots, x_N\}, u, t)\, dt\, d\omega(u)\, dv(x_1) \dots dv(x_N) \\ &= \int_U \int_{-\infty}^{\infty} \left[\int \dots \int f(\operatorname{conv}\{x_1, \dots, x_N\}, u, t)\, dv(x_1) \dots dv(x_N)\right] dt\, d\omega(u), \end{aligned} \tag{4}$$

where the application of Fubini's theorem is easily justified.

Suppose that the hyperplane $H = \{x \in \mathbb{R}^d \colon \langle x, u \rangle = t\}$ intersects K and that the two parts of K into which K is divided by H have volumes v $(= v(u, t))$ and $V - v$, respectively, where $v \leqq V/2$. Further, let

$$h(u) = \max\{\langle x, u \rangle : x \in K\} \qquad \text{for } u \in U.$$

We have $f(\operatorname{conv}\{x_1, \dots, x_N\}, u, t) = 1$ if and only if not all of the points $x_1, \dots, x_N$ lie on one side of H, hence the integral in brackets in (4) is equal to

$$1 - \left(\frac{v}{V}\right)^N - \left(1 - \frac{v}{V}\right)^N$$

if $-h(-u) \leqq t \leqq h(u)$, and equal to 0 otherwise. Thus we get

$$E_N(\bar{b}) = \bar{b}(K) - 2 \int_U \int_0^{h(u)} \left[\left(\frac{v}{V}\right)^N + \left(1 - \frac{v}{V}\right)^N\right] dt\, d\omega(u). \tag{5}$$

Following Rényi-Sulanke we observe that

$$\int_U \int_0^{h(u)} \left(\frac{v}{V}\right)^N dt\, d\omega(u) = O(2^{-N}) \tag{6}$$

and, writing $w(u, \eta) := v(u, h(u) - \eta)$, that

$$\int_0^{h(u)} \left(1 - \frac{v}{V}\right)^N dt = \int_0^c \left(1 - \frac{w}{V}\right)^N d\eta + O((1 - \varepsilon)^N) \tag{7}$$

if $c > 0$ (small) is given and $\varepsilon > 0$ and the origin of $\mathbb{R}^d$ are chosen such that $w/V \geqq \varepsilon$ for $c \leqq \eta \leqq h(u)$. Clearly such a choice is possible independent of u.

For given unit vector u, let $y \in \partial K$ be the point at which K has u as exterior normal vector (y is unique because we assume $\kappa > 0$). We choose $c > 0$ (independent of u, which is clearly possible) so small that every boundary point x of K lying in a hyperplane

$$H_{u,\eta} = \{z \in \mathbb{R}^d \colon \langle z, u \rangle = h(u) - \eta\}$$

with $0 \leqq \eta \leqq c$ can be represented in the form $x = y + h - f(u)u$ with $\langle h, u \rangle = 0$, where f is a real C^3 function defined in a convex neighbourhood of 0 in the linear subspace L parallel to the tangent plane $H_{u,0}$. For given $\eta \in]0, c]$ let $h' \in L$ be a point at which the function

$$h \mapsto \left| f(h) - \frac{1}{2} \sum_{i,j=1}^{d-1} f_{ij}(0)\, h_i h_j \right| \qquad (h \in L)$$

attains its maximum, say $b(\eta)$, under the condition $f(h)=\eta$. Here $h_1, \dots, h_{d-1}$ are cartesian coordinates of h, and $f_{ij}=\partial^2 f/\partial h_i \partial h_j$. In the following, $c_1, c_2, \dots$ denote positive constants which can be chosen independent of u. It follows from Taylor's theorem that

$$b(\eta) \leqq c_1 \|h'\|^3. \tag{8}$$

The eigenvalues of the matrix $(f_{ij}(0))_{i,j=1}^{d-1}$ are the principal curvatures k_i ($i=1, \dots, d-1$) of ∂K at y and hence have a positive lower bound c_2. We have

