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On the Theory of Probabilistic Metric Spaces with Applications

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1. Introduction

The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis. Recently, some fixed point theorems for mappings in a probabilistic metric space (PM-space) were proved by several authors [1–5, 10, 13]. The purpose of this paper is to give some new fixed point theorems which generalize and unify the corresponding theorems stated above. As an example of applications, in § 5 we use the results of the type considered in § 3 to study the existence and uniqueness of solutions of nonlinear Volterra integral equations on probabilistic metric spaces.

2. Preliminaries

For the sake of convenience, following [2], we first introduce some basic definitions and concepts.

Throughout this paper R denotes the real, $R^+ = [0, \infty)$, Z^+ is the set of all positive integers.

Definition 1. A mapping F: $R \to R^+$ is called a distribution function if it is nondecreasing left-continuous with $\inf_{t \in R} F(t) = 0$, $\sup_{t \in R} F(t) = 1$.

In the sequel, we always denote by $\mathscr{D}$ the set of all distribution functions, and H will always denote the specific distribution function defined by

$$H(T) = \begin{cases} 0, & t \leqq 0, \\ 1 & t > 0. \end{cases}$$

Definition 2. A mapping Δ: $[0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(Δ-1) $\qquad \Delta(a, 1) = a, \quad \Delta(0,0) = 0;$

(Δ-2) $\qquad \Delta(a, b) = \Delta(b, a);$

(Δ-3) $$\Delta(c,d) \geqq \Delta(a,b) \text{ for } c \geqq a,\ d \geqq b;$$

(Δ-4) $$\Delta(\Delta(a,b),c) = \Delta(a, \Delta(b,c)).$$

We list here three of the simplest t-norms which will be used.

$\Delta_1 = \max\{\text{sum} - 1,\ 0\}$, i.e. $\Delta_1(a,b) = \max\{a+b-1,\ 0\}$, $\forall\, a, b \in [0,1]$.

$\Delta_2 = \text{product}$, i.e. $\Delta_2(a,b) = a \cdot b$, $\forall\, a, b \in [0,1]$.

$\Delta_3 = \min$, i.e. $\Delta_3(a,b) = \min\{a,b\}$, $\forall\, a, b \in [0,1]$.

Definition 3. A Menger PM-space (briefly Menger space) is a triplet $(E, \mathscr{F}, \Delta)$, where E is an abstract set of elements, $\mathscr{F}$ is a mapping of $E \times E \to \mathscr{D}$ and Δ is a t-norm satisfying the following conditions (we shall denote the distribution function $\mathscr{F}(x,y)$ by $F_{x,y}$):

(PM-1) $$F_{x,y} = H \text{ if and only if } x = y;$$

(PM-2) $$F_{x,y}(0) = 0;$$

(PM-3) $$F_{x,y} = F_{y,x};$$

(PM-4) $$F_{x,z}(t_1 + t_2) \geqq \Delta(F_{x,y}(t_1),\ F_{y,z}(t_2)),\ \forall\, x, y, z \in E,\ t_1, t_2 \geqq 0.$$

Schweizer, Sklar [8] have pointed out that if $(E, \mathscr{F}, \Delta)$ is a Menger space with continuous t-norm Δ, then $(E, \mathscr{F}, \Delta)$ is a Hausdorff space in the topology $\mathscr{T}$ induced by the family of neighborhoods

$$\{U_p(\varepsilon, \lambda):\ p \in E,\ \varepsilon > 0,\ \lambda > 0\}, \tag{2.1}$$

where $U_p(\varepsilon, \lambda) = \{x \in E, F_{x,p}(\varepsilon) > 1 - \lambda\}$. From the topology $\mathscr{T}$ we can derive some notions as follows:

Definition 4. Let $(E, \mathscr{F}, \Delta)$ be a Menger space with continuous t-norm Δ. A sequence $\{x_n\} \subset E$ is said to be $\mathscr{T}$-convergent to $x \in E$ (we write $x_n \xrightarrow{\mathscr{T}} x$) if for any $\varepsilon > 0$ and $\lambda > 0$, there exists a $N = N(\varepsilon, \lambda) \in Z^+$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ whenever $n \geqq N$.

