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A Generalization of Strassen's Functional Law of Iterated Logarithm

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Dedicated to Professor Leopold Schmetterer on his 60th birthday

Summary. Let $\{a_T; T \geqq 0\}$ be a non-decreasing function of T with $0 < a_T \leqq T$ and let $\{W(t); t \geqq 0\}$ be a Wiener process. The limit functions of the processes

$$\Gamma_{t,T}(x) = \beta_T(W(t + x a_T) - W(t)) \qquad (0 \leqq x \leqq 1)$$

where $0 \leqq t \leqq T - a_T$, $T > 0$ and

$$\beta_T = (2 a_T (\log T a_T^{-1} + \log \log T))^{-\frac{1}{2}}$$

are characterized. In the case $a_T = T$ Strassen's law is obtained as a special case.

1. Introduction

Let $\mathscr{S} \subset C(0, 1)$ be the Strassen's class of absolutely continuous functions defined on (0, 1) that is to say the absolutely continuous function $S(x)$ $(0 \leqq x \leqq 1)$ belongs to $\mathscr{S}$ if and only if $\int_0^1 (S'(x))^2 \, dx \leqq 1$. Further let $\{W(x); x \geqq 0\} = \{W(x, \omega); \omega \in \Omega, x \geqq 0\}$ be a Wiener process (here Ω is the underlying probability space). Finally for any $A \subset C(0, 1)$ and $\varepsilon > 0$ denote $U(A, \varepsilon)$ be the ε-neighbourhood of A in $C(0, 1)$ metrics, that is a continuous function $\alpha(x)$ is an element of $U(A, \varepsilon)$ if there exists an $a(x) \in A$ such that $\sup_{0 \leqq x \leqq 1} |\alpha(x) - a(x)| \leqq \varepsilon$.

Then the celebrated Strassen's functional law of iterated logarithm can be formulated as follows:

Theorem A (Strassen, 1964). *Put*

$$W_T^*(x) = \frac{W(xT)}{(2T \log \log T)^{\frac{1}{2}}} \qquad (0 \leqq x \leqq 1).$$

Then for almost all $\omega \in \Omega$ and for all $\varepsilon > 0$ there exists a $T_0 = T_0(\omega, \varepsilon)$ such that

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$$W_T^*(x) \in U(\mathcal{S}, \varepsilon)$$

if $T \geqq T_0$. *Further for any* $s = s(x) \in \mathcal{S}$, $\varepsilon > 0$ *and for almost all* $\omega \in \Omega$ *there exists a* $T = T(\omega, \varepsilon, s)$ *such that*

$$\sup_{0 \leqq x \leqq 1} |W_T^*(x) - s(x)| < \varepsilon.$$

The meaning of this Theorem is the following:

a) for any T big enough the process $W_T^*(x)$ can be approximated by a suitable element of $\mathcal{S}$,

b) any $s(x) \in \mathcal{S}$ will be approximated by $W_T^*(x)$ for a suitable T.

This Theorem clearly implies:

Consequence A. *For any* $\varepsilon > 0$ *and for almost all* $\omega \in \Omega$ *there exists a* $T_0 = T_0(\varepsilon, \omega)$ *such that if*

$$W(T) \geqq (1-\varepsilon)(2T \log\log T)^{\frac{1}{2}} \quad \textit{for some } T \geqq T_0$$

then

$$\sup_{0 \leqq t \leqq T} \left| W(t) - t \left(\frac{2 \log\log T}{T} \right)^{\frac{1}{2}} \right| \leqq 2\varepsilon (2T \log\log T)^{\frac{1}{2}}.$$

Consequence A tells us that if $W(t)$ "wants" to be as big in point T as it can be at all then it has to increase in $(0, T)$ nearly linearly (that is to say it has to minimize the used energy).

A related question is: How big can the increment of a Wiener process be in an interval of size $a_T < T$ if we observe the process in $[0, T]$. Introducing the following notations:

$$W(x + \Delta x) - W(x) = V(x; \Delta x) = V(x; \Delta x, \omega),$$
$$\beta_T = (2 a_T (\log T a_T^{-1} + \log\log T))^{-\frac{1}{2}},$$

an answer to the above question is:

Theorem B (Csörgő-Révész, 1978, 1979). *Let* $0 < a_T \leqq T$ *be a function of* $T \geqq 0$ *satisfying the following conditions:*

(i) a_T *is non-decreasing,*

(ii) $T a_T^{-1}$ *is non-decreasing.*

Then we have

$$\limsup_{T \to \infty} \sup_{0 \leqq t \leqq T - a_T} \beta_T V(t; a_T) = 1 \quad a.s.$$