$$\sum_{i,j=1}^{d-1} f_{ij}(0)\, h'_i h'_j \geqq c_2 \|h'\|^2,$$

hence

$$\eta = f(h') \geqq \frac{1}{2} \sum_{i,j=1}^{d-1} f_{ij}(0)\, h'_i h'_j - b(\eta) \geqq c_2 \|h'\|^2 - c_1 \|h'\|^3 \geqq c_3 \|h'\|^2, \tag{9}$$

provided that c has been chosen sufficiently small. Inequalities (8) and (9) together yield

$$b(\eta) \leqq c_4\, \eta^{3/2}. \tag{10}$$

By the definition of $b(\eta)$, every point $h \in L$ with $f(h)=\eta$ satisfies

$$\eta - b(\eta) \leqq \frac{1}{2} \sum_{i,j=1}^{d-1} f_{ij}(0)\, h_i h_j \leqq \eta + b(\eta).$$

Defining the $(d-1)$-dimensional ellipsoids

$$E_{\pm} = \{y + h - \eta u \colon h \in L \text{ and } \frac{1}{2} \sum_{i,j=1}^{d-1} f_{ij}(0)\, h_i h_j = \eta \pm b(\eta)\}$$

we deduce that $E_- \subset K \cap H_{u,\eta} \subset E_+$. Hence the $(d-1)$-dimensional volume $\lambda_{d-1}(K \cap H_{u,\eta})$ of the intersection $K \cap H_{u,\eta}$ lies between the $(d-1)$-volumes of these ellipsoids, which have semi-axes $\sqrt{2(\eta \pm b(\eta))/k_i}$ ($i=1, \dots, d-1$). This yields

$$\lambda_{d-1}(K \cap H_{u,\eta}) = \beta_{d-1}\, \kappa^{-1/2} (2\eta)^{\frac{d-1}{2}} (1 + \varphi(\eta))$$

with $|\varphi(\eta)| \leqq c_5\, \eta^{1/2}$, and hence

$$w = \int_0^{\eta} \lambda_{d-1}(K \cap H_{u,\rho})\, d\rho = \frac{\beta_{d-1}}{d+1}\, \kappa^{-1/2} (2\eta)^{\frac{d+1}{2}} (1 + \varphi_1(\eta))$$

with $|\varphi_1(\eta)| \leqq c_6\, \eta^{1/2}$. Now from (5), (6), (7), assuming $\varepsilon \leqq 1/2$, we get

$$E_N(\bar{b}) = \bar{b}(K) - 2 \int_U J\, d\omega(u) + O((1-\varepsilon)^N)$$

where

$$J=\int_0^c \left[1-\frac{2^{\frac{d+1}{2}}\beta_{d-1}\kappa^{-1/2}}{(d+1)V}\eta^{\frac{d+1}{2}}(1+\varphi_1(\eta))\right]^N d\eta.$$

Here we substitute

$$t=Na\eta^{\frac{d+1}{2}} \quad \text{with } a=\frac{2^{\frac{d+1}{2}}\beta_{d-1}\kappa^{-1/2}}{(d+1)V}$$

to get

$$J=\frac{2}{d+1}(aN)^{-\frac{2}{d+1}}\int_0^{\tau N}\left[\left(1-\frac{t}{N}\right)-\frac{t}{N}\psi\left(\frac{t}{N}\right)\right]^N t^{\frac{1-d}{d+1}}dt$$

with $\tau=ac^{\frac{d+1}{2}}$ and $\left|\psi\left(\frac{t}{N}\right)\right|\leqq c_7\left(\frac{t}{N}\right)^{\frac{1}{d+1}}$. We assume, as we may, that $\tau\leqq 1$. Developing the integrand by the binomial theorem and using

$$\int_0^{\tau N}\left(1-\frac{t}{N}\right)^N t^\alpha dt=\Gamma(\alpha+1)+O\left(\frac{1}{N}\right),$$

which may be proved by an obvious modification of the argument given in Whittaker-Watson [7], p. 242, we obtain

$$J=\frac{2}{d+1}(aN)^{-\frac{2}{d+1}}\left[\Gamma\left(\frac{2}{d+1}\right)+O\left(N^{-\frac{1}{d+1}}\right)\right].$$

Now we integrate over U, equivalently over ∂K using $d\omega=\kappa\, dS/d\beta_d$, thus completing the proof of (3).

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