$\{x_n\} \subset E$ is called a $\mathscr{T}$-Cauchy sequence if for any $\varepsilon > 0$, $\lambda > 0$, there exists a $N = N(\varepsilon, \lambda) \in Z^+$ such that $F_{x_n, x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geqq N$.

A Menger space $(E, \mathscr{F}, \Delta)$ is said to be $\mathscr{T}$-complete if each $\mathscr{T}$-Cauchy sequence in E converges in $\mathscr{T}$ to an element in E.

Definition 5. Let $(E, \mathscr{F}, \Delta)$ be a Menger space with continuous t-norm Δ, T a self-mapping on $(E, \mathscr{F}, \Delta)$. T is said to be $\mathscr{T}$-continuous if, whenever $\{x_n\} \subset E$ converges in $\mathscr{T}$ to a point $x \in E$, then $Tx_n \xrightarrow{\mathscr{T}} Tx$ $(n \to \infty)$.

Lemma 1 [8]. *Let $(E, \mathscr{F}, \Delta)$ be a Menger space with continuous t-norm. Then $\{x_n\} \subset E$ is $\mathscr{T}$-convergent to $x \in E$ if and only if for each $t \in R$*

$$\lim_{n \to \infty} F_{x_n, x}(t) = H(t).$$

3. Common Fixed Point Theorems for a Sequence of Mappings

Definition 6. A set $B \subset (E, \mathscr{F}, \Delta)$ is called probabilistically bounded if

$$\sup_{t>0} \inf_{p,q \in B} F_{p,q}(t) = 1.$$

Throughout this section we always assume that $(E, \mathscr{F}, \Delta)$ is a $\mathscr{T}$-complete Menger space, Δ is a continuous t-norm which is stronger than $\Delta_1 = \max\{\text{sum} - 1, 0\}$, i.e. $\Delta(a,b) \geqq \Delta_1(a,b)$, $\forall a, b \in [0,1]$, and the function $\Phi(t)$ satisfies the following condition (Φ):

(Φ): $\Phi(t)$: $R^+ \to R^+$ is strictly increasing, $\Phi(0) = 0$, and $\Phi^n(t) \to \infty$ $(n \to \infty)$, $\forall t > 0$, where $\Phi^n(t)$ denotes the n-th iteration of $\Phi(t)$.

Lemma 2 [8]. Let $(E, \mathscr{F}, \Delta)$ and Δ be the same as above. Suppose that $x_n \xrightarrow{\mathscr{T}} x$, $y_n \xrightarrow{\mathscr{T}} y$ and that $F_{x,y}$ is continuous at $t_0 \in R$. Then

$$\lim_{n \to \infty} F_{x_n, y_n}(t_0) = F_{x,y}(t_0).$$

In particular, under the hypotheses of Lemma 2, if $y_n \xrightarrow{\mathscr{T}} y$, then

$$\lim_{n \to \infty} F_{x, y_n}(t_0) = F_{x,y}(t_0).$$

Theorem 1. *Let $(E, \mathscr{F}, \Delta), \Delta$, and Φ be the same as above. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of self-mappings on $(E, \mathscr{F}, \Delta)$. Suppose that there exists a functional sequence $\{m_n(x)\}_{n=1}^{\infty}$: $(E, \mathscr{F}, \Delta) \to Z^+$ such that for each $n \in Z^+$ and each $x \in E$, $m_n(x) | m_n(T_n x)$ and that for any $i, j \in Z^+$, $i \neq j$ and $x, y \in E$ the following holds:*

$$F_{T_i^{m_i(x)}x,\, T_j^{m_j(y)}y}(t) \geqq \min_{p,q \in \{x, y, T_i^{m_i(x)}x, T_j^{m_j(y)}y\}} F_{p,q}(\Phi(t)), \ \forall t \geqq 0. \tag{3.1}$$

Suppose further that there exists some $x_0 \in E$ such that the set $\{x_n\}_{n=0}^{\infty} \subset E$,

$$x_n = T_n^{m_n(x_{n-1})} x_{n-1}, \quad n = 1, 2, \ldots \tag{3.2}$$

is probabilistically bounded. Then there exists a unique common fixed point $x_* \in E$, and $x_n \xrightarrow{\mathscr{T}} x_*$.