If we also assume that

$$\text{(iii)} \quad \frac{\log T a_T^{-1}}{\log\log T} \to \infty \quad \textit{as } T \to \infty$$

then we also have

$$\lim_{T\to\infty} \sup_{0\leqq t\leqq T-a_T} \beta_T V(t;a_T)=1 \quad a.s.$$

This Theorem clearly implies

Consequence B. *For almost all $\omega\in\Omega$ and for all $\varepsilon>0$ there exists a $T_0=T_0(\varepsilon,\omega)$ such that for all $T\geqq T_0$ there is a corresponding $0\leqq t=t(\omega,\varepsilon,T)\leqq T-a_T$ such that*

$$V(t;a_T)\geqq(1-\varepsilon)\beta_T^{-1}\approx(1-\varepsilon)(2a_T\log Ta_T^{-1})^{\frac{1}{2}} \tag{1}$$

provided that a_T satisfies conditions (i), (ii), (iii).

Knowing Consequence A we can pose the following question: does inequality (1) imply that $W(x)$ is increasing nearly linearly in $(t,t+a_T)$? The answer to this question is positive in the same sense as in the case of Consequence A, and we also can formulate the following more general result.

Theorem 1. *For all $\omega\in\Omega$ define the set $V_T(\omega)\subset C(0,1)$ as follows:*

$$V_T(\omega)=\{\Gamma_{t,T}(x,\omega):\ 0\leqq t\leqq T-a_T\}$$

where

$$\Gamma_{t,T}(x,\omega)=\Gamma_{t,T}(x)=\beta_T V(t;xa_T) \qquad (0\leqq x\leqq 1;\ 0\leqq t\leqq T-a_T).$$

Then for almost all $\omega\in\Omega$ and for all $\varepsilon>0$ there exists a $T_0=T_0(\omega,\varepsilon)$ such that

$$U(V_T(\omega),\varepsilon)\supset\mathscr{S} \tag{2}$$

and

$$U(\mathscr{S},\varepsilon)\supset V_T(\omega) \tag{3}$$

if $T\geqq T_0$ provided that a_T satisfies conditions (i), (ii), (iii).

On the meaning of this Theorem let us mention that it says that:

(a) for all T big enough and for all $s(x)\in\mathscr{S}$ there exists a $0<t<T$ such that $\Gamma_{t,T}(x)$ $(0\leqq x\leqq 1)$ will approximate the given $s(x)$,

(b) for all T big enough and for every $0<t<T-a_T$ the function $\Gamma_{t,T}(x)$ $(0\leqq x\leqq 1)$ can be approximated by a suitable element $s(x)\in\mathscr{S}$.

We have to emphasize that in Theorem 1 we assumed all the conditions (i), (ii), (iii). If we only assume conditions (i) and (ii) then we get a weaker result which contains Theorem A in case $a_T=T$.

Theorem 2. *Assume that a_T satisfies conditions* (i) *and* (ii). *Then for almost all $\omega\in\Omega$ and for all $\varepsilon>0$ there exists a $T_0=T_0(\varepsilon,\omega)$* such that

$$V_T(\omega)\subset U(\mathscr{S},\varepsilon) \tag{3*}$$

if $T\geqq T_0$. Further for any $s=s(x)\in\mathscr{S}$, $\varepsilon>0$ and for almost all $\omega\in\Omega$ there exist a $T=T(\varepsilon,\omega,s)$ and a $0\leqq t=t(\varepsilon,\omega,s)\leqq T-a_T$ such that

$$\sup_{0\leqq x\leqq 1}|\Gamma_{t,T}(x)-s(x)|\leqq\varepsilon. \tag{2*}$$

Remark. The important difference between Theorems 1 and 2 is the fact that in Theorem 1 we stated that for *every* T big enough and for *every* $s(x)\in\mathscr{S}$ there exists a $0\leqq t\leqq T-a_T$ such that $\Gamma_{t,T}(x)$ approximates the given $s(x)$ while in Theorem 2 we only stated that for every $s(x)\in\mathscr{S}$ there exists a T (in fact there exist infinitely many T but not all T are suitable as in Theorem 1) and a $0\leqq t\leqq T-a_T$ such that $\Gamma_{t,T}(x)$ approximates the given $s(x)$.

In other words if a_T is small (condition (iii) holds true), then for every T (big enough) the random functions $\Gamma_{t,T}(x)$ will approximate every element of $\mathscr{S}$ as t runs over the interval $[0, T-a_T]$. However if a_T is large then for any fixed T the random functions $\Gamma_{t,T}(x)$ $(0\leqq t\leqq T-a_T)$ will approximate some elements of $\mathscr{S}$ but not all of them; all of them will be approximated when T is also allowed to vary. (An analogue of Theorem 2 was also given in Chan, Csörgö, Révész, 1978).