Proof. For given $x_0 \in E$ we prove that the sequence $\{x_n\}$ defined by (3.2) is a $\mathscr{T}$-Cauchy sequence of E.

In fact, for any $i, j \in Z^+$, $i \neq j$ it follows from (3.1) that, $\forall t \geqq 0$,

$$F_{x_i, x_j}(t) = F_{T_i^{m_i(x_{i-1})}x_{i-1},\, T_j^{m_j(x_{j-1})}x_{j-1}}(t) \geqq \min_{p,q \in \{x_{i-1}, x_{j-1}, x_i, x_j\}} F_{p,q}(\Phi(t)). \tag{3.3}$$

Therefore for any $m, n \in Z^+$ $(m < n)$ from (3.3) we have

$$\inf_{m \leqq i, j \leqq n} F_{x_i, x_j}(t) \geqq \inf_{m-1 \leqq i, j \leqq n} F_{x_i, x_j}(\Phi(t)).$$

By the arbitrariness of $n \in Z^+$ $(n>m)$ we have

$$\inf_{i,j \geqq m} F_{x_i,x_j}(t) \geqq \inf_{i,j \geqq m-1} F_{x_i,x_j}(\Phi(t)) \geqq \sup_{u<\Phi(t)} \inf_{i,j \geqq m-1} F_{x_i,x_j}(u). \tag{3.4}$$

By induction, it is easy to prove

$$\inf_{i,j \geqq m} F_{x_i,x_j}(t) \geqq \inf_{i,j \geqq 0} F_{x_i,x_j}(\Phi^m(t)) \geqq$$
$$\geqq \sup_{u<\Phi^m(t)} \inf_{i,j \geqq 0} F_{x_i,x_j}(u), \quad m=1,2,\ldots, \forall t \geqq 0. \tag{3.5}$$

In view of condition (Φ) and the probabilistic boundedness of the set $\{x_n\}$ defined by (3.2) it follows that

$$\lim_{m\to\infty} \inf_{i,j \geqq m} F_{x_i,x_j}(t) \geqq \lim_{m\to\infty} \big(\sup_{u<\Phi^m(t)} \inf_{i,j \geqq 0} F_{x_i,x_j}(u)\big)$$
$$= \begin{cases} 0, & t=0, \\ \sup_{u>0} \inf_{i,j \geqq 0} F_{x_i,x_j}(u), & t>0, \end{cases}$$
$$= \begin{cases} 0, & t=0, \\ 1, & t>0, \end{cases} = H(t), \quad \forall t \geqq 0. \tag{3.6}$$

Consequently for any given $\varepsilon>0$ and $\lambda>0$ there exists $N=N(\varepsilon,\lambda) \in Z^+$ such that

$$\inf_{i,j \geqq m} F_{x_i,x_j}(\varepsilon) > 1-\lambda, \quad \forall m \geqq N.$$

Therefore we have

$$F_{x_i,x_j}(\varepsilon) > 1-\lambda, \quad \forall i, j \geqq N. \tag{3.7}$$

This implies that the sequence $\{x_n\}$ defined by (3.2) is a $\mathscr{T}$-Cauchy sequence in E. By the $\mathscr{T}$-completeness of E we can suppose that $x_n \xrightarrow{\mathscr{T}} x_* \in E$.

Now we prove that x_* is a common periodic point of $\{T_n\}_{n=1}^{\infty}$, i.e.

$$T_n^{m_n(x_*)} x_* = x_*, \quad n=1,2,\ldots.$$

Indeed, for any $i \in Z^+$, it follows from (3.1) that for any $n>i$ we have

$$F_{x_n, T_i^{m_i(x_*)} x_*}(t) = F_{T_n^{m_n(x_{n-1})} x_{n-1}, T_i^{m_i(x_*)} x_*}(t)$$
$$\geqq \min_{p,q \in \{x_{n-1}, x_*, x_n, T_i^{m_i(x_*)} x_*\}} F_{p,q}(\Phi(t)), \quad \forall t \geqq 0. \tag{3.8}$$

Let G_0 be the set of all discontinuity points of $F_{x_*, T_i^{m_i(x_*)} x_*}(t)$. Since Φ^m is strictly increasing, we know that $\Phi^{-m}(G_0)$ is the set of all discontinuity points of $F_{x_*, T_i^{m_i(x_*)} x_*}(\Phi^m(t))$, $m=1,2,\ldots,$. Moreover, G_0, $\Phi^{-m}(G_0)$, $m=1,2,\ldots$ are all countable sets, therefore

$$G = G_0 \cup \Big(\sum_{m=1}^{\infty} \Phi^{-m}(G_0) \Big)$$

is also countable. Let $\tilde{G} = R^+ \setminus G$. When $t=0$ or $t \in \tilde{G}$ (i.e. t is a common continuity point of $F_{x_*, T_i^{m_i(x_*)} x_*}(t)$, and $F_{x_*, T_i^{m_i(x_*)} x_*}(\Phi^m(t))$, $m=1,2,\ldots$) it follows from (3.8) and Lemma 2 that

$$F_{x_* T_i^{m_i(x_*)} x_*}(t) = \lim_{n\to\infty} F_{x_n, T_i^{m_i(x_*)} x_*}(t)$$
$$\geqq F_{x_*, T_i^{m_i(x_*)} x_*}(\Phi(t)).$$

Repeating this procedure we can prove that

$$F_{x_*, T_i^{m_i(x_*)} x_*}(t) \geqq F_{x_*, T_i^{m_i(x_*)} x_*}(\Phi(t)) \geqq \dots \geqq F_{x_*, T_i^{m_i(x_*)} x_*}(\Phi^n(t)).$$

Letting $n\to\infty$ and noting condition (Φ) we have

$$F_{x_*, T_i^{m_i(x_*)} x_*}(t) = H(t), \quad \forall\, t \in \tilde{G} \text{ or } t=0. \tag{3.9}$$

When $t\in G$ with $t>0$, by the density of real numbers there exist $t_1, t_2 \in \tilde{G}$ such that $0<t_1<t<t_2$. A distribution function being nondecreasing, we have from (3.9):

$$1 = H(t_1) = F_{x_*, T_i^{m_i(x_*)} x_*}(t_1) \leqq F_{x_*, T_i^{m_i(x_*)} x_*}(t) \leqq F_{x_*, T_i^{m_i(x_*)} x_*}(t_2) = 1.$$

This shows that for all $t\in G$ with $t>0$

$$F_{x_*, T_i^{m_i(x_*)} x_*}(t) = H(t). \tag{3.10}$$

Combining (3.9) with (3.10) we have

$$F_{x_*, T_i^{m_i(x_*)} x_*}(t) = H(t), \qquad \forall\, t \geqq 0. \tag{3.11}$$

i.e.

$$x_* = T_i^{m_i(x_*)} x_*, \qquad i=1,2,\dots.$$

To prove that x_* is the unique periodic point of $\{T_n\}_{n=1}^{\infty}$ we proceed as follows.

Suppose that $y_* \in E$ is another periodic point of some T_j, i.e.

$$y_* = T_j^{m_j(y_*)} y_*.$$

Hence for any $i\in Z^+$, $i\neq j$, we have

$$F_{x_*, y_*}(t) = F_{T_i^{m_i(x_*)} x_*, T_j^{m_j(y_*)} y_*}(t) \geqq \min_{p,q\in\{x_*, y_*, x_*, y_*\}} F_{p,q}(\Phi(t))$$
$$= F_{x_*, y_*}(\Phi(t)), \qquad \forall\, t\geqq 0.$$

Repeating this procedure we can prove

$$F_{x_*, y_*}(t) \geqq F_{x_*, y_*}(\Phi^n(t)), \qquad n=1,2,\dots, \forall\, t\geqq 0.$$

Letting $n\to\infty$ and noting condition (Φ) we get

$$F_{x_*, y_*}(t) = H(t), \qquad \forall\, t\geqq 0, \text{ i.e. } x_* = y_*.$$

Furthermore, by the assumption that $m_n(x) | m_n(T_n(x))$, $\forall\, x\in E$ and for all $n\in Z^+$, there exists for each $i=1,2,\dots$ some $k_i\in Z^+$ such that

$$m_i(T_i x_*) = k_i m_i(x_*). \tag{3.11}$$

Hence we have for each $i=1,2,\dots$

$$T_i x_* = T_i T_i^{m_i(x_*)} x_* = T_i T_i^{2m_i(x_*)} x_* = \dots = T_i T_i^{k_i m_i(x_*)} x_*$$
$$= T_i T_i^{m_i(T_i x_*)} x_* = T_i^{m_i(T_i x_*)} T_i x_*.$$

This implies that $T_i x_*$ is also a periodic point of T_i. Since x_* is the unique periodic point of T_i, we have $x_* = T_i x_*$, $i=1,2,\dots$. This means that x_* is the desired unique common fixed point of $\{T_n\}_{n=1}^\infty$.