2. Some Lemmas

Lemma 1. *Let m be a given integer, $\lambda_1, \lambda_2, \ldots, \lambda_m$ be real numbers with $\sum_{i=1}^{m}\lambda_i^2=m$ and define*

$$\begin{aligned}\mathscr{G}_h(t)&=\mathscr{G}_h(t;\lambda_1,\lambda_2,\ldots,\lambda_m)\\&=\sum_{i=1}^{m}\lambda_i\left(W\left(t+i\frac{h}{m}\right)-W\left(t+(i-1)\frac{h}{m}\right)\right).\end{aligned}$$

Then there exists a universal constant C such that

$$P\{\sup_{0\leqq t\leqq 1-h}|\mathscr{G}_h(t)|\geqq uh^{\frac{1}{2}}\}\leqq Cmh^{-1}ue^{-\frac{u^2}{2}}$$

for every $0<h<1$, $u>1$.

The proof of this Lemma is based on

Slepian's Lemma (Slepian, 1962). *Let $\mathscr{G}(t)$ and $\mathscr{G}^*(t)$ be Gaussian processes (possessing continuous sample functions). Suppose that these are standardized so that $E\Gamma(t)=E\Gamma^*(t)=0$, $E\Gamma^2(t)=E(\Gamma^*(t))^2=1$ and write $\rho(t,s)$ and $\rho^*(t,s)$ for their covariance functions. Suppose that for some $T>0$ we have $\rho(t,s)\geqq\rho^*(t,s)$ when $0\leqq t, s\leqq T$. Then*

$$P\{\sup_{0\leqq t\leqq T}|\Gamma(t)|\leqq u\}\geqq P\{\sup_{0\leqq t\leqq T}|\Gamma^*(t)|\leqq u\}$$

for all $u>0$.

Proof of Lemma 1. Put

$$\mathscr{G}_h^*(t)=m^{\frac{1}{2}}(W(t+h/m)-W(t))=\mathscr{G}_h(t;m^{\frac{1}{2}},0,0,\ldots,0).$$

Then the Slepian's lemma implies that

$$\begin{aligned}&P\{\sup_{0\leqq t\leqq 1}|\mathscr{G}_h(t;\lambda_1,\lambda_2,\ldots,\lambda_m)|\geqq uh^{\frac{1}{2}}\}\\&\quad\leqq P\{\sup_{0\leqq t\leqq 1-h}|\mathscr{G}_h^*(t)|\geqq uh^{\frac{1}{2}}\}.\end{aligned}$$

The covariance function $\rho^*(t,s)$ of $\mathscr{G}_h^*(t)$ is

$$\rho^*(t,s)=\begin{cases}1-\dfrac{|t-s|}{h} & \text{if } |t-s|\leqq h,\\[2mm] 0 & \text{otherwise.}\end{cases}$$

Hence applying the result of Qualls and Watanabe (1972, Theorem 2.1) for the Gaussian process $\mathscr{G}_h^*(t)$ we get our Lemma 1.

Lemma 2. *Let m be a given positive integer, $\mu_1, \mu_2, \ldots, \mu_m$ be a sequence of real numbers with $m\sum_{i=1}^{m}\mu_i^2=1$. Then*

$$\lim_{T\to\infty}\ \inf_{0\leqq t\leqq T-aT}\ \max_{1\leqq i\leqq m}\left|\frac{V\left(t+(i-1)\dfrac{a_T}{m};\dfrac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}}-\mu_i\right|=0 \quad a.s. \tag{4}$$

provided that a_T satisfies conditions (i), (ii), (iii).

Proof. In fact the following stronger relations will be proved

$$\lim_{T\to\infty}\ \min_{0\leqq j\leqq[Ta_T^{-1}-1]}\ \max_{1\leqq i\leqq m}\left|\frac{V\left(ja_T+(i-1)\dfrac{a_T}{m};\dfrac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}}-\mu_i\right|=0 \quad \text{a.s.}$$

As a first step let us evaluate the probability

$$p=P\left\{\left|\frac{W\left(\dfrac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}}-\mu\right|\leqq\varepsilon\right\}.$$

In case $\mu>0$ we have

$$p\geqq P\left\{(\mu-\varepsilon)(2a_T\log Ta_T^{-1})^{\frac{1}{2}}\leqq W\left(\frac{a_T}{m}\right)\leqq\left(\mu-\frac{\varepsilon}{2}\right)(2a_T\log Ta_T^{-1})^{\frac{1}{2}}\right\}$$
$$\geqq\frac{\varepsilon}{2}(2a_T\log Ta_T^{-1})^{\frac{1}{2}}\exp\left\{-\left(\mu-\frac{\varepsilon}{2}\right)^2 m\ \log Ta_T^{-1}\right\}$$

and hence

$$p\geqq\exp\left\{-\left(1-\frac{\varepsilon}{4}\right)\mu^2 m\log Ta_T^{-1}\right\} \tag{5}$$

if ε is small enough and T is big enough. One can similarly see that (5) holds true for $\mu<0$.