This completes the proof of Theorem 1.

Taking $\Phi(t)=\frac{t}{h}$, $h\in(0,1)$, $t\geqq 0$ it is easy to see that $\Phi(t)$ satisfies the condition (Φ). From Theorem 1 we get the following

Corollary 1. *Let $\{T_n\}_{n=1}^\infty$, $(E,\mathscr{F},\Delta)$ and $\{m_n(x)\}_{n=1}^\infty$ be the same as in Theorem 1. Suppose that for any $i,j\in Z^+$, $i\neq j$, and any $x,y\in E$ the following holds*

$$F_{T_i^{m_i(x)}x,\, T_j^{m_j(y)}y}(t) \geqq \min_{p,q\in\{x,\,y,\,T_i^{m_i(x)}x,\,T_j^{m_j(y)}y\}} F_{p,q}\left(\frac{t}{h}\right), \quad \forall\, t\geqq 0. \tag{3.11}$$

Suppose further that there exists $x_0\in E$ such that the set $\{x_n\}_{n=0}^\infty$ defined by (3.2) *is probabilistically bounded. Then the conclusions of Theorem* 1 *still hold.*

Remark 1. A particular case of Corollary 1 with $T_n=T$, $n=1,2,\dots,m(x)=m_i(x)$, $i=1,2,\dots$, appears in Istrăţescu [4].

Corollary 2. *Let $\{T_n\}_{n=1}^\infty$ be a sequence of self-mappings on $(E,\mathscr{F},\Delta)$. Suppose that there exists a sequence $\{m_n\}_{n=1}^\infty\subset Z^+$ such that for any x, $y\in E$ and any $i,j\in Z^+$, $i\neq j$ the following holds*

$$F_{T_i^{m_i}x,\, T_j^{m_j}y}(t) \geqq \min_{p,q\in\{x,\,y,\,T_i^{m_i}x,\,T_j^{m_j}y\}} F_{p,q}\left(\frac{t}{h}\right), \quad \forall\, t\geqq 0. \tag{3.12}$$

Suppose further that there exists $x_0\in E$ such that the set $\{x_n\}_{n=0}^\infty$ defined by (3.2) *is probabilistically bounded. Then the conclusion of Theorem* 1 *still holds.*

Remark 2. The main result of Sehgal, Bharucha-Reid [10] is a particular case of Corollary 2.

4. Common Fixed Point Theorems for a Pair of Mappings

Throughout this section we suppose that $(E,\mathscr{F},\Delta)$ is a $\mathscr{T}$-complete Menger space with continuous t-norm Δ, and that the function $\Phi(t)$ satisfies the condition (Φ). Suppose further that S, T are self-mappings on $(E,\mathscr{F},\Delta)$ and that they are commutable and $\mathscr{T}$-continuous. Furthermore we denote

$$O_{S,T}(x;0,\infty)=\{S^i T^j x\}_{i,j=0}^\infty, \qquad \forall\, x\in E.$$

Theorem 2. *Let $(E,\mathscr{F},\Delta)$, S, T and Φ be the same as above. Suppose that for each $x\in E$ the set $O_{S,T}(x;0,\infty)$ is probabilistically bounded. Suppose further that there* *exist $m, m' \in Z^+$, $m'+m \geqq 1$ such that for each $x \in E$ and all $t \geqq 0$ the following holds*

$$\inf_{p,q \in O_{ST}(S^m T^{m'} x;\, 0,\infty)} F_{p,q}(t) \geqq \inf_{p,q \in O_{S,T}(x;\, 0,\infty)} F_{p,q}(\Phi(t)). \tag{4.1}$$

Then for each $x_0 \in E$, the sequence $\{S^n T^n x_0\}_{n=0}^{\infty}$ converges in $\mathscr{T}$ to some common fixed point $x_* \in E$ of S and T.

Proof. Letting $h = \max\{m, m'\}$ for any given $x_0 \in E$, from (4.1) we have

$$\begin{aligned}\inf_{p,q \in O_{S,T}(S^h T^h x_0;\, 0,\infty)} F_{p,q}(t) &\geqq \inf_{p,q \in O_{S,T}(S^m T^{m'} x_0;\, 0,\infty)} F_{p,q}(t) \\ &\geqq \inf_{p,q \in O_{S,T}(x_0;\, 0,\infty)} F_{p,q}(\Phi(t)) \geqq \sup_{u<\Phi(t)} \inf_{p,q \in O_{S,T}(x_0;\, 0,\infty)} F_{p,q}(u), \quad \forall t \geqq 0.\end{aligned}$$

By induction we can prove that the following inequality holds:

$$\begin{aligned}\inf_{p,q \in O_{S,T}(S^{nh} T^{nh} x_0;\, 0,\infty)} F_{p,q}(t) &\geqq \inf_{p,q \in O_{S,T}(S^{(n-1)h} T^{(n-1)h} x_0;\, 0,\infty)} F_{p,q}(\Phi(t)) \\ &\geqq \ldots \geqq \inf_{p,q \in O_{S,T}(x_0;\, 0,\infty)} F_{p,q}(\Phi^n(t)) \\ &\geqq \sup_{u<\Phi^n(t)} \inf_{p,q \in O_{S,T}(x_0;\,0,\infty)} F_{p,q}(u), \quad \forall t \geqq 0.\end{aligned} \tag{4.2}$$

Invoking condition (Φ) from (4.2) we have

$$\begin{aligned}&\lim_{n\to\infty} \inf_{p,q \in O_{S,T}(S^{nh} T^{nh} x_0;\, 0,\infty)} F_{p,q}(t) \\ &\quad\geqq \lim_{n\to\infty} \sup_{u<\Phi^n(t)} \inf_{p,q \in O_{S,T}(x_0;\, 0,\infty)} F_{p,q}(u) \\ &\quad= \begin{cases} \sup\limits_{u>0} \inf\limits_{p,q \in O_{S,T}(x_0;\, 0,\infty)} F_{p,q}(u), & t>0 \\ 0, & t=0 \end{cases} = H(t), \ \forall t \geqq 0.\end{aligned}$$

Therefore for any given $\varepsilon>0$ and $\lambda>0$ there exists a positive integer $N = N(\varepsilon, \lambda)$ such that

$$\inf_{p,q \in O_{S,T}(S^{nh} T^{nh} x_0;\, 0,\infty)} F_{p,q}(\varepsilon) > 1-\lambda, \quad \forall n \geqq N.$$

This implies that all subsequences of $\{S^i T^j x_0\}_{i,j=0}^{\infty}$ in which the indexes i and j are both convergent to ∞, are $\mathscr{T}$-Cauchy sequences. By the $\mathscr{T}$-completeness of E they all converge in $\mathscr{T}$ to the same limit $x_* \in E$. In particular, the following three subsequences converge in $\mathscr{T}$ to the limit x_*:

$$\{S^n T^n x_0\}_{n=0}^{\infty}, \quad \{S(S^n T^n) x_0\}_{n=0}^{\infty}, \quad \{T(S^n T^n x_0)\}_{n=0}^{\infty}.$$

By the $\mathscr{T}$-continuity of S and T we get

$$x_* = S x_* = T x_*,$$

i.e. x_* is a common fixed point of S and T.

This completes the proof of Theorem 2.

Theorem 3. *Let $(E, \mathscr{F}, \Delta)$, S, T and Φ be the same as in Theorem 2. Suppose that for each $x \in E$ the set $O_{S,T}(x; 0, \infty)$ is probabilistically bounded. Suppose further that there exist $m, m', n, n' \in Z^+$ with $m+m' \geqq 1$, $n+n' \geqq 1$ such that for any $x, y \in E$ and any $t \geqq 0$ the following holds:*

$$\inf_{\substack{p,q \in \{O_{S,T}(S^m T^{m'} x; 0, \infty) \\ \cup O_{S,T}(S^n T^{n'} y; 0, \infty)\}}} F_{p,q}(t) \geqq \inf_{\substack{p,q \in \{O_{S,T}(x; 0, \infty) \\ \cup O_{S,T}(y; 0, \infty)\}}} F_{p,q}(\Phi(t)). \tag{4.3}$$

Then there exists a unique common fixed point x_* of S and T in E and for any $x_0 \in E$ the sequence $\{S^n T^n x_0\}_{n=0}^{\infty}$ converges in $\mathscr{T}$ to x_*.