This implies

$$P\left\{\max_{1\leqq i\leqq m}\left|\frac{V\left((i-1)\dfrac{a_T}{m};\dfrac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}}-\mu_i\right|\leqq\varepsilon\right\}\geqq(a_T T^{-1})^{1-\varepsilon/4}$$

if ε is small enough and T is big enough. Hence we have

$$P\left\{\min_{0\leqq j\leqq[Ta_T^{-1}-1]}\max_{1\leqq i\leqq m}\left|\frac{V\left(ja_T+(i-1)\frac{a_T}{m};\frac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}}-\mu_i\right|\geqq\varepsilon\right\}$$
$$\leqq(1-(a_TT^{-1})^{1-\varepsilon/4})^{[Ta_T^{-1}-1]}\leqq\exp\{-(Ta_T^{-1})^{\varepsilon/4}\}.$$

Let $T_n=n$ then by condition (iii):

$$\sum_{n=1}^{\infty}\exp\{-(Ta_T^{-1})^{\varepsilon/4}\}<\infty$$

and by the Borel-Cantelli Lemma we get (4) for $T=T_n$. The general case (the case of arbitrary T) follows simply from this special case.

Lemma 3. *Let m be a given positive integer, $\lambda_1,\lambda_2,\ldots,\lambda_m$ be real numbers with $\sum_{i=1}^{m}\lambda_i^2=m$. Then*

$$\limsup_{T\to\infty}\sup_{0\leqq t\leqq T-a_T}\beta_T|\mathcal{G}_{a_T}(t;\lambda_1,\lambda_2,\ldots,\lambda_m)|\leqq 1\quad a.s.$$

provided a_T satisfies conditions (i), (ii).

Proof. By Lemma 1 we have

$$P\{\sup_{0\leqq t\leqq T-a_T}\beta_T|\mathcal{G}_{a_T}(t)|\geqq(1+\varepsilon)^{\frac{1}{2}}\}$$
$$\leqq CmTa_T^{-1}(1+\varepsilon)^{\frac{1}{2}}\beta_T^{-1}a_T^{-\frac{1}{2}}\exp\{-(1+\varepsilon)(\log Ta_T^{-1}+\log\log T)\}$$
$$\leqq Cm(1+\varepsilon)^{\frac{1}{2}}(2(\log Ta_T^{-1}+\log\log T))^{\frac{1}{2}}(a_TT^{-1})^{\varepsilon}(\log T)^{-(1+\varepsilon)}.$$

Put $T_k=\Theta^k\,(\Theta>1)$. Then we get

$$\limsup_{k\to\infty}\sup_{0\leqq t\leqq T_k-a_{T_k}}\beta_{T_k}|\mathcal{G}_{a_{T_k}}(t)|\leqq 1\quad\text{a.s.}$$

Now, let $T_k\leqq T<T_{k+1}$. Then

$$\sup_{0\leqq t\leqq T-a_T}\beta_T|\mathcal{G}_{a_T}(t)|\leqq\sup_{0\leqq t\leqq T_{k+1}-a_{T_{k+1}}}\beta_{T_k}|\mathcal{G}_{a_{T_{k+1}}}(t)|$$
$$+\sum_{i=1}^{m}\lambda_i\sup_{0\leqq t\leqq T_{k+1}-a_{T_{k+1}}}\sup_{0\leqq u\leqq a_{T_{k+1}}-a_{T_k}}\beta_{T_k}|V(t;u)|.$$

Since by condition (ii) the inequality $a_{T_{k+1}}-a_{T_k}\leqq(\Theta-1)a_{T_k}$ holds true, choosing Θ near enough to one and applying Theorem B we get Lemma 3.