Proof. Letting $h = \max\{m, m' n, n'\}$ and taking $y = x$ in (4.3) we have

$$\inf_{p,q \in O_{S,T}(S^h T^h x; 0, \infty)} F_{p,q}(t) \geqq \inf_{p,q \in O_{S,T}(x; 0, \infty)} F_{p,q}(\Phi(t)), \quad \forall x \in E, \ t \geqq 0.$$

By Theorem 2 we know that for each $x_0 \in E$ the sequence $\{S^n T^n x_0\}$ converges in $\mathscr{T}$ to some common fixed point $x_* \in E$ of S and T.

Now we prove that x_* is the unique common fixed point of S and T in E.

Suppose this is not the case, hence there exists another common fixed point $y_* \in E$ of S and T. From (4.3) we have

$$\begin{aligned} F_{x_*, y_*}(t) &= \inf_{p,q \in \{O_{S,T}(S^m T^{m'} x_*; 0, \infty) \cup O_{S,T}(S^n T^{n'} y_*; 0, \infty)\}} F_{p,q}(t) \\ &\geqq \inf_{p,q \in \{O_{S,T}(x_*; 0, \infty) \cup O_{S,T}(y_*; 0, \infty)\}} F_{p,q}(\Phi(t)) = F_{x_*, y_*}(\Phi(t)), \quad \forall t \geqq 0. \end{aligned}$$

Repeating this procedure we can prove

$$F_{x_*, y_*}(t) \geqq F_{x_*, y_*}(\Phi^n(t)), \quad \forall t \geqq 0, \ n = 1, 2, \dots.$$

Letting $n \to \infty$ and using the condition (Φ) we get

$$F_{x_*, y_*}(t) = H(t), \quad \forall t \geqq 0, \text{ i.e. } x_* = y_*.$$

This completes the proof of Theorem 3.

Remark 3. The special cases of Theorem 3 with $S = T$ and $m' = n' = 0$ appear in Chang [3].

As a consequence of Theorem 3 we have the following result.

Corollary 3. *Let $(E, \mathscr{F}, \Delta)$ and Φ be the same as in Theorem 3. Suppose that T is a $\mathscr{T}$-continuous self-mapping on $(E, \mathscr{F}, \Delta)$ and that for each $x \in E$ the set $O_T(x; 0, \infty) = \{T^n x\}_{n=0}^{\infty}$ is probabilistically bounded. Suppose further that*

(i) *there exists $m \in Z^+$ such that for all $t \geqq 0$*

$$\inf_{p,q \in O_T(T^m x; 0, \infty)} F_{p,q}(t) \geqq \inf_{p,q \in O_T(x; 0, \infty)} F_{p,q}(\Phi(t)), \quad \forall x \in E.$$

Then for any $x_0 \in E$ the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges in $\mathscr{T}$ to some fixed point in E.

(ii) *there exist $m, n \in Z^+$ such that for all $x, y \in E$ and all $t \geqq 0$*

$$\inf_{p,q \in \{O_T(T^m x; 0, \infty) \cup O_T(T^n y; 0, \infty)\}} F_{p,q}(t) \geqq \inf_{p,q \in \{O_T(x; 0, \infty) \cup O_T(y; 0, \infty)\}} F_{p,q}(\Phi(t)).$$

Then for any $x_0 \in E$ the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges in $\mathcal{T}$ to some fixed point of T in E.

Proof. Conclusion of (i) is a special case of Theorem 2 with $S=I$ (the identical mapping), and conclusion (ii) is a special case of Theorem 3 with $S=I$.

5. Application

As an application, in this section we use some results stated in Sect. 3 to study the existence and uniqueness of the solution of nonlinear Volterra integral equations on a kind of particular probabilistic metric spaces.

Definition 7. Let (E,d) be a metric space. The space $(E,\mathcal{F},\Delta)$ is called an induced Menger space if $\Delta=\Delta_3=\min$(i.e. $\Delta_3(a,b)=\min\{a,b\}$, $\forall\, a,b\in[0,1]$) and $\mathcal{F}$ is a mapping from $E\times E\to\mathcal{D}$ defined via $\mathcal{F}(x,y)=F_{x,y}$, where

$$F_{x,y}(t)=H(t-d(x,y)), \qquad \forall\, x,\ y\in E,\ t\in R.$$

It is well-known that if (E,d) is a complete metric space, then the induced Menger space $(E,\mathcal{F},\Delta_3)$ is a $\mathcal{T}$-complete Menger space (see [10], Theorem 2), and that the sequence $\{x_n\}\subset E$ converges in $\mathcal{T}$ to a point $x_*\in E$ if and only if $\{x_n\}\subset E$ converges in the metric d to x_*.

In what follows let $[0,a]$ be a fixed real interval $(0<a<\infty)$ and $(E,\|\cdot\|_E)$ a real Banach space. We denote by $C([0,a];E)$ the Banach space of all E-valued continuous functions defined on $[0,a]$ with norm defined by

$$\|x\|_C=\sup_{0\le t\le a}\|x(t)\|_E, \qquad x(t)\in C([0,a];E). \tag{5.1}$$

Besides the norm $\|\cdot\|_C$, the space $C([0,a];E)$ can be endowed with another norm $\|\cdot\|_*$ which is defined as follows:

$$\|x\|_*=\sup_{0\le t\le a}(e^{-Lt}\|x(t)\|_E), \tag{5.2}$$

where L is any positive number. It is clear that the norm $\|\cdot\|_*$ is equivalent to the norm $\|\cdot\|_C$.

In the sequel we also denote by $(C([0,a];E),\mathcal{F},\Delta_3)$ the induced Menger space where $\mathcal{F}$ is the mapping from $C([0,a];E)\times C([0,a];E)$ into $\mathcal{D}$ defined by

$$F_{x,y}(t)=H(t-\|x-y\|_*), \qquad x(s),\ y(s)\in C([0,a];E),\ t\in R.$$

Now we study the existence and uniqueness of solutions of the following kind of nonlinear Volterra integral equations

$$x(t)=y(t)+\int_0^t K(t,s,x(s))\,ds, \qquad 0\le t\le a, \tag{5.3}$$

where $y(t)\in C([0,a],E)$ is any given function.

Theorem 4. *Let $(E,\|\cdot{}_E)$, $C([0,a];E)$ and $(C([0,a];E),\mathcal{F},\Delta_3)$ be the same as stated as above. Suppose that the following conditions are satisfied*

(i) $K(t,s,x(s))\in C([0,a]\times[0,a]\times C([0,a];E);E)$, *and*

$$\|K\|_C = \sup_{t,s\in[0,a],x\in E} \|K(s,t,x)\|_E < \infty;$$

(ii) *there exist* $m\in Z^+$ *and a constant* $\beta\in(0,1)$ *such that*

$$F_{T^m x, T^m y}(t) = \min_{p,q\in\{x,y,T^m x,T^m y\}} F_{p,q}\left(\frac{t}{\beta}\right), \quad \forall x,\ y\in C([0,a];E),\ t\in R^+,$$

where the mappings T and T^m *are defined as follows:*

$$(Tx)(t) = y(t) + \int_0^t K(t,s,x(s))\,ds,$$

$$(T^m x)(t) = y(t) + \int_0^t K(t,s,T^{m-1}x(s))\,ds;$$

(iii) *for any* $x(t)\in C([0,a];E)$ *the set* $O_T(x;0,\infty)=\{T^n x(t)\}_{n=0}^{\infty}$ *is bounded. Then for any* $x_0(t)\in C([0,a];E)$ *the sequence* $\{T^n x_0(t)\}_{n=0}^{\infty}$ *converges in the norm* $\|\cdot\|_C$ *to a solution* $x_*(t)\in C([0,a];E)$ *of equation* (5.3).

Proof. The conclusion follows immediately from Corollary 2.

For similar results, see [6].

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