Lemma 4. *Let m be a given positive integer, $\mu_1,\mu_2,\ldots,\mu_m$ be a sequence of real numbers with $m\sum_{i=1}^{m}\mu_i^2=1$. Suppose also that a_T satisfies conditions* (i), (ii). *Then*

$$\liminf_{T\to\infty}\inf_{0\leqq t\leqq T-a_T}\max_{1\leqq i\leqq m}\left|\beta_TV\left(t+(i-1)\frac{a_T}{m};\frac{a_T}{m}\right)-\mu_i\right|=0\quad a.s.\tag{6}$$

Proof. Put

$$\lim_{T\to\infty} a_T T^{-1} = \rho$$

and assume at first that $\rho<1$. Then we can define the sequence $\{T_k;\ k=1,2,\ldots\}$ by $T_1=1$ and

$$T_{k+1}-a_{T_{k+1}}=T_k.$$

Instead of (6) the following stronger relation will be proved

$$\liminf_{T\to\infty} \max_{1\leq i\leq m} \left|\beta_{T_{k+1}} V\left(T_k+(i-1)\frac{a_{T_{k+1}}}{m};\frac{a_{T_{k+1}}}{m}\right)-\mu_i\right|=0 \quad \text{a.s.} \tag{7}$$

Along the line of the proof of Lemma 2 we get

$$P\left\{\max_{1\leq i\leq m}\left|\beta_{T_{k+1}} V\left(T_k+(i-1)\frac{a_{T_{k+1}}}{m};\frac{a_{T_{k+1}}}{m}\right)-\mu_i\right|\leq\varepsilon\right\}$$
$$\geq\left(\frac{a_{T_{k+1}}}{T_{k+1}\log T_{k+1}}\right)^{1-\varepsilon/4}.$$

Our conditions imply that

$$\sum_{k=1}^{\infty}\left(\frac{a_{T_{k+1}}}{T_{k+1}\log T_{k+1}}\right)^{1-\varepsilon/4}=\infty$$

which proves (7).

We have not yet covered the case $\rho=1$. Let us mention that in this case our conditions (i), (ii) imply that $a_T=T$ and (6) follow from Theorem A.

3. Proof of Theorem 1

Proof of (2). Let $s(x)\in\mathcal{S}$ be given. Then by Lemma 2 for any fixed integer m we have

$$\lim_{T\to\infty}\ \inf_{0\leq t\leq T-a_T}\ \max_{1\leq i\leq m}\left|\frac{V\left(t+(i-1)\frac{a_T}{m};\frac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}}-\left(s\left(\frac{i}{m}\right)-s\left(\frac{i-1}{m}\right)\right)\right|=0 \quad \text{a.s.}$$

Now, (2) follows from Theorem B, choosing a large enough m.

Proof of (3). Let $C(m,\rho)$ be the sphere of radius ρ around the origin of R^m that is $x=(x_1,x_2,\ldots,x_m)\in C(m,\rho)$ if $\|x\|=(x_1^2+x_2^2+\ldots+x_m^2)^{\frac{1}{2}}\leq\rho$. Then Lemma 3 implies that for all $\varepsilon>0$, $m=1,2,\ldots$ and for almost all $\omega\in\Omega$ there exists a $T_0=T_0(\varepsilon,m,\omega)$ such that the vector

$$\left(\frac{V\left(t;\frac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}},\frac{V\left(t+\frac{a_T}{m};\frac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}},\ldots,\frac{V\left(t+\frac{m-1}{m}a_T;\frac{a_T}{m}\right)}{(2a_T\log Ta_T^{-1})^{\frac{1}{2}}}\right)$$

belongs to $C(m,(1+\varepsilon)^{\frac{1}{2}})$ for all $0 \leqq t \leqq T - a_T$ if $T \geqq T_0$. Now, (3) follows again from Theorem B.

4. Proof of Theorem 2

Proof of (3*) is the same as that of (3).

Proof of (2*) is the same as that of (2) applying Lemma 4 instead of Lemma 2.

References

Csörgö, M., Révész, P.: How big are the increments of a Wiener process? Ann. Probability 7, - (1979)

Csörgö, M., Révész, P.: How big are the increments of a multiparameter Wiener process? Z. Wahrscheinlichkeitstheorie verw. Gebiete **42**, 1–12 (1978)

Qualls, P., Watanabe, H.: Asymptotic properties of Gaussian processes. Ann. Math. Statist. **43**, 580–596 (1972)

Slepian, P.: The one-sided barrier problem for Gaussian noise. Bell. System Tech. J. **41**, 463–501 (1962)

Chan, A.H.C., Csörgö, M., Révész, P.: Strassen type limit points for moving averages of a Wiener process. The Canad. J. Statist. **6**, 57–75 (1978)

Strassen, V.: An invariance principle for the law of iterated logarithm. Z. Wahrscheinlichkeitstheorie verw. Gebiete **3**, 211–226 (1964)